Fuzzy Sets and Fuzzy Techniques

Lecture 8 – Operations on Fuzzy Sets

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Outline

Standard fuzzy operations

Recall:
for fuzzy sets A, B on a reference set X, given by the corresponding membership functions $A(x)$ and $B(x)$:

$$\bar{A}(x) = 1 - A(x) \quad \text{– fuzzy complement}$$
$$(A \cap B)(x) = \min[A(x), B(x)] \quad \text{– fuzzy intersection}$$
$$(A \cup B)(x) = \max[A(x), B(x)] \quad \text{– fuzzy union}$$

for all $x \in X$.

Properties of the standard operations

- They are **generalizations** of the corresponding (uniquely defined!) classical set operations.
- They satisfy the **cutworthy and strong cutworthy** properties. They are the only ones that do.
- The standard fuzzy **intersection** of two sets **contains** (is bigger than) all other fuzzy intersections of those sets.
- The standard fuzzy **union** of two sets **is contained in** (is smaller than) all other fuzzy unions of those sets.
- They inherently **prevent the compound of errors** of the operands.

Other generalizations of the set operations

Aggregation operators

Recall:
Aggregation operators are used to combine several fuzzy sets in order to produce a single fuzzy set.

Associative aggregation operations

- (general) fuzzy intersections - t-norms
- (general) fuzzy unions - t-conorms

Non-associative aggregation operations

- averaging operations - **idempotent** aggregation operations

Fuzzy complements

Definition

- A fuzzy complement cA of a fuzzy set A is given by a function $c : [0,1] \to [0,1]$
- Function c assigns a value to each membership value $A(x)$ of $x \in X$ to the fuzzy set A.
- A membership function of the set cA is defined as $cA(x) = c(A(x))$.

Note:

- the value $cA(x)$ is interpreted not only as the degree to which $x \in X$ belongs to the fuzzy set cA, but also as *the degree to which x does not belong to the fuzzy set A*;
- the value $cA(x)$ does not depend on x, but only on $A(x)$.

Fuzzy complements

Axiomatic requirements

Ax c1. $c(0) = 1$ and $c(1) = 0$. **boundary condition**

Ax c2. For all $a, b \in [0,1]$, if $a \leq b$, then $c(a) \geq c(b)$. **monotonicity**

c1 and **c2** are called *axiomatic skeleton for fuzzy complements*

Ax c3. c is a **continuous** function.

Ax c4. c is **involutive**, i.e., $c(c(a)) = a$, for each $a \in [0,1]$.

Theorem

Let a function $c : [0,1] \to [0,1]$ satisfy **Ax c2** *and* **Ax c4**. *Then c satisfies Axioms* **Ax c1** *and* **Ax c3** *too. Moreover, the function c is a bijection.*

Nested structure of the basic classes of fuzzy complements

- All functions $c : [0,1] \to [0,1]$

  $IsNotC(a) = a$

- All fuzzy complements (**Ax c1 and Ax c2)**

  $$c(a) = \begin{cases} 1 & \text{for } a \leq t \\ 0 & \text{for } a > t \end{cases}$$

- All continuous fuzzy complements (**Ax c1- Ax c3)**

  $c(a) = \frac{1}{2}(1 + \cos \pi a) \qquad c(\frac{1}{3}) = ?$

- All involutive fuzzy complements (**Ax c1- Ax c4)**

  $c_\lambda(a) = \dfrac{1-a}{1+\lambda a}, \quad \lambda > -1$ (Sugeno class)

  $c_\omega(a) = (1 - a^\omega)^{\frac{1}{\omega}} \quad \omega > 0$ (Yager class)

  $cA(x) = 1 - A(x)$ **Classical fuzzy complement**

Generators

Increasing generators

- **Increasing generator**
  is a strictly increasing continuous function
  $g : [0,1] \to R$, such that $g(0) = 0$.
- A **pseudo-inverse** of increasing generator g is defined as

$$g^{(-1)} = \begin{cases} 0 & \text{for } a \in (-\infty, 0) \\ g^{-1}(a) & \text{for } a \in [0, g(1)] \\ 1 & \text{for } a \in (g(1), \infty) \end{cases}$$

- An example:

$$g(a) = a^p, \quad p > 0$$

$$g^{(-1)}(a) = \begin{cases} 0 & \text{for } a \in (-\infty, 0) \\ a^{\frac{1}{p}} & \text{for } a \in [0, 1] \\ 1 & \text{for } a \in (1, \infty) \end{cases}$$

Generators

Decreasing generators

- **Decreasing generator**
  is a strictly decreasing continuous function
  $f : [0,1] \to R$, such that $f(1) = 0$.
- A **pseudo-inverse** of increasing generator f is defined as

$$f^{(-1)} = \begin{cases} 1 & \text{for } a \in (-\infty, 0) \\ f^{-1}(a) & \text{for } a \in [0, f(0)] \\ 0 & \text{for } a \in (f(0), \infty) \end{cases}$$

- An example:

$$f(a) = 1 - a^p, \quad p > 0$$

$$f^{(-1)}(a) = \begin{cases} 1 & \text{for } a \in (-\infty, 0) \\ (1-a)^{\frac{1}{p}} & \text{for } a \in [0, 1] \\ 0 & \text{for } a \in (1, \infty) \end{cases}$$

Generators

Making one from another

Theorem

Let f be a decreasing generator f. The function g, defined by

$$g(a) = f(0) - f(a), \qquad \text{for } a \in [0,1]$$

is an increasing generator, with $g(1) = f(0)$. Its pseudo-inverse is then given by

$$g^{(-1)}(a) = f^{(-1)}(f(0) - a), \qquad \text{for } a \in R.$$

Theorem

Let g be an increasing generator. The function f, defined by

$$f(a) = g(1) - g(a), \qquad \text{for } a \in [0,1]$$

is a decreasing generator, with $f(0) = g(1)$. Its pseudo-inverse is then given by

$$f^{(-1)}(a) = g^{(-1)}(g(1) - a), \qquad \text{for } a \in R.$$

Generating fuzzy complements

Theorem

(First Characterization Theorem of Fuzzy Complements.)
*Let c be a function from $[0,1]$ to $[0,1]$. Then c is a (involutive) fuzzy complement iff there exists an **increasing generator** g such that, for all $a \in [0,1]$*

$$c(a) = g^{-1}(g(1) - g(a)).$$

Theorem

(Second Characterization Theorem of Fuzzy Complements.)
*Let c be a function from $[0,1]$ to $[0,1]$. Then c is a (involutive) fuzzy complement iff there exists an **decreasing generator** f such that, for all $a \in [0,1]$*

$$c(a) = f^{-1}(f(0) - f(a)).$$

Generating fuzzy complements

Examples

Increasing generators

Standard fuzzy complement: $g(a) = a$.

Sugeno class of fuzzy complements:

$$g_\lambda(a) = \frac{1}{\lambda}\ln(1+\lambda a), \text{ for } \lambda > -1$$

Yager class of fuzzy complements: $g_\omega(a) = a^\omega$, for $\omega > 0$.

Decreasing generators

Standard fuzzy complement: $f(a) = -ka + k$ for $k > 0$.

Yager class of fuzzy complements: $f(a) = 1 - a^\omega$.

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Fuzzy intersections

Definition

An **intersection** of two fuzzy sets A and B is given by a function of the form

$$i : [0,1] \times [0,1] \to [0,1].$$

A value is assigned to a pair of membership values $A(x)$ and $B(x)$ of an element x of the universal set X. It represents membership of x to the intersection of A and B:

$$(A \cap B)(x) = i(A(x), B(x)), \qquad \text{for } x \in X.$$

Note:

- Intuitive requirements to be fulfilled by a function i to qualify as an intersection of fuzzy sets are those of well known and extensively studied t-norms (triangular norms); the names *fuzzy intersection* and *t-norm* are therefore used interchangeably in the literature.
- The value $(A \cap B)(x)$ does not depend on x, but only on $A(x)$ and $B(x)$.

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Fuzzy intersections

Axiomatic requirements

For all $a, b, d \in [0,1]$,

Ax i1. $i(a, 1) = a$. **boundary condition**

Ax i2. $b \le d$ implies $i(a, b) \le i(a, d)$. **monotonicity**

Ax i3. $i(a, b) = i(b, a)$. **commutativity**

Ax i4. $i(a, i(b, d)) = i(i(a, b), d)$. **associativity**

Axioms **i1** - **i4** are called **axiomatic skeleton for fuzzy intersections**.

If the sets are crisp, i becomes the classical (crisp) intersection.

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Fuzzy intersections

Additional (optional) requirements

For all $a, b, d \in [0,1]$,

Ax i5. i is a continuous function. **continuity**

Ax i6. $i(a, a) \le a$. **subidempotency**

Ax i7. $a_1 < a_2$ and $b_1 < b_2$ implies $i(a_1, b_1) < i(a_2, b_2)$. **strict monotonicity**

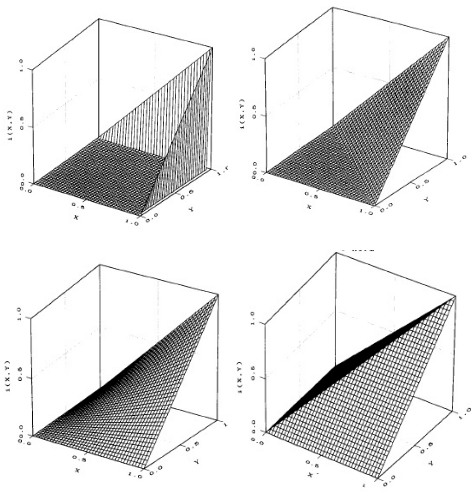
Note:

Subidempotency is a weaker requirement than idempotency, $i(a, a) = a$.
A continuous subidempotent t-norm is called **Archimedean *t*-norm**

The standard fuzzy intersection, $i(a, b) = \min[a, b]$, is the only idempotent t-norm.

Fuzzy intersections

Examples of t-norms frequently used



- **Drastic intersection**
  $$i(a,b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$
- **Bounded difference**
  $i(a,b) = \max[0, a+b-1]$
- **Algebraic product**
  $i(a,b) = ab$
- **Standard intersection**
  $i(a,b) = \min[a,b]$

Fuzzy intersections

Properties

- $i_{\min}(a,b) \leq \max(0, a+b-1) \leq ab \leq \min(a,b)$.
- For all $a, b \in [0,1]$, $\quad i_{\min}(a,b) \leq i(a,b) \leq \min[a,b]$.

Fuzzy intersections

How to generate t-norms

Theorem

(Characterization Theorem of t-norms) *Let i be a binary operation on the unit interval. Then, i is an Archimedean t-norm iff there exists a decreasing generator f such that*

$$i(a,b) = f^{(-1)}(f(a) + f(b)), \qquad \text{for } a, b \in [0,1].$$

Example: A class of decreasing generators $f_\omega(a) = (1-a)^\omega, \ \omega > 0$ generates a Yager class of t-norms

$$i_\omega(a,b) = 1 - \min[1, ((1-a)^\omega + (1-b)^\omega)^{\frac{1}{\omega}}], \quad \omega > 0.$$

It can be proved that $i_{\min}(a,b) \leq i_\omega(a,b) \leq \min[a,b]$.

Fuzzy unions

Definition

A **union** of two fuzzy sets A and B is given by a function of the form

$$u : [0,1] \times [0,1] \to [0,1].$$

A value is assigned to a pair of membership values $A(x)$ and $B(x)$ of an element x of the universal set X. It represents membership of x to the union of A and B:

$$(A \cup B)(x) = u(A(x), B(x)), \qquad \text{for } x \in X.$$

Note:

- Intuitive requirements to be fulfilled by a function u to qualify as a union of fuzzy sets are those of well known and extensively studied t-conorms (triangular conorms); the names *fuzzy union* and *t-conorm* are therefore used interchangeably in the literature.
- The value $(A \cup B)(x)$ does not depend on x, but only on $A(x)$ and $B(x)$.

Fuzzy unions

Axiomatic requirements

For all $a, b, d \in [0, 1]$,

Ax u1. $u(a, 0) = a$. **boundary condition**

Ax u2. $b \le d$ implies $u(a, b) \le u(a, d)$. **monotonicity**

Ax u3. $u(a, b) = u(b, a)$. **commutativity**

Ax u4. $u(a, u(b, d)) = u(u(a, b), d)$. **associativity**

Axioms **u1** - **u4** are called **axiomatic skeleton for fuzzy unions**.
They differ from the axiomatic skeleton of fuzzy intersections only in boundary condition.
For crisp sets, u behaves like a classical (crisp) union.

Fuzzy unions

Additional (optional) requirements

For all $a, b, d \in [0, 1]$,

Ax u5. u is a continuous function. **continuity**

Ax u6. $u(a, a) \ge a$. **superidempotency**

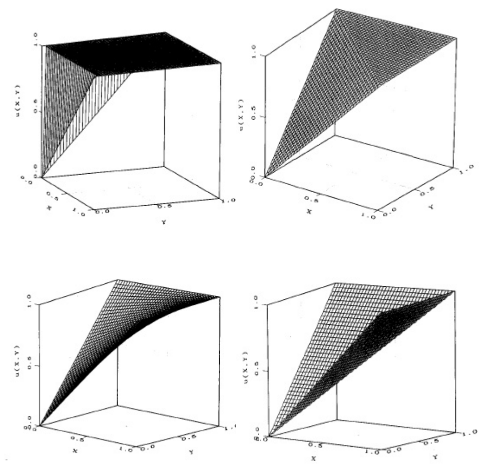
Ax u7. $a_1 < a_2$ and $b_1 < b_2$ implies $u(a_1, b_1) < u(a_2, b_2)$. **strict monotonicity**

Note:

Requirements **u5** - **u7** are analogous to Axioms **i5** - **i7**.
Superidempotency is a weaker requirement than idempotency.
A continuous superidempotent t-conorm is called **Archimedean t-conorm**.
The standard fuzzy union, $u(a, b) = \max[a, b]$, is the only idempotent t-conorm.

Fuzzy unions

Examples of t-conorms frequently used



- **Drastic union**
  $$u(a,b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$$
- **Bounded sum**
  $u(a, b) = \min[1, a + b]$
- **Algebraic sum**
  $u(a, b) = a + b - ab$
- **Standard intersection**
  $u(a, b) = \max[a, b]$

Fuzzy unions

Properties

- $\max[a, b] \le a + b - ab \le \min(1, a + b) \le u_{\max}(a, b)$.
- For all $a, b \in [0, 1]$, $\max[a, b] \le u(a, b) \le u_{\max}(a, b)$.

Fuzzy unions

How to generate t-conorms

Theorem

(Characterization Theorem of t-conorms) *Let u be a binary operation on the unit interval. Then, u is an Archimedean t-conorm iff there exists an increasing generator g such that*

$$u(a,b) = g^{(-1)}(g(a)+g(b)), \qquad \text{for } a,b \in [0,1].$$

Example: A class of increasing generators $f_\omega(a) = a^\omega$, $\omega > 0$ generates a Yager class of t-conorms

$$u_\omega(a,b) = \min[1, (a^\omega + b^\omega)^{\frac{1}{\omega}}], \quad \omega > 0.$$

It can be proved that $\max[a,b] \le u_\omega(a,b) \le u_{\max}(a,b)$.

Combinations of set operations

De Morgan laws and duality of fuzzy operations

De Morgan laws in classical set theory:

$$\overline{A \cap B} = \bar{A} \cup \bar{B} \qquad \text{and} \qquad \overline{A \cup B} = \bar{A} \cap \bar{B}.$$

The union and intersection operation are **dual** with respect to the complement.

De Morgan laws for fuzzy sets:

$$c(i(A,B)) = u(c(A), c(B)) \quad \text{and} \quad c(u(A,B)) = i(c(A), c(B))$$

for a t-norm i, a t-conorm u, and fuzzy complement c.

Notation: $\langle i, u, c\rangle$ denotes a **dual triple**.

Duality of fuzzy set operations

Examples of dual triples

Dual triples with respect to the **standard** fuzzy complement

$$\langle \min(a,b), \max(a,b), c_s\rangle$$
$$\langle ab, a+b-ab, c_s\rangle$$
$$\langle \max(0, a+b-1), \min(1, a+b), c_s\rangle$$
$$\langle i_{\min}(a,b), u_{\max}(a,b), c_s\rangle$$

Dual triples - Six theorems (1)

Theorem

The triples $\langle \min, \max, c\rangle$ and $\langle i_{\min}, u_{\max}, c\rangle$ are dual with respect to any fuzzy complement c.

Theorem

Given a t-norm i and an involutive fuzzy complement c, the binary operation u on $[0,1]$, defined for all $a,b \in [0,1]$ by

$$u(a,b) = c(i(c(a), c(b)))$$

is a t-conorm such that $\langle i, u, c\rangle$ is a dual triple.

Theorem

Given a t-conorm u and an involutive fuzzy complement c, the binary operation i on $[0,1]$, defined for all $a,b \in [0,1]$ by

$$i(a,b) = c(u(c(a), c(b)))$$

is a t-norm such that $\langle i, u, c\rangle$ is a dual triple.

Dual triples - Six theorems (2)

Theorem
Given an involutive fuzzy complement c and an increasing generator g of c, the t-norm and the t-conorm generated by g are dual with respect to c.

Theorem
Let $\langle i, u, c\rangle$ be a dual triple generated by an increasing generator g of the involutive fuzzy complement c. Then the fuzzy operations i, u, c satisfy the law of excluded middle, and the law of contradiction.

Theorem
Let $\langle i, u, c\rangle$ be a dual triple that satisfies the law of excluded middle and the law of contradiction. Then $\langle i, u, c\rangle$ does not satisfy the distributive laws.

Aggregation operations

Definition

Aggregations on fuzzy sets are operations by which several fuzzy sets are **combined** in a desirable way to produce a single fuzzy set.

Definition
Aggregation operation on n fuzzy sets ($n \geq 2$) is a function

$$h : [0,1]^n \to [0,1].$$

Applied to fuzzy sets $A_1, A_2, \ldots, A_n$, function h produces an aggregate fuzzy set A, by operating on membership grades to these sets for each $x \in X$:

$$A(x) = h(A_1(x), A_2(x), \ldots, A_n(x)).$$

Axiomatic requirements

Ax h1 $h(0,0,\ldots,0) = 0$ and $h(1,1,\ldots,1) = 1$. **boundary conditions**

Ax h2 For any $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, such that $a_i, b_i \in [0,1]$ and $a_i \leq b_i$ for $i = 1, \ldots, n$,

$$h(a_1, a_2, \ldots, a_n) \leq h(b_1, b_2, \ldots, b_n).$$

h is **monotonic increasing** in all its arguments.

Ax h3 h is **continuous**.

Ax h4 h is a **symmetric** function in all its arguments; for any permutation p on $\{1, 2, \ldots, n\}$

$$h(a_1, a_2, \ldots, a_n) = h(a_{p(1)}, a_{p(2)}, \ldots, a_{p(n)}).$$

Ax h5 h is an **idempotent** function; for all $a \in [0,1]$

$$h(a, a, \ldots, a) = a.$$

Averaging operations

- If an aggregation operator h is monotonic and idempotent (**Ax h2** and **Ax h5**), then for all $(a_1, a_2, \ldots, a_n) \in [0,1]^n$

  $$\min(a_1, a_2, \ldots, a_n) \leq h(a_1, a_2, \ldots, a_n) \leq \max(a_1, a_2, \ldots, a_n).$$

- All aggregation operators between the standard fuzzy intersection and the standard fuzzy union are idempotent.
- The only idempotent aggregation operators are those between standard fuzzy intersection and standard fuzzy union.

Idempotent aggregation operators are called **averaging operations**.

Averaging operations

Generalized means:

$$h_\alpha(a_1, a_2, \dots, a_n) = \left(\frac{a_1^\alpha + a_2^\alpha + \dots + a_n^\alpha}{n} \right)^{\frac{1}{\alpha}},$$

for $\alpha \in R$, and $\alpha \neq 0$, and for $\alpha < 0 \;\; a_i \neq 0$.

- **Geometric mean**: For $\alpha \to 0$,
  $$\lim_{\alpha \to 0} h_\alpha(a_1, a_2, \dots, a_n) = (a_1 \cdot a_2 \cdot \dots \cdot a_n)^{\frac{1}{n}};$$
- **Harmonic mean**: For $\alpha = -1$,
  $$h_{-1}(a_1, a_2, \dots, a_n) = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}};$$
- **Arithmetic mean**: For $\alpha = 1$,
  $$h_1(a_1, a_2, \dots, a_n) = \frac{1}{n}(a_1 + a_2 + \dots a_n).$$

Functions h_α satisfy axioms **Ax h1** - **Ax h5**.

Ordered Weighted Averaging Operations - OWA

For

- a given weighting vector $w = (\omega_1, \omega_2, \dots, \omega_n)$,
  $\omega_i \in [0,1]$ for $i = 1, \dots, n$
  $$\sum_{i=1}^{n} \omega_i = 1$$
- a permutation $(b_1, b_2, \dots, b_n)$ of a given vector $(a_1, a_2, \dots, a_n)$
  $b_i \geq b_j$ if $i < j$, for any pair $i, j \in \{1, 2, \dots, n\}$

an **OWA operation** associated with w is defined as

$$h_w(a_1, a_2, \dots, a_n) = \omega_1 b_1 + \omega_2 b_2 + \dots \omega_n b_n.$$

Example: For $w = (0.3, 0.1, 0.2, 0.4)$,

$h_w(0.6, 0.9, 0.2, 0.7) = 0.3 \cdot 0.9 + 0.1 \cdot 0.7 + 0.2 \cdot 0.6 + 0.4 \cdot 0.2 = 0.54$.

- OWA operations satisfy axioms **Ax h1** - **Ax h5**
- OWA operations cover the whole range between min and max operations.

Norm operations

Aggregation operations h on $[0,1]^2$ which are

monotonic

commutative

associative

fulfil boundary conditions $h(0,0) = 0$ and $h(1,1) = 1$

are called **norm operations**.

A class of norm operations

- contains t-norms and t-conorms, as special cases, which fulfill stronger boundary conditions;
- covers the whole range of aggregation operators, from i_{min} to u_{max};

Norm operations

Example: λ-averages

For a given t-norm i, and given t-conorm u
for $a, b \in [0,1]$ and $\lambda \in (0,1)$

$$h_{(a,b)} = \begin{cases} \min[\lambda, u(a,b)] & \text{when } a, b \in [0, \lambda] \\ \max[\lambda, i(a,b)] & \text{when } a, b \in [\lambda, 1] \\ \lambda & \text{otherwise} \end{cases}$$

is a parametrized class of norm operations which are neither t-norms, nor t-conorms.

These operations are called **λ-averages**.

Averaging operations

Do we need more than standard operations?

Making decisions in fuzzy environment:

Taking into account objectives and constraints expressed by fuzzy sets, make a new fuzzy set representing a decision.

(H.-J. Zimmermann, P. Zysno, "Latent Connectivities in Human Decision Making", FSS 4, 1980)

1. "The board of directors tries to find the 'optimal' dividend to be paid to the shareholders. For financial reasons it should be attractive **and** for reasons of wage negotiations it should be modest."
   The decision about optimal dividend is based on the fuzzy set

$$\mu_{Optimal}(x) = \min[\mu_{Attractive}(x), \mu_{Modest}(x)].$$

2. "The teacher has to decide a student's grade on a written test. The given task was supposed to be solved by Method1 **or** by Method2. The student provided two solutions, using both methods."
   The decision about his mark is based on a fuzzy set

$$\mu_{GoodSol}(x) = \max[\mu_{Good1stSol}(x), \mu_{Good2ndSol}(x)].$$

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Decision as a fuzzy set

Intersection: No positive compensation (trade-off) between the memberships of the fuzzy sets observed.

Union: Full compensation of lower degrees of membership by the maximal membership.

In reality of decision making, rarely either happens.

(non-verbal) "merging connectives" → (language) connectives {'and', 'or',...,}.

Aggregation operations called **compensatory and** are needed to model fuzzy sets representing to, e.g., managerial decisions.

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An example

- Two independent criteria, cr_1 and cr_2, are used to determine the quality of a product.
- For a number of products, the memberships to three fuzzy sets are experimentally determined:
  μ_{cr_1}, μ_{cr_2}, and μ_{Ideal};
- Numbers: 60 people rating 24 products by assigning a value between 0 and 100, expressing memberships to :
  "FulfilledCr1", "FulfilledCr2", "IdealProduct";
- Different **theoretical** aggregations (minimum, maximum, arithmetic mean, geometric mean) are computed and compared with the **experimentally** obtained "Ideal".

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Theory vs. Experiments

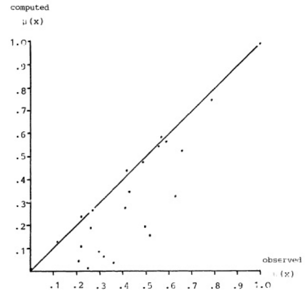


Fig. 4. Min-operator: Observed versus computed grades of membership.

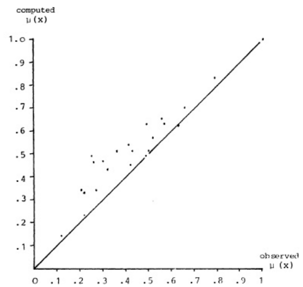


Fig. 6. Arithmetic mean: Observed versus computed grades of membership.

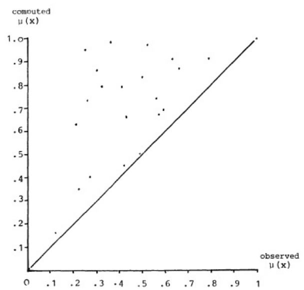


Fig. 5. Max-operator: Observed versus computed grades of membership.

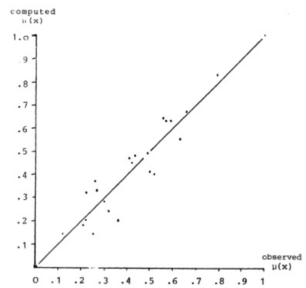


Fig. 7. Geometric mean: Observed versus computed grades of membership.

Conclusions

- Geometric mean gives the best prediction of the empirical data, among the operations tested.
- Humans use other connectives than “and” or “or”; they need other aggregation operations than min and max.
- Even if the criteria are independent, and the corresponding fuzzy sets do not “interact”, the aggregation operation itself can “put them into interaction”.
- Every concrete decision set may require a specific aggregation operation.
- A general connective is required to involve some parameter corresponding to “grade of compensation”.
- The suggested operation is a weighted combination of non-compensatory ’and’ and fully compensatory ’or’ (in this work, they are interpreted as the product and the algebraic sum).

Aggregation by γ-operator

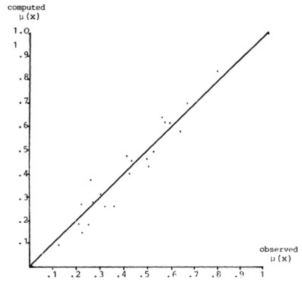


Fig. 10. γ-operator: Observed versus comp..ted grades of membership.

γ-operation, for $\gamma \in [0, 1]$, is defined as

$$\mu_{A\theta B} = \mu_{A\cap B}^{1-\gamma} \cdot \mu_{A\cup B}^{\gamma}.$$

The value $\gamma = 0.562$ is determined from the experimental data, for the chosen interpretation of intersection and union.

An Application: Fuzzy morphologies

Morphological operations

- Mathematical morphology is completely based on set theory. Fuzzification started in 1980s.
- Basic morphological operations are **dilation and erosion**. Many others can be derived from them.
- Dilation and erosion are, in crisp case, **dual operations** with respect to the complementation: $D(A) = c(E(cA))$.
- In crisp case, dilation and erosion fulfil a certain number of properties.

Dilation and erosion

Definitions

Definition

Dilation of a set A by a structuring element B is

$$D_B(A) = A \oplus B = \{x \in X | \tau_x(\hat{B}) \cap A \neq \emptyset\}.$$

$\tau_a(S) = \{y | y = a + s, s \in S\}$
$\hat{S} = \{y | y = -s, s \in S\}.$

Definition

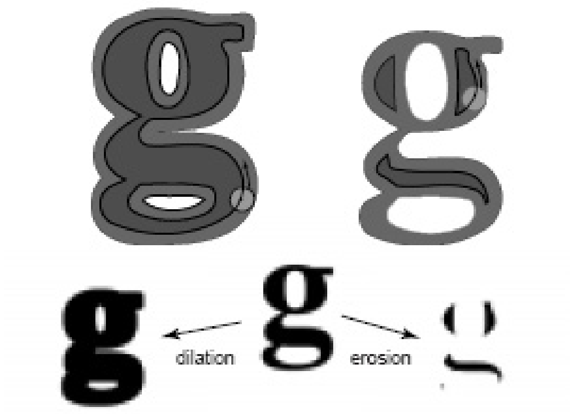
Erosion of a set A by a structuring element B is

$$E_B(A) = A \ominus B = \{a \in X | \tau_a(B) \subseteq A\}.$$

Dilation and erosion

An example

Dilation and erosion of a shape by a circular structuring element.



How to construct fuzzy mathematical morphology

- Infinitely many fuzzy mathematical morphologies can be constructed.
- It is desirable to understand the differences and to be able to make choices (of operations, structuring elements,...) in the way that fits the task the best.
- Main construction principles:
  α-cut decomposition;
  fuzzification of set operations.

How to construct fuzzy mathematical morphology

α-cut decomposition

For fuzzy sets $\mu(x)$ and $\nu(x)$ it holds that

$$[\mu(x) \oplus \nu(x)]_\alpha(x) = \mu_\alpha(x) \oplus \nu_\alpha(x).$$

Steps to take:

- α-cut decomposition of both the set and the structuring element;
- performance of (crisp) morphological operations;
- reconstruction of a resulting fuzzy set from its obtained α-cuts.

How to construct fuzzy mathematical morphology

α-cut decomposition

It is shown that this approach leads to definitions:

$$D_\nu(\mu)(x) = \int_0^1 \sup_{y \in (\nu_\alpha)_x} \mu(y)\, d\alpha$$

$$E_\nu(\mu)(x) = \int_0^1 \inf_{y \in (\nu_\alpha)_x} \mu(y)\, d\alpha$$

if a reconstruction of a fuzzy set is performed by using

$$\mu(x) = \int_0^1 \mu_\alpha(x)\, d\alpha$$

...

How to construct fuzzy mathematical morphology

α-cut decomposition

... or

$$\begin{aligned} D_\nu(\mu)(x) &= \sup_{y\in X}[\nu(y-x)\mu(y)] \\ E_\nu(\mu)(x) &= \inf_{y\in X}[\mu(y)\nu(y-x)+1-\nu(y-x)] \end{aligned}$$

if a reconstruction of a fuzzy set is performed by using $\mu(x) = \sup_{\alpha\in(0,1]} [\alpha\mu_\alpha(x)]$.

How to construct fuzzy mathematical morphology

Fuzzification of set operations

Perform "fuzzification" of all (set) operations and relations used for definitions of morphological operations.

- Replace unions and intersections by t-conorms and t-norms,
- Use fuzzy complementation
- Replace subsethood relation by inclusion indicators
- Use duality of morphological operations
- ...

How to construct fuzzy mathematical morphology

Fuzzification of set operations

Most general definitions of fuzzy dilation and fuzzy erosion obtained by fuzzification of set operations are:

$$\begin{aligned} D_\nu(\mu)(x) &= \sup_{y\in X} i(\nu(y-x),\mu(y)) \\ E_\nu(\mu)(x) &= \inf_{y\in X} u(c(\mu(y),\nu(y-x))), \end{aligned}$$

where i is any t-norm and u is its dual t-conorm with respect to a fuzzy complement c.

How to construct fuzzy mathematical morphology

Fuzzification of set operations

- A fuzzy erosion can be defined as

  $$E_\nu(\mu)(x) = I(\nu+x,\mu)$$

  where $I(\nu,\mu)$ is an **inclusion indicator**, a function that fulfils a certain set of axioms. (Several exist in the literature.)
  Then, dilation is defined by duality principle.
- Alternatively, a fuzzy dilation is defined as

  $$D_\nu(\mu)(x) = O(\nu+x,\mu)$$

  where O is an **overlapping indicator.**

A task on fuzzy morphology

Use the paper:
I. Bloch and H. Maitre: "Fuzzy Mathematical Morphologies: A Comparative Study", *Pattern Recognition*, 28(9), 1995.

- Design fuzzy morphological operations. Use α-cut decomposition principle, and fuzzification of set operation by t-norms and t-conorms.
- Implement at least three different morphological dilations and corresponding erosions; use 2-3 different structuring elements; take simple (one-dimensional) examples; look at the examples in the paper.
- Comment the results. Observe visual differences and think of their effect in possible applications. Study the properties of the morphological operations, using your obtained results and the paper.
- Make a short summary.