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Tentamen i matematik
Linjär algebra II
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Inga hjälpmedel förutom skrivdon. Lösningarna skall åtföljas av förklaranda text. Endast svar ger 0 p. Tentamen består av åtta uppgifter värda 5 poäng vardera, d.v.s. maximalt kan man få 40 poäng på tentamen. Ett resultat om minst 18, 25 och 32 poäng ger betyg 3,4 respektive 5.

1. Avgör om följande delmängder är delvektorrum eller ej. Motivera ditt svar.

(a) $U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 0\}$,

(b) $V = \left\{A \in M_2(\mathbb{R}) \mid A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$,

(c) $W = \{(3t^3, t^3, 5t^3) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}$.

Solution A subset of a vector space is a subspace if and only if it is not empty, it is closed under addition, and under scalar multiplication.

(a) For $(x, y, z) \in \mathbb{R}^3$ it follows from $x^2 + y^2 = 0$ and $x, y \in \mathbb{R}$ that $x = y = 0$. Therefore $U = \{(0, 0, z) \in \mathbb{R}^3\}$. We check the three conditions: U is not empty since $(0, 0, 0) \in U$. It is closed under addition since for $(0, 0, z_1), (0, 0, z_2) \in U$ also $(0, 0, z_1) + (0, 0, z_2) = (0, 0, z_1 + z_2) \in U$ and it is closed under scalar multiplication since for $(0, 0, z) \in U$ and $\lambda \in \mathbb{R}$ also $\lambda(0, 0, z) = (0, 0, \lambda z) \in U$.

(b) A subspace of a vector space necessarily has to contain the zero vector of that space. The zero vector of $M_2(\mathbb{R})$ is $0_{M_2(\mathbb{R})} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. But this is not contained in V since

$$0_{M_2(\mathbb{R})}^T 0_{M_2(\mathbb{R})} = 0_{M_2(\mathbb{R})} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(c) Note that for every $s \in \mathbb{R}$ there exists a unique $t \in \mathbb{R}$ such that $t^3 = s$ (i.e. the map $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto x^3$ is bijective). It follows that $W = \{(3s, s, 5s) \in \mathbb{R}^3 | s \in \mathbb{R}\}$. We check the three conditions: W is not empty since $(0, 0, 0) = (3 \cdot 0, 1 \cdot 0, 5 \cdot 0) \in W$. It is closed under addition since for $(3s_1, s_1, 5s_1), (3s_2, s_2, 5s_2) \in W$ also $(3s_1, s_1, 5s_1) + (3s_2, s_2, 5s_2) = (3(s_1 + s_2), (s_1 + s_2), 5(s_1 + s_2)) \in W$, and it is closed under scalar multiplication since for $(3s, s, 5s) \in W$ and $\lambda \in \mathbb{R}$ also $\lambda(3s, s, 5s) = (3(\lambda s), \lambda s, 5(\lambda s)) \in W$.

2. Låt A vara en 5×7-matris med kolonnrum av dimension 3.

(a) Hur många parametrar behövs för att beskriva nollrummet?

(b) Vad är radrummets dimension?

(c) Hur många nollrader har matrisen efter fullständig Gausselimination?

Solution (a) The dimension of the column space equals the rank of the matrix, which is therefore 3. According to the rank-nullity theorem, the number of columns of a matrix A equals the sum of the rank of the matrix and the dimension of the nullspace of the matrix. The dimension of a space is equal to the number of parameters needed to describe the space. It is therefore equal to $4 = 7 - 3$.

(b) The dimension of the row space is also equal to the rank of the matrix, it is therefore equal to 3.

(c) The number of zero rows after complete Gaussian elimination is equal to the difference of the number of rows of the matrix and the rank of the matrix (which is equal to the number of leading 1's). It is therefore equal to $5 - 3 = 2$.

3. Låt $F : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ vara en linjär avbildning som ges av

$$F(A) = A^T + A.$$

Låt $e_1 = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$ och $e_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(a) Visa att $(e_1, ..., e_4)$ är en bas för M_{22}.

(b) Ange matrisen för F i denna bas.

Solution (a) We know that $M_2(\mathbb{R})$ has dimension 4. Since there are 4 vectors given it thus suffices to prove that e_1, e_2, e_3, e_4 are linearly independent. To show this, we have to show that whenever $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 = 0$, then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. We compute that

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 = \begin{pmatrix} \lambda_1 + \lambda_2 & -\lambda_1 + \lambda_2 + 2\lambda_3 - \lambda_4 \\ -\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4 & \lambda_3 \end{pmatrix}$$

In order for the latter matrix to be the zero matrix we conclude from comparing the lower right entry that $\lambda_3 = 0$. Subtracting the lower left entry from the upper right entry we obtain that $2\lambda_4 = 0$ and therefore $\lambda_4 = 0$. Now the comparison for the upper left entry reads $\lambda_1 + \lambda_2 = 0$ while the comparison for the upper right entry reads $-\lambda_1 + \lambda_2 = 0$. Adding these two equalities gives $\lambda_2 = 0$ and thus also $\lambda_1 = 0$. Therefore e_1, e_2, e_3, e_4 are linearly independent and (since $\dim M_2(\mathbb{R}) = 4$), they form a basis of $M_2(\mathbb{R})$.

(b) To compute the matrix of F with respect to this basis we first apply F to every basis vector and write the result as a linear combination of the basis vectors:

$$F(e_1) = \begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix} = 2e_1 + 0e_2 + 0e_3 + 0e_4$$
$$F(e_2) = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} = 0e_1 + 2e_2 + 0e_3 + 0e_4$$
$$F(e_3) = \begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix} = 0e_1 + 0e_2 + 2e_3 + 0e_4$$
$$F(e_4) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0e_1 + 0e_2 + 0e_3 + 0e_4$$

Since the columns of the matrix of F are the coordinate vectors of the images of the basis vectors of the domain with respect to the basis of the codomain it follows that

$$[F]_{B \leftarrow B} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $B = \{e_1, e_2, e_3, e_4\}$.

4. Lös systemet av differentialekvationer:

$$\begin{cases} y_1' &= 5y_1 &+ 4y_2 \\ y_2' &= 3y_1 &+ 6y_2 \end{cases}$$

med begynnelsevillkoren $y_1(0) = 2$ och $y_2(0) = 5$.

Solution Writing the system of differential equations in matrix form we obtain that $y' = Ay$ where $A = \begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}$.The first step is to compute the eigenvalues for A. The characteristic polynomial of A is equal to

$$\chi_A(\lambda) = \det(\lambda I_2 - A) = (\lambda - 5)(\lambda - 6) - (-4)(-3) = \lambda^2 - 11\lambda + 18 = (\lambda - 2)(\lambda - 9).$$

The eigenvalues of A are the zeroes of the characteristic polynomial of A, i.e. they are 2 and 9. The next step is to compute corresponding eigenvectors:

For $\lambda = 2$ we obtain that $E(2) = \ker(A - 2I_2) = \ker \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}$. This is a one-dimensional space with basis vector $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

For $\lambda = 9$ we obtain that $E(9) = \ker(A - 9I_2) = \ker \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}$. This is also a one-dimensional space with basis vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Setting $S = \begin{pmatrix} -4 & 1 \\ 3 & 1 \end{pmatrix}$ and $y = Sz$ we obtain the system of equations

$$z' = S^{-1}ASz = \begin{pmatrix} 2 & 0 \\ 0 & 9 \end{pmatrix} z.$$

It follows that $z_1 = c_1e^{2t}$ and $z_2 = c_2e^{9t}$ and therefore

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1e^{2t} \\ c_2e^{9t} \end{pmatrix} = \begin{pmatrix} -4c_1e^{2t} + c_2e^{9t} \\ 3c_1e^{2t} + c_2e^{9t} \end{pmatrix}.$$

Using the intial value condition we obtain that $2 = -4c_1 + c_2$ and $5 = 3c_1 + c_2$, thus $c_1 = \frac{3}{7}$ and $c_2 = \frac{26}{7}$ and we obtain that

$$y_1(t) = -\frac{12}{7}e^{2t} + \frac{26}{7}e^{9t}$$
$$y_2(t) = \frac{9}{7}e^{2t} + \frac{26}{7}e^{9t}$$

5. Låt $V = C[0, \pi]$ vara mängden av de kontinuerliga funktioner $f(x)$ på intervallet $[0, \pi]$ som är noll i ändpunkterna: $f(0) = f(\pi) = 0$.

(a) Visa att V är ett vektorrum.

(b) Bestäm vinkeln mellan funktionerna $\sin x$ och $\sin 2x$ med avseende på den inre produkten $\langle f|g\rangle = \int_0^\pi f(x)g(x)\,dx$, samt beräkna deras längder.

Ledtråd: Minns att $\sin^2 x = \frac{1 - \cos 2x}{2}$, *samt att* $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

Solution (a) We know from the lectures that the space of continuous functions is a vector space. For V to be a vector space we therefore only have to show that it is a subspace of the space of continuous functions. Thus, we have to show the three conditions given in **1.**. To show that V is non-empty observe that $\sin(x) \in V$ since $\sin(0) = \sin(\pi) = 0$. To show that V is closed under addition assume that $f, g \in V$. We check that also $f + g \in V$: We compute that $(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$ and $(f + g)(\pi) = f(\pi) + g(\pi) = 0 + 0 = 0$. Therefore $f + g \in V$. To show that V is closed under scalar multiplication assume that $f \in V$ and $\lambda \in \mathbb{R}$. We check that also $\lambda f \in V$: We compute that $(\lambda f)(0) = \lambda(f(0)) = \lambda \cdot 0 = 0$ and $(\lambda f)(\pi) = \lambda(f(\pi)) = \lambda \cdot 0 = 0$ and thus $\lambda f \in V$.

(b) The length of a vector f in an inner product space is defined as $\|f\| = \sqrt{\langle f|f\rangle}$ while the angle α between two vectors f, g in an inner product space is defined via $\cos \alpha = \frac{\langle f|g\rangle}{\|f\| \cdot \|g\|}$.

We first compute the length of the two given functions:

$$\langle \sin(x)|\sin(x)\rangle = \int_0^\pi \sin^2(x)dx = \int_0^\pi \frac{1-\cos(2x)}{2}dx = \left[\frac{1}{2}x - \frac{1}{4}\sin(2x)dx\right]_0^\pi = \frac{\pi}{2}$$

and therefore $\|\sin(x)\| = \sqrt{\frac{\pi}{2}}$.

$$\langle \sin(2x)|\sin(2x)\rangle = \int_0^\pi \sin^2(2x)dx = \int_0^{2\pi} \frac{1}{2}\sin^2(u)du = \frac{\pi}{2}$$

and therefore $\|\sin(2x)\| = \sqrt{\frac{\pi}{2}}$.

To compute the angle we compute

$$\langle \sin(x)|\sin(2x)\rangle = \int_0^\pi \sin(x)\sin(2x)dx = \int_0^\pi \sin(x)(2\sin(x)\cos(x))dx = \int_0^0 2u^2du = 0$$

Therefore $\sin(x)$ and $\sin(2x)$ are orthogonal and the angle between them is therefore equal to $\frac{\pi}{2} = 90°$.

6. Låt F vara den linjära avbildning som ges av spegling i planet $x + 2y = 0$ (koordinater i standardbas S).

(a) Finn en ON-bas B som består av egenvektorer till F.

(b) Bestäm F:s matris $[F]_S$.

(c) Bestäm $F(1, 0, 1)$.

Solution (a) For a reflection at a plane in $\mathbb{R}^3$, the eigenspaces are the plane of reflection, which is the eigenspace with respect to the eigenvalue 1 as well as the line through the normal vector, which is an eigenspace with respect to the eigenvalue -1. The normal vector of the plane is $\begin{pmatrix}1\\2\\0\end{pmatrix}$ while a basis of the plane of reflection is given by $\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. The vectors are already orthogonal, thus to compute an orthonormal basis of eigenvectors we just have to normalise them. A basis of $\mathbb{R}^3$ consisting of eigenvectors of F is therefore given by $\left\{\begin{pmatrix}\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\\0\end{pmatrix}, \begin{pmatrix}-\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

(b) The matrix $[F]_B$ of F with respect to the orthonormal basis B given in (a) is given by $\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$. Using the base change formula $[F]_S = [\mathrm{id}]_{S\leftarrow B}[F]_B[\mathrm{id}]_{B\leftarrow S}$, the fact that $[\mathrm{id}]_{S\leftarrow B} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $[\mathrm{id}]_{B\leftarrow S} = [\mathrm{id}]_{S\leftarrow B}^{-1} = [\mathrm{id}]_{S\leftarrow B}^{T}$ (since B is an orthonormal basis) we obtain

$$[F]_B = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ -\frac{4}{5} & -\frac{3}{5} & 0 \\ & & 1 \end{pmatrix}$$

7. För vilka a är matrisen

$$A = \begin{pmatrix} 2 & a & 0 \\ a & 4 & a \\ 0 & a & 2 \end{pmatrix}$$

positivt definit?

Solution A matrix is positive definite if and only if all its eigenvalues are positive.

We compute that the characteristic polynomial of A:

$$\begin{aligned} \chi_A(\lambda) = \det(\lambda I_3 - A) &= \det \begin{pmatrix} \lambda-2 & -a & 0 \\ -a & \lambda-4 & -a \\ 0 & -a & \lambda-2 \end{pmatrix} \\ &= (\lambda-2)\det\begin{pmatrix} \lambda-4 & -a \\ -a & \lambda-2 \end{pmatrix} + a\det\begin{pmatrix} -a & 0 \\ -a & \lambda-2 \end{pmatrix} \\ &= (\lambda-2)((\lambda-4)(\lambda-2)-a^2) + a((-a)(\lambda-2)) \\ &= (\lambda-2)(\lambda^2-6\lambda+8-2a^2) \end{aligned}$$

The zeroes of the characteristic polynomial of A are therefore 2 and $3 \pm \sqrt{2a^2+1}$. The only one of these which is not necessarily positive is $3 - \sqrt{2a^2+1}$. It is positive if and only if $3 > \sqrt{2a^2+1}$ which is if and only if $a^2 < 4$, i.e. $-2 < a < 2$.

8. Låt $\mathcal{F}(\mathbb{R},\mathbb{R})$ vara vektorrummet bestående av funktioner och låt V vara det delrum som spänns upp av $g(x) = (\sin(x))^2$ and $h(x) = (\cos(x))^2$. Låt $P_{\leq 1}$ vara vektorrummet av polynom av grad ≤ 1.

(a) Visa att V innehåller värderummet av den linjära avbildning $\varphi\colon P_{\leq 1} \to \mathcal{F}(\mathbb{R},\mathbb{R})$ som ges av $\varphi(p) = \frac{d}{dx}p$.

(b) Betrakta den linjära avbildningen $\varphi : P_{\leq 1} \to V$. Bestäm avbildningsmatrisen $[\varphi]_{B'\leftarrow B}$ of φ med avseende på basen $B = \{1, x\}$ i $P_{\leq 1}$ och basen $B' = \{g, h\}$ i V.

Solution (a) An element of $P_{\leq 1}$ is of the form $a + bx$ for $a, b \in \mathbb{R}$. It follows that $\varphi(a + bx) = b = b\sin^2(x) + b\cos^2(x)$. Therefore V contains the codomain of the linear map φ.

(b) To compute the matrix $[\varphi]_{B' \leftarrow B}$ of φ we compute the images of the basis vectors of B and write them as linear combinations of the basis vectors in B'.

$$\begin{aligned}\varphi(1) &= 0 = 0 \cdot \sin^2(x) + 0 \cdot \cos^2(x) \\ \varphi(x) &= 1 = 1 \cdot \sin^2(x) + 1 \cdot \cos^2(x)\end{aligned}$$

As the columns of the matrix $[\varphi]_{B' \leftarrow B}$ are the coordinate vectors of the images of the basis vectors of B with respect to the basis B' we obtain that

$$[\varphi]_{B' \leftarrow B} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$