TENTAMEN - LINEAR ALGEBRA II 2018/01/09

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1. (i) For which values of $a \in \mathbb{R}$ do the following polynomials form a basis $B = \{p_1(x), p_2(x), p_3(x)\}$ of $P_2(\mathbb{R})$:

$$p_1(x) = 2 + x^2$$
$$p_2(x) = (7 + a)x - 3x^2$$
$$p_3(x) = 4x + ax^2$$

Justify your answer.

(ii) Let $B' = \{1, x, x^2\}$. In case that B is a basis provide the transition matrix $P_{B' \leftarrow B}$ from B to B'.

Possible solution 1a: (i) Since $\dim P_2(\mathbb{R}) = 3$, it suffices to determine when $p_1(x), p_2(x), p_3(x)$ are linearly independent.
For this we have to check when

$$\lambda_1 p_1(x) + \lambda_2 p_2(x) + \lambda_3 p_3(x) = 0$$

has only the trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Plugging in p_1, p_2, and p_3, we obtain

$$\lambda_1(2 + x^2) + \lambda_2((7 + a)x - 3x^2) + \lambda_3(4x + ax^2) = 0.$$

Comparing coefficients we see that this is the case if and only if

$$2\lambda_1 = 0$$
$$(7 + a)\lambda_2 + 4\lambda_3 = 0$$
$$\lambda_1 - 3\lambda_2 + a\lambda_3 = 0$$

We solve this system using Gaussian elimination.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 7+a & 4 \\ 1 & -3 & a \end{pmatrix} \overset{III-\frac{1}{2}I}{\rightsquigarrow} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 7+a & 4 \\ 0 & -3 & a \end{pmatrix} \overset{II\leftrightarrow III}{\rightsquigarrow} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & a \\ 0 & 7+a & 4 \end{pmatrix} \overset{III+\frac{7+a}{3}II}{\rightsquigarrow} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & a \\ 0 & 0 & 4+\frac{7+a}{3}a \end{pmatrix}$$

We see that this has only the trivial solution if and only if $4 + \frac{7+a}{3}a \neq 0$, i.e. if and only if $a^2 + 7a + 12 \neq 0$. With the help of the pq-formula it follows that the zeroes of this equation are given by $a_{1/2} = -\frac{7}{2} \pm \sqrt{\frac{49}{4} - \frac{48}{4}} = \begin{cases} -4 \\ -3. \end{cases}$
It follows that $p_1(x), p_2(x), p_3(x)$ form a basis if and only if $a \neq -4$ and $a \neq -3$.

(ii) Since

$$\begin{aligned} 2 + x^2 &= \mathbf{2} \cdot 1 + \mathbf{0} \cdot x + \mathbf{1} \cdot x^2 \\ (7+a)x - 3x^2 &= \mathbf{0} \cdot 1 + (\mathbf{7} + \mathbf{a}) \cdot x + (-\mathbf{3}) \cdot x^2 \\ 4x + ax^2 &= \mathbf{0} \cdot 1 + \mathbf{4} \cdot x + \mathbf{a} \cdot x^2 \end{aligned}$$

we obtain that

$$P_{B' \leftarrow B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 7+a & 4 \\ 1 & -3 & a \end{pmatrix}.$$

Possible solution 1b:

only for part (i) We know that $B' = \{1, x, x^2\}$ is a basis of $P_2(\mathbb{R})$. Therefore $c_{B'}$ is an isomorphism and hence B is a basis of $P_2(\mathbb{R})$ if and only if $\{c_{B'}(p_1), c_{B'}(p_2), c_{B'}(p_3)\}$ is a basis of $\mathbb{R}^3$. We have that

$$c_{B'}(p_1) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, c_{B'}(p_2) = \begin{pmatrix} 0 \\ 7+a \\ -3 \end{pmatrix}, c_{B'}(p_3) = \begin{pmatrix} 0 \\ 4 \\ a \end{pmatrix}$$

We know that these vectors in $\mathbb{R}^3$ form a basis if and only if the matrix with these vectors as columns is invertible if and only if its determinant is non-zero.

$$\det \begin{pmatrix} 2 & 0 & 0 \\ 0 & 7+a & 4 \\ 1 & -3 & a \end{pmatrix} = 2 \det \begin{pmatrix} 7+a & 4 \\ -3 & a \end{pmatrix} = 2((7+a)a + 12) = 2a^2 + 14a + 12$$

Using the pq-formula as in Solution 1a we obtain that this determinant is non-zero if and only if $a \neq -4$ and $a \neq -3$.

2. (i) Which of the following functions are linear? Justify your answer.

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x^2 \\ y \end{pmatrix}$$

$$g\colon \mathbb{R}^3 \to \mathbb{R}^2, \quad g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 3x + z \\ x + y + z \end{pmatrix}$$

$$h\colon P_2(\mathbb{R}) \to P_2(\mathbb{R}), \quad h(a_0 + a_1 x + a_2 x^2) = a_0 x^2$$

(ii) Choose one of the functions in (i) which is linear and determine a basis of its kernel, and a basis of its image.

Possible solution 2a: (i) The function f is not linear: We saw in the lecture that the function $\tilde{f}\colon \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is linear. Furthermore since they are given by multiplication with matrices it follows that the following functions are linear: $\hat{f}\colon \mathbb{R} \to \mathbb{R}^2, x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x$ and $\overline{f}\colon \mathbb{R}^2 \to \mathbb{R}, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Therefore if f were linear, then also $\tilde{f} = \overline{f} \circ f \circ \hat{f}$ would be linear, which we know it is not.

The function g is linear: It is given by multiplication with the matrix $\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and therefore we know from the lecture that it is linear.

The function h is linear: We know that the function $\tilde{h}\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by multiplication with the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is linear. We also know that $B = \{1, x, x^2\}$ is a basis of $P_2(\mathbb{R})$. Therefore the function $h = c_B^{-1} \circ \tilde{h} \circ c_B$ is linear.

(ii) We choose the function h. It is easy to see that $\mathrm{Im}(h) = \mathrm{span}(x^2)$ and since $x^2 \neq 0$ a basis of $\mathrm{Im}(h)$ is given by $\{x^2\}$. By the rank-nullity theorem we therefore know that $\dim \ker(f) = 2$ (since $\dim P_2(\mathbb{R}) = 3$). It is easy to see that $h(x) = h(x^2) = 0$ and therefore $\{x, x^2\}$ gives a basis of $\ker(h)$.

Possible solution 2b: (i) The function f is not linear. We can compute that

$$f\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = f\left(\begin{pmatrix} 4 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 16 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 8 \\ 0 \end{pmatrix} = f\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) + f\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right)$$

The function g is linear. We compute that

$$g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\right) = g\left(\begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix}\right) = \begin{pmatrix} 3(x + x') + z + z' \\ (x + x') + (y + y') + (z + z') \end{pmatrix}$$

$$= \begin{pmatrix} 3x + z \\ x + y + z \end{pmatrix} + \begin{pmatrix} 3x' + z' \\ x' + y' + z' \end{pmatrix} = g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) + g\left(\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\right)$$

and

$$g\left(\lambda\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = g\left(\begin{pmatrix}\lambda x\\\lambda y\\\lambda z\end{pmatrix}\right) = \begin{pmatrix}3(\lambda x)+(\lambda z)\\\lambda x+\lambda y+\lambda z\end{pmatrix} = \lambda\begin{pmatrix}3x+z\\x+y+z\end{pmatrix} = \lambda g\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)$$

Therefore, g is linear.

The function h is linear:

$$\begin{aligned}h((a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2)) = h((a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2) = (a_0+b_0)x^2\\ = a_0x^2+b_0x^2 = h(a_0+a_1x+a_2x^2)+h(b_0+b_1x+b_2x^2)\end{aligned}$$

and

$$h(\lambda(a_0+a_1x+a_2x^2)) = h((\lambda a_0)+(\lambda a_1)x+(\lambda a_2)x^2) = (\lambda a_0)x^2 = \lambda(a_0x^2) = \lambda h(a_0+a_1x+a_2x^2)$$

(ii) We choose the function g. Since g is given by multiplication with the matrix $A = \begin{pmatrix}3&0&1\\1&1&1\end{pmatrix}$, the kernel of g is given by the null space of A. We perform Gaussian elimination to A to obtain a basis

$$\begin{pmatrix}3&0&1\\1&1&1\end{pmatrix} \overset{I-3II}{\rightsquigarrow} \begin{pmatrix}0&-3&-2\\1&1&1\end{pmatrix}$$

We see that a basis of the null space of g is given by $\left\{\begin{pmatrix}-1\\-2\\3\end{pmatrix}\right\}$. Furthermore, the image of g is equal to the column space of A. The leading 1's are in the first and second column, therefore $\left\{\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a basis of $\mathrm{Im}(g)$.

3. Let

$$A = \begin{pmatrix} -3 & -6 & 0 & 4 \\ -1 & -4 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ -3 & -9 & 0 & 5 \end{pmatrix}.$$

(i) Show that the eigenvalues of A are 0 and -1.
(ii) For each eigenvalue, determine a basis of the corresponding eigenspace.
(iii) Is A diagonalisable? Justify your answer.

Possible solution 3a: (i) We know that λ is an eigenvalue of A if and only if $\chi_A(\lambda) = \det(A - \lambda I_4) = 0$. We compute

$$\begin{aligned}
\chi_A(\lambda) &= \det \begin{pmatrix} -3-\lambda & -6 & 0 & 4 \\ -1 & -4-\lambda & 0 & 2 \\ 0 & 0 & -1-\lambda & 0 \\ -3 & -9 & 0 & 5-\lambda \end{pmatrix} \\
&= (-1-\lambda) \det \begin{pmatrix} -3-\lambda & -6 & 4 \\ -1 & -4-\lambda & 2 \\ -3 & -9 & 5-\lambda \end{pmatrix} \\
&= (-1-\lambda) \left((-3-\lambda) \det \begin{pmatrix} -4-\lambda & 2 \\ -9 & 5-\lambda \end{pmatrix} - (-1) \det \begin{pmatrix} -6 & 4 \\ -9 & 5-\lambda \end{pmatrix} - 3 \det \begin{pmatrix} -6 & 4 \\ -4-\lambda & 2 \end{pmatrix} \right) \\
&= (-1-\lambda) \left((-3-\lambda)((-4-\lambda)(5-\lambda) + 18) + (-6(5-\lambda) + 36) - 3(-12 + 4(4+\lambda)) \right) \\
&= (-1-\lambda)(-\lambda^3 - 2\lambda^2 - \lambda) \\
&= (-1-\lambda)\lambda(-\lambda^2 - 2\lambda - 1) \\
&= (\lambda+1)^3 \lambda
\end{aligned}$$

It follows that the eigenvalues of A are 0 and -1.

(ii) We compute bases of the corresponding eigenspaces. For $\lambda = 0$ we obtain $E(0, A) = N(A)$ which we compute using Gaussian elimination:

$$\begin{pmatrix} -3 & -6 & 0 & 4 \\ -1 & -4 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ -3 & -9 & 0 & 5 \end{pmatrix} \overset{I \leftrightarrow II}{\rightsquigarrow} \begin{pmatrix} -1 & -4 & 0 & 2 \\ -3 & -6 & 0 & 4 \\ 0 & 0 & -1 & 0 \\ -3 & -9 & 0 & 5 \end{pmatrix} \overset{II-3I, IV-3I}{\rightsquigarrow} \begin{pmatrix} -1 & -4 & 0 & 2 \\ 0 & 6 & 0 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}$$

From this it is easy to see that a basis for the eigenspace is given by $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\}$.

For $\lambda = -1$ we obtain that $E(-1, A) = N(A + I_4)$ which we compute using Gaussian elimination:

$$\begin{pmatrix} -2 & -6 & 0 & 4 \\ -1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ -3 & -9 & 0 & 6 \end{pmatrix} \overset{I \leftrightarrow II}{\rightsquigarrow} \begin{pmatrix} -1 & -3 & 0 & 2 \\ -2 & -6 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ -3 & -9 & 0 & 6 \end{pmatrix} \overset{II-2I, IV-3I}{\rightsquigarrow} \begin{pmatrix} -1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that a basis for this eigenspace is given by $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

(iii) Yes, A is diagonalisable since for both eigenvalues the algebraic multiplicity coincides with the geometric multiplicity. For $\lambda = 0$ both are equal to 1 while for $\lambda = -1$ both are equal to 3.

Possible solution 3b: Note that in this solution we changed the order in which we solve the three parts of the exercise.

(ii) We compute the eigenspaces of the eigenvalues 0 and -1 as in Solution 3a.

(iii) Since by (ii) the geometric multiplicities of 0 and -1 are 1 and 3, respectively, we see that $1 + 3 = 4$, thus the sum of the geometric multiplicities is equal to the size of the matrix. Therefore the matrix is diagonalisable.

(i) We know that eigenvectors corresponding to different eigenvalues are linearly independent. But according to (iii), $\mathbb{R}^4$ has a basis consisting of eigenvectors for A. Therefore, an eigenvector to a different eigenvalue cannot exist and therefore 0 and -1 are the only eigenvalues (we already checked in (ii) that they are indeed eigenvalues).

4. Let $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be a basis of $M_{2\times 2}(\mathbb{R})$.

(i) Determine the coordinate vector of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ with respect to the basis B.

(ii) Let $f\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear function given by $f(A) = \begin{pmatrix} 8 & 3 \\ 5 & 6 \end{pmatrix} A$. Determine the matrix $[f]_{B\leftarrow B}$ with respect to the basis B of $M_{2\times 2}(\mathbb{R})$.

Possible solution 4a: (i) Since $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \mathbf{1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{3}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{4}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, it follows that the coordinate vector of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ with respect to the basis B is $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

(ii) We compute the image $f(b_i)$ of each of the basis vectors b_i in B and express them in the basis B:

$$f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 8 & 0 \\ 5 & 0 \end{pmatrix} = \mathbf{8}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{0}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{5}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{0}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 8 \\ 0 & 5 \end{pmatrix} = \mathbf{0}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{8}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{0}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{5}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 3 & 0 \\ 6 & 0 \end{pmatrix} = \mathbf{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{0}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{6}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{0}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} 0 & 3 \\ 0 & 6 \end{pmatrix} = \mathbf{0}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{3}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{0}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{6}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore,

$$[f]_{B'\leftarrow B} = \begin{pmatrix} 8 & 0 & 3 & 0 \\ 0 & 8 & 0 & 3 \\ 5 & 0 & 6 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}.$$

5. Let V be a vector space. Let $\{b_1, b_2, b_3\}$ be a basis of V.

(i) Let $f\colon V \to \mathbb{R}^2$ be a function such that $f(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}$. Prove that f is linear.

(ii) Determine the matrix $[f]_{E\leftarrow B}$ of f with respect to the basis B of V and the standard basis E of $\mathbb{R}^2$.

Possible solution 5a: (i) We know from the lecture that the function $c_B\colon V \to \mathbb{R}^3, \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 \mapsto \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}$ is well-defined and linear. Furthermore we know that the function $g\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by multiplication with $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$ is linear. Therefore the function $f = g \circ c_B$ is linear.

(ii) We have that

$$f(b_1) = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$f(b_2) = \begin{pmatrix}0\\ 1\end{pmatrix}$$

$$f(b_3) = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Therefore $[f]_{E\leftarrow B} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$.

Possible solution 5b:

only for part (i) Since $\{b_1, b_2, b_3\}$ is a basis of V we know that every vector $v \in V$ has a unique expression as $v = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$. Let $v, v' \in V$. Write

$$v = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$$
$$v' = \mu_1 b_1 + \mu_2 b_2 + \mu_3 b_3$$

Then

$$f(v + v') = f((\lambda_1 + \mu_1)b_1 + (\lambda_2 + \mu_2)b_2 + (\lambda_3 + \mu_3)b_3) = \begin{pmatrix}\lambda_1 + \mu_1\\ \lambda_2 + \mu_2\end{pmatrix} = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} + \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = f(v) + f(v')$$

and

$$f(\lambda v) = f((\lambda\lambda_1)b_1 + (\lambda\lambda_2)b_2 + (\lambda\lambda_3)b_3) = \begin{pmatrix}\lambda\lambda_1\\ \lambda\lambda_2\end{pmatrix} = \lambda\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \lambda f(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) = \lambda f(v)$$

Therefore, f is linear.

6. Let $U = \operatorname{span}\left(\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\-2\end{pmatrix}\right)$.

(i) Give the definition of when a basis of an inner product space V is called orthonormal.
(ii) Find an orthonormal basis of U.

Possible solution 6a: (i) Let V be an inner product space with inner product $\langle -, - \rangle$. Then a basis $\{b_1, \dots, b_n\}$ of V is called ***orthonormal*** if $\langle b_i, b_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

(ii) We compute an orthonormal basis of U using the Gram–Schmidt process starting from the given basis of U. Call the given vectors $v_1 = \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, v_2 = \begin{pmatrix}1\\-2\\1\\0\end{pmatrix}, v_3 = \begin{pmatrix}1\\0\\1\\-2\end{pmatrix}$.

Then,

$$b'_1 = v_1, \qquad \|b'_1\| = \sqrt{1+1+1+1} = 2, \qquad b_1 = \frac{1}{\|b'_1\|} b'_1 = \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$$

$$b'_2 = v_2 - \frac{\langle v_2, b'_1 \rangle}{\langle b'_1, b'_1 \rangle} b'_1 = v_2 - \frac{-2}{4} b'_1 = \begin{pmatrix}\frac{3}{2}\\-\frac{3}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}, \qquad \|b'_2\| = \sqrt{\frac{9}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4}} = \sqrt{5}$$

$$b_2 = \frac{1}{\|b'_2\|} b'_2 = \frac{1}{2\sqrt{5}}\begin{pmatrix}3\\-3\\1\\-1\end{pmatrix}$$

$$b'_3 = v_3 - \frac{\langle v_3, b'_1 \rangle}{\langle b'_1, b'_1 \rangle} b'_1 - \frac{\langle v_3, b'_2 \rangle}{\langle b'_2, b'_2 \rangle} b'_2 = v_3 - \frac{2}{4} b'_1 - \frac{3}{5} b'_2 = \begin{pmatrix}-\frac{2}{5}\\\frac{2}{5}\\\frac{6}{5}\\-\frac{6}{5}\end{pmatrix}$$

$$\|b'_3\| = \sqrt{\frac{4}{25} + \frac{4}{25} + \frac{36}{25} + \frac{36}{25}} = \sqrt{\frac{80}{25}} = \frac{4\sqrt{5}}{5}$$

$$b_3 = \frac{1}{\|b'_3\|} b'_3 = \begin{pmatrix}-\frac{1}{2\sqrt{5}}\\\frac{1}{2\sqrt{5}}\\\frac{3}{2\sqrt{5}}\\\frac{-3}{2\sqrt{5}}\end{pmatrix}$$

7. Solve the following system of differential equations

$$y_1'(t) = 4y_1(t) + 2y_2(t)$$
$$y_2'(t) = -y_1(t) + y_2(t)$$

with the initial condition $y_1(0) = 2, y_2(0) = 3$.

Possible solution 7a: We write the system in matrix form:

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

As a next step we compute the eigenvalues for $\begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$ as the zeroes of its characteristic polynomial:

We obtain $\chi_A(\lambda) = \det \begin{pmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{pmatrix} = (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6$. Using the pq-formula we see that $\lambda_{1/2} = \frac{5}{2} \pm \sqrt{\frac{25}{4} - \frac{24}{4}} = \begin{cases} 3 \\ 2 \end{cases}$

The next step is to compute basis of the corresponding eigenspaces (which we know to be 1-dimensional as the geometric multiplicity is between 1 and the algebraic multiplicity which is also 1).

We have $A - 3I_2 = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$ and therefore a basis of $E(3, A)$ is given by $\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$.

We have that $A - 2I_2 = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$ and therefore a basis of $E(2, A)$ is given by $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

With the substitution $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ we obtain the system $u_1'(t) = 3u_1(t), u_2'(t) = 2u_2(t)$ which has the solution $u_1 = c_1 e^{3t}, u_2 = c_2 e^{2t}$. Substituting back we obtain

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 2c_1 e^{3t} + c_2 e^{2t} \\ -c_1 e^{3t} - c_2 e^{2t} \end{pmatrix}$$

Taking into account the initial condition $y_1(0) = 2, y_2(0) = 3$ we obtain the additional condition that $\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ -c_1 - c_2 \end{pmatrix}$ which one sees to have the solution $c_1 = 5, c_2 = -8$.

Therefore a solution to the above system of differential equations with the above initial condition is given by

$$y_1(t) = 10e^{3t} - 8e^{2t}$$
$$y_2(t) = -5e^{3t} + 8e^{2t}$$

8. (i) On $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, x, y \in \mathbb{R}, x + y = 2 \right\}$ define an addition $\boxplus$ and a scalar multiplication $\boxdot$ via

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxplus \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + x' - 1 \\ y + y' - 1 \end{pmatrix}$$

$$\lambda \boxdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x - \lambda + 1 \\ \lambda y - \lambda + 1 \end{pmatrix}$$

(We checked in the lecture that this defines a vector space.)

Let W be the subspace of $\mathbb{R}^2$ given by $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, x + y = 0 \right\}$.

Let $g \colon V \to W$ be the function defined by $g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix}$. Show that g is an isomorphism.

(ii) What is $\dim V$? Justify your answer.

Possible solution 8a: (i) To show that g is an isomorphism we have to show that it is linear, injective, and surjective.

To show that it is linear we show that $g\left(\begin{pmatrix} x \\ y \end{pmatrix} \boxplus \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) + g\left(\begin{pmatrix} x' \\ y' \end{pmatrix}\right)$ and $g\left(\lambda \boxdot \begin{pmatrix} x \\ y \end{pmatrix}\right) = \lambda g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$.

We have that

$$g\left(\begin{pmatrix} x \\ y \end{pmatrix} \boxplus \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = g\left(\begin{pmatrix} x + x' - 1 \\ y + y' - 1 \end{pmatrix}\right) = \begin{pmatrix} x + x' - 2 \\ y + y' - 2 \end{pmatrix} = \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} + \begin{pmatrix} x' - 1 \\ y' - 1 \end{pmatrix} = g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) + g\left(\begin{pmatrix} x' \\ y' \end{pmatrix}\right)$$

$$g\left(\lambda \boxdot \begin{pmatrix} x \\ y \end{pmatrix}\right) = g\left(\begin{pmatrix} \lambda x - \lambda + 1 \\ \lambda y - \lambda + 1 \end{pmatrix}\right) = \begin{pmatrix} \lambda x - \lambda \\ \lambda y - \lambda \end{pmatrix} = \lambda \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} = \lambda g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

Therefore g is linear.

To show that it is injective we show that $\ker(f) = \{0_V\}$. Assume that $g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. It follows from the definition of g that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We have shown in the lecture that this is the zero vector of V. Therefore f is injective.

To show that g is surjective let $\begin{pmatrix} x \\ y \end{pmatrix} \in W$. It is easy to see that $g\left(\begin{pmatrix} x + 1 \\ y + 1 \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$. Therefore g is surjective.

(ii) We know that W is one-dimensional since it is given as the null space of a rank 1 matrix with 2 columns. By a result in the lecture we know that isomorphic spaces have the same dimension. Therefore $\dim V = 1$.

Possible solution 8b: (ii) The dimension of a vector space V is defined to be the number of basis vectors in a basis for V. We claim that $\left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$ is a basis for V. Note that the zero vector in V is $\begin{pmatrix}1\\1\end{pmatrix}$, therefore $\begin{pmatrix}0\\2\end{pmatrix}$ is not the zero vector, and this set is linearly independent. We show that it is also spanning. Let $\begin{pmatrix}x\\y\end{pmatrix} \in V$. We know that $y = 2 - x$. Therefore $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\2-x\end{pmatrix} = (1-x) \boxdot \begin{pmatrix}0\\2\end{pmatrix}$. Therefore the set is also spanning and hence a basis. It follows that $\dim V = 1$.

(i) We show that g is linear in the same way as in Solution 8a. Since we know that $\dim W = 1 = \dim V$ (see Solution 8a and part (ii) of Solution 8b) it suffices to prove that g is injective to show that g is an isomorphism. Assume that $g\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = g\left(\begin{pmatrix}x'\\y'\end{pmatrix}\right)$. Then $\begin{pmatrix}x-1\\y-1\end{pmatrix} = \begin{pmatrix}x'-1\\y'-1\end{pmatrix}$ and therefore $x = x'$ and $y = y'$. Thus, $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x'\\y'\end{pmatrix}$ and it follows that g is injective (and therefore an isomorphism).