ADAPTIVE MORPHOLOGICAL OPERATORS DEFINED ON GRAPHS

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ABSTRACT

Adaptive mathematical morphology has recently become a popular topic in the mathematical morphology community. In particular, the construction of adaptive structuring elements that adjust their size and shape to the local image structures have increased a lot of attention in recent years. Apart from that, there is a growing interest for representation of digital objects as graph structures. This review aims to give an overview of the methods that have been used to define adaptive morphological operators on graphs. Among the methods discussed is the classical one that is a counterpart of the adaptive structuring elements often called morphological amoebas, while the other uses partial differential equations on graphs to describe morphological operators. The review concludes by assessing the prospectives of these two different approaches for adaptive morphology on graph spaces, and by identifying possible paths for future research.

1. INTRODUCTION

Mathematical morphology has been introduced by Matheron and Serra [1], [2], and it provides a set of methods for filtering and segmentation that are very useful for various applications in image analysis, such as shape analysis, texture analysis, biomedical applications, text recognition, etc. Mathematical morphology is well defined theory, where morphological operators are defined on complete lattices [3]. Such operators are defined utilizing patterns, which are often called structuring elements, that are used to probe the image and enhance desirable features in it. When mathematical morphology was introduced the size and shape of structuring elements remain fixed for every point in the image.

Let D be a subset of the Euclidean space $\mathbb{Z}^2$ that corresponds to the support of the image, and let $T \subset \mathbb{R}$ be a set that corresponds to the gray level values in the image. Then, a gray valued image can be represented by a function $f : D \to T$. Let $\mathcal{L}$ be a lattice of gray valued functions with domain D and range T. For a fixed structuring element $B \subset D$, an erosion $\varepsilon : \mathcal{L} \to \mathcal{L}$ and a dilation $\delta : \mathcal{L} \to \mathcal{L}$ of a function f can be represented, respectively, as

$$\varepsilon(f)(x) = \bigwedge_{z \in B} f(x - z),\ \delta(f)(x) = \bigvee_{z \in B} f(x + z),$$

where $\bigwedge$ and $\bigvee$ denote the infimum and supremum, respectively. Other morphological operators, such as opening and closing can be constructed using these basic morphological operators. Here, it should be stressed that an erosion and a dilation should be adjunct operators in order to compute morphological opening and closing, and consequently alternating sequential filters. Only if an erosion and a dilation are adjunct operators, their superpositions satisfy properties of openings and closings. In particular, an operator is a morphological opening if it is idempotent, increasing and anti-extensive. Two operators ε and δ are adjunct operators if the following relation is satisfied

$$\delta(f) \leq g \Leftrightarrow f \leq \varepsilon(g), \tag{1}$$

where $f, g : D \to T$.

An alternative way to define morphological operators is to use a framework based on partial differential equations [4]. Morphological operators, an erosion and a dilation of an input function f^0 can be described, respectively, by the following PDEs

$$\partial_t f(x,t) = -\|\nabla f(x,t)\|_p, x \in D, \tag{2}$$

$$\partial_t f(x,t) = \|\nabla f(x,t)\|_p, x \in D, \tag{3}$$

where the initial condition is $f(\cdot, 0) = f^0(\cdot)$, and with respect to structuring element tB, where $B = \{z \in D : \|z\|_p \leq 1\}$. Here, $\|\cdot\|_p$ denotes the classical l_p norm. Morphological operators defined in this way can be used for non-digitally scalable structuring elements, where shape of the structuring element cannot be correctly represented on a digital grid. Furthermore, it allows subpixel accuracy, and more important for the context of this paper, it allows incorporating adaptability directly into the PDEs that model morphological erosions and dilations [5], [6].

In this paper, we present a review on adaptive morphological operators defined in graph spaces. First, in Section 2, we revisit a recent work on adaptive mathematical morphology. Adaptive morphological operators defined on graph are presented in Section 3. Section 4 discusses possible future extensions of adaptive morphology to hypergraphs and cell complexes, and concludes the paper.

2. ADAPTIVE MATHEMATICAL MORPHOLOGY

Adaptive mathematical morphology has been recently an area of intense interest [7], [8], [9], [10], [11], [12], [13], [14], [15], which deals with adaptive structuring elements that adapt their shape and size according to the local image structures. Structuring elements can be spatially variant, but also can adapt to the intensity levels. A good overview on this topic can be found in [16].

Most known spatially adaptive structuring elements are called morphological amoebas [7]. A morphological amoeba is defined as a geodesic ball in a metric space determined by the path-based amoeba distance. Let $\mathcal{P}_{xy}$ is a path that connects pixels x and y, and consists of points $\{x_1, x_2, ..., x_{n+1}\}$, such that $x_1 = x$ and $x_{n+1} = y$, and x_i, x_{i+1}, $i = 1, ..., n$ are two adjacent pixels. The amoeba distance, d_A, is defined as

$$d_A(x, y) = \min_{\mathcal{P}_{xy}} \sum_{i=1}^{n} \Big(1 + \lambda|f(x_i) - f(x_{i+1})|\Big),$$

where 1 stands for a spatial distance $\|x_i - x_{i+1}\|$ and $\lambda > 0$ is a constant. Then, a morphological amoeba centered in a point $x \in D$ is defined as [7]

$$A_r(x) = \{y \in D : d(x, y) < r\},$$

where r is the radius of the morphological amoeba. Instead of using $\|x_i - x_{i+1}\| = 1$ for a spatial distance, other distances can be used, such as the Euclidean distance $\|x_i - x_{i+1}\|_2$, or $\langle 3, 4\rangle$ weighted distance.

As pointed in [7], the morphological amoebas should be computed on a smoothed version of the input image f, in order to reduce the influence of noise that can be present in the image. This smoothed version of the input image is often called a pilot image. Moreover, the same pilot image should be used for the construction of the erosion and its adjunct dilation, in order to obtain corresponding opening and closing [13].

3. ADAPTIVE MORPHOLOGICAL OPERATORS DEFINED ON GRAPHS

Recently, an increasing interest is to consider digital object as graphs structures. In this representation of digital objects, not only the object points are considered, but also adjacency relations between these image points can be included. Therefore, the image domain is considered as a graph whose vertices are pixels and whose edges are adjacency relations between these pixels. Let $G = (V, E, w)$ is a graph, where V is the set of vertices and $E \subset V \times V$ is the set of edges. The edge (u, v) between two neighbouring vertices $u, v \in V$ we will denote with e_{uv}. A neighbourhood of $v \in V$ is defined as $N(v) = \{u \in V : (u, v) \in E\}$, and $\bar{N}(v) = N(v) \cup \{v\}$. The weight function $w : E \to \mathbb{R}^+$ can be defined as $w(uv) = w_{uv}$ if $e_{uv} \in E$, otherwise $w_{uv} = 0$. In this paper, we consider non-oriented graphs, i.e., E is a set of non-ordered pairs of edges (u, v), i.e., $w_{uv} = w_{vu}$. Also, graphs have no loop, i.e., $(\forall u \in V)$ $(u, u) \notin E$.

A pioneer work of mathematical morphology on graphs has been undertaken by Vincent [17], where morphological operators are proposed for weighed and unweighted graphs. These operators are defined on the set of vertices V. For instance, the erosion and the dilation of graph G are defined by

$$\varepsilon(G)(v) = \min\{f(v) : v \in \bar{N}(u)\}$$

$$\delta(G)(v) = \max\{f(v) : v \in \bar{N}(u)\},$$

where function $f : V \to \mathbb{R}^+$ corresponds to the gray level values in the image. Note that, these morphological operators are defined only for the set of vertices V. Apart from this initial work on mathematical morphology on graphs, a recent work of Meyer et al. [19], [18] as well as Cousty et al. [20], [21] proposed morphological operators as operators that are derived on a set of edges and produce a set of vertices, and opposite, morphological operators transform a set of vertices to a set of edges. A graph representation allows that adjacency relation be spatially invariant, but also to be spatially variant, which is the focus of this paper.

In this paper, we consider two approaches for spatially adaptive mathematical morphology on weighted graphs. First, we consider the approach based on discrete morphological operators that are counterpart of morphological amoebas for images [22]. Second, we review the approach based on PDE mathematical morphology defined on graphs [23].

3.1. Morphological amoebas for graphs [22]

Morphological amoebas are computed for each pixel in the image, and hence the resulting computational cost is relatively high. To overcome this problem, a set of predefined paths computed from a minimal spanning tree was considered [22].

Isotropic structuring elements of size s that is centered at a node u is defined as

$$B(u) = \{v \in V : d_0(u, v) \le s\},$$

where the number of the edges in the shortest path $\mathcal{P}_{uv} = \{e_{u\,u+1}, e_{u+1\,u+2}, ..., e_{u+s-1\,v}\}$ from the node u to the node v is computed according to the distance

$$d_0(u, v) = \min_{\mathcal{P}_{uv}} \left(\sum_{e_{ij} \in \mathcal{P}_{uv}} 1 \right).$$

In this case, the shape of structuring elements depends only on the considered distance measure d_0.

Accordingly, a morphological amoeba in the node u with the size of s and radius r can be defined as

$$A^r_s(u) = \{v \in V : d_0(u, v) \le s,\ \lambda d_1(u, v) \le r\},$$

where $d_1(u,v)$ is the length of the shortest path $\mathcal{P}_{uv}$ with respect to weights from node u to node v using l_1 norm, i.e.,

$$d_1(u,v) = \min_{\mathcal{P}_{uv}} \left(\sum_{e_{uv} \in \mathcal{P}_{uv}} w_{uv} \right).$$

Opposite from the isotropic structuring elements, for adaptive structuring elements the edges of the graph have weights and the weights are depends on the image structure. The weights for morphological amoebas [7] on graphs are defined as

$$w_{uv} = |f(u) - f(v)|, \quad e_{uv} \in E.$$

One can consider other distance measures for the construction of morphological amoebas as well. For instance, morphological amoebas using l_∞ norm

$$d_\infty(u,v) = \min_{\mathcal{P}_{uv}} \Big(\max_{e_{uv} \in \mathcal{P}_{uv}} w_{uv} \Big),$$

are defined as

$$A_s^r(u) = \{v \in V : d_0(u,v) \le s,\ \lambda d_\infty(u,v) \le r\}.$$

Similarly to morphological amoebas, an erosion ε and dilation δ for adaptive structuring elements $A_s^r(u)$, which satisfy adjunction property (1) can be defined, respectively, as

$$(\varepsilon(f))(u) = \bigwedge_{v \in A_s^r(u)} f(v), \quad u \in V,$$

$$(\delta(f))(u) = \bigvee_{v \in \check{A}_s^r(u)} f(v), \quad u \in V,$$

where $\check{A}_s^r(x)$ is the reflected neighbourhood defined as

$$v \in A_s^r(u) \Leftrightarrow u \in \check{A}_s^r(v). \tag{4}$$

Then, the corresponding opening and closing are defined by $\gamma(f)(u) = (\delta(\varepsilon))(f)(u)$ and $\psi(f)(u) = (\varepsilon(\delta))(f)(u)$, $u \in V$, respectively.

3.2. PDE-based adaptive mathematical morphology [23]

As presented in Section 2, morphological operators can be described by partial differential equations (2), (3). Nevertheless, the graphs are discrete structures and these equations should be adapted to them. Since basic morphological operator, an erosion and a dilation can be modelled by the differential equations that contain the gradient of the image, we should find a way how to define the gradient on a graph. It is a logical choice to choose the weighted difference $\sqrt{w_{uv}}(f(v)-f(u))$ as an approximation of the first derivative in a point $u \in V$ and the direction to a vertex v. Then, the gradient of a function f at a vertex u can be defined as

$$(\nabla f)(u) = \sum_{v \in N(u)} \sqrt{w_{uv}} \big(f(v) - f(u)\big).$$

To use the equations (2), (3) for a weighted graph $G = (V,E,w)$, we consider the following sets: $\partial^+ U = \{u \in V \setminus U : \exists v \in U,\ v \in N(u)\}$ and $\partial^- U = \{u \in U : \exists v \in E \setminus U,\ v \in N(u)\}$ be the external and internal boundary of a set U.

Then, morphological operators can be considered as

$$\text{erosion}: \ \varepsilon : \partial_t f(u,t) = -\|(\nabla^- f)(u,t)\|_p$$

$$\text{dilation}: \ \delta : \partial_t f(u,t) = \|(\nabla^+ f)(u,t)\|_p,$$

for every $u \in V$ and for the initial condition $f(\cdot,0) = f^0(\cdot)$, and where

$$(\nabla^- f)(u,t) = \sum_{v \in N(u)} \sqrt{w_{uv}} \left| \min(0, f(v) - f(u)) \right|$$

$$(\nabla^+ f)(u,t) = \sum_{v \in N(u)} \sqrt{w_{uv}} \left| \max(0, f(v) - f(u)) \right|.$$

To use these equations, it is necessary to consider their composition into level sets, denoted here for a function f as $f^l = \kappa(f-l)$, where $\kappa : V \to \{0,1\}$ is the indicator function. Hence, morphological operators can be defined as

$$\text{erosion}: \ \varepsilon : \partial_t f^l(u,t) = -\|(\nabla^- f^l)(u,t)\|_p$$

$$\text{dilation}: \ \delta : \partial_t f^l(u,t) = \|(\nabla^+ f^l)(u,t)\|_p,$$

for all level sets l. Intuitively, a dilation (resp. erosion) over $U \subset V$ can be interpreted as a growth (resp. contraction) process that adds (resp. removes) vertices from $\partial^+ U$ (resp. $\partial^- U$) to U. These resulting morphological operators corresponds to the definitions of morphological operators proposed by Vincent [17] for unweighted graphs. Dilation of f^l over U^l corresponds only to the set $\partial^+ U^l$. Similarly, erosion of f^l over U^l corresponds only to the set $\partial^- U^l$.

Adaptive morphological operators can be consider through adaptivity of graph weights and graph topology. Weights can be used for the edges of the graph. The simplest choice is to have $w(uv) = 1$, i.e., to consider unweighted graphs. Nevertheless, a typical weighting is often obtained using one of the following functions:

$$w(uv) = \exp\left(\frac{-\rho(f(u), f(v))^2}{\sigma^2} \right),$$

or

$$w(uv) = \frac{1}{1 + \rho(f(u), f(v))},$$

where ρ is a distance measure. A weighting function w can be seen as a local speed function that controls the morphological operators.

The other approach to adapt morphological operators, is to consider different vertex neighbourhood as a structuring element. Then, a structuring element can be the same neighbourhood for each vertex, or can be adaptive according the position and weights of the vertices and its edges. The adaptive structuring elements on the graphs can be defined as $\tau-$neighbourhood as $N(u) = \{v \in V : d(u,v) \leq \tau\}$, where $\tau > 0$ is the radius of the adaptive structuring element. Here, $d : V \times V \rightarrow \mathbb{R}^+$ is a pairwise distance function. Similarly, it can be defined $k-$nearest neighbour graph where adaptive structuring elements is determined by the k-nearest neighbour of the considered vertex. More details on different neighbourhood on graphs can be found in [17].

Interestingly enough, Ta et al. [23] did not consider, nor even mention when an erosion and a dilation for adaptive structuring elements are adjunct morphological operators. As pointed by Roerdink [13], this is a common misunderstanding of adaptive morphological operators, since to construct an erosion and its adjunct dilation (1), the condition (4) has to be satisfied.

4. DISCUSSION AND PROSPECTIVES

It has been shown, in a number of experiments, that adaptive morphological operators have advantages over using the classic ones [7, 8, 15]. Nevertheless, there is still a problem how to find a superior and efficient method to construct adaptive structuring elements, and consequently adaptive morphological operators for a particular application. One of the possibles approaches to overcome these issue is to construct morphological operators using a graph representation of the image. In this paper, we have revisited two different approaches to define adaptive morphological operators on graphs, one that mimic morphological amoebas, and one that is based on PDEs which describe morphological operators.

Morphology operators has been recently defined on the other structures than graphs. Recent work of Dias et al. [24] and Bloch and Bretto [25] have proposed morphological operators on cell complexes and hypergraphs, respectively. Initial experiments with these two approaches have promising results since they provide morphological operators that have good performance, and they can be less computationally expensive in some cases. Hence, here, we speculate that graph-based mathematical morphology defined on complex structures, such as cell complexes and hypergraphs will be in a focus of the morphological community in the following years.

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