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Multilevel Monte Carlo Methods for failure Probabilities

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Introduction: **Failure Probabilities**

- $X(u(\omega)) : V \to \mathbb{R}$ – A quantity of interest (functional) of the solution u to some model problem with stochastic input ω

Definition: failure probability

The failure probability p given y is:

$$p = \Pr(X \leq y) \quad \text{or}$$
$$p = F(y),$$

where $F(\cdot)$ is the cdf associated with X.

- **Goal** – Estimate the probability $p \approx \widehat{Q}$ to a given root mean square error (RMSE), $e(\widehat{Q}) \leq \varepsilon$, using minimal computational cost

Multilevel Monte Carlo for failure probabilities

- Let $Q(\omega) = \mathbb{1}(X(\omega) < y)$ and $Q_\ell^\delta(\omega) = \mathbb{1}(X_\ell^\delta(\omega) < y)$ be binomial distributed random variables
- Let $Q_{-1}^\delta(\omega) = 0$ then the MLMC estimator reads

$$\widehat{Q}^{ML}_{\{N_\ell\},\delta} = \sum_{\ell=0}^{L} N_\ell^{-1} \sum_{i=1}^{N_\ell} \left(Q_\ell^\delta(\omega_\ell^i) - Q_{\ell-1}^\delta(\omega_\ell^i) \right)$$

Assumption

We have that

M1 $|\mathbb{E}\left[Q_\ell^\delta(\omega) - Q(\omega)\right]| \leq C_1 \delta_\ell$,

M2 $\mathbb{V}\left[Q_\ell^\delta(\omega) - Q_{\ell-1}^\delta(\omega)\right] \leq C_2 \delta_\ell$ for $\ell \geq 1$,

M3 $\mathcal{C}(Q_\ell^\delta(\omega)) = C_3 \delta_\ell^\alpha$,

are satisfied where C_1, C_2, and C_3 do not depend on the sample or the underlying discretization, and α is some constant.

Theorem

Then there exist a constant L and a sequence $\{N_\ell\}$ such that the RMSE is less then ε, with the required work in terms of ε,

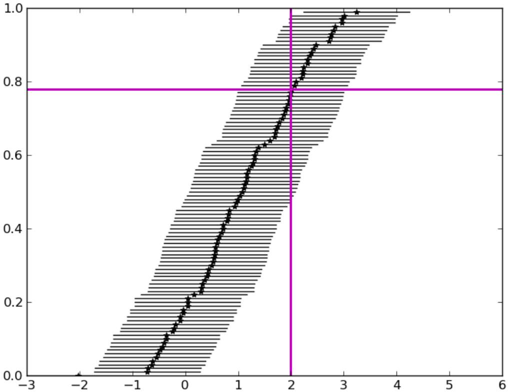
$$\mathbb{E}\left[\mathcal{C}_q\left(\widehat{Q}^{ML}_{\{N_\ell\},\delta}\right)\right] \lesssim \begin{cases} \varepsilon^{-2} & \alpha < 1 \\ \varepsilon^{-2}(\log \varepsilon)^2 & \alpha = 1 \\ \varepsilon^{-1-\alpha} & \alpha > 1 \end{cases} .$$

Selective algorithm

For each sample $\omega_\ell^i \in \Omega$ we have

$$|X(\omega_\ell^i) - X_\ell^\delta(\omega_\ell^i)| \leq \delta_\ell^i \quad \text{and} \quad \mathcal{C}(X_\ell^\delta(\omega_\ell^i)) = (\delta_\ell^i)^{-q}.$$

- $N = 100$
- $y = 2$
- $\delta_\ell^i = 1$
- $\# I_0 = 100$

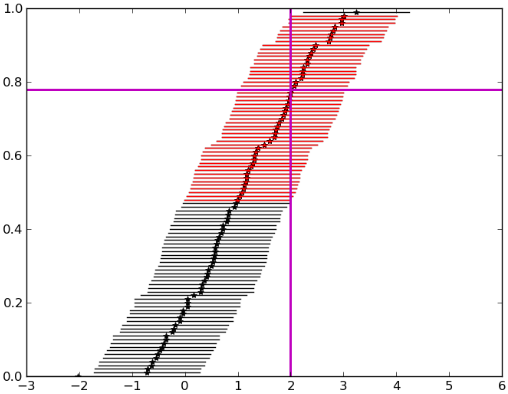


Selective algorithm

For each sample $\omega_\ell^i \in \Omega$ we have

$$|X(\omega_\ell^i) - X_\ell^\delta(\omega_\ell^i)| \leq \delta_\ell^i \quad \text{and} \quad \mathcal{C}(X_\ell^\delta(\omega_\ell^i)) = (\delta_\ell^i)^{-q}.$$

- $N = 100$
- $y = 2$
- $\delta_\ell^i = 1$
- $\# I_1 = 51$

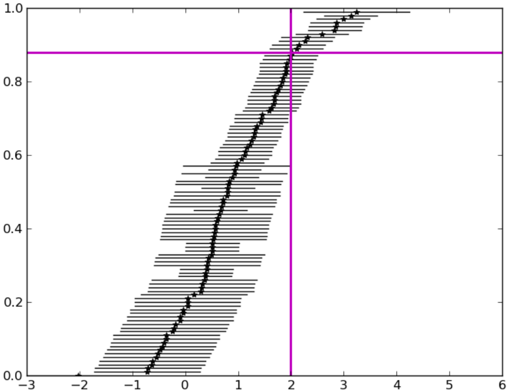


Selective algorithm

For each sample $\omega_\ell^i \in \Omega$ we have

$$|X(\omega_\ell^i) - X_\ell^\delta(\omega_\ell^i)| \leq \delta_\ell^i \quad \text{and} \quad \mathcal{C}(X_\ell^\delta(\omega_\ell^i)) = (\delta_\ell^i)^{-q}.$$

- $N = 100$
- $y = 2$
- $\delta_\ell^i = 0.5$
- $\# I_1 = 51$

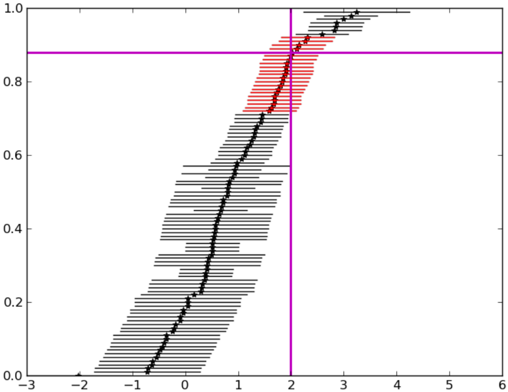


Selective algorithm

For each sample $\omega_\ell^i \in \Omega$ we have

$$|X(\omega_\ell^i) - X_\ell^\delta(\omega_\ell^i)| \le \delta_\ell^i \quad \text{and} \quad \mathcal{C}(X_\ell^\delta(\omega_\ell^i)) = (\delta_\ell^i)^{-q}.$$

- $N = 100$
- $y = 2$
- $\delta_\ell^i = 0.5$
- $\# I_2 = 21$

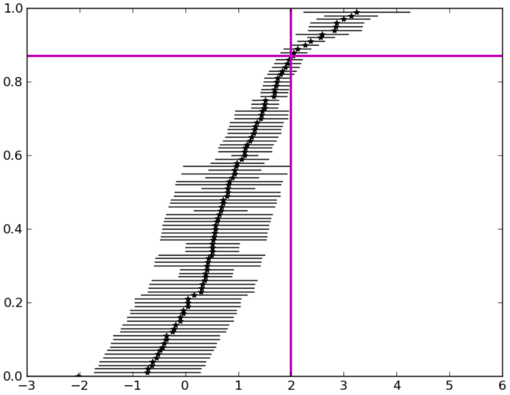


Selective algorithm

For each sample $\omega_\ell^i \in \Omega$ we have

$$|X(\omega_\ell^i) - X_\ell^\delta(\omega_\ell^i)| \leq \delta_\ell^i \quad \text{and} \quad \mathcal{C}(X_\ell^\delta(\omega_\ell^i)) = (\delta_\ell^i)^{-q}.$$

- $N = 100$
- $y = 2$
- $\delta_\ell^i = 0.25$
- $\# I_2 = 21$

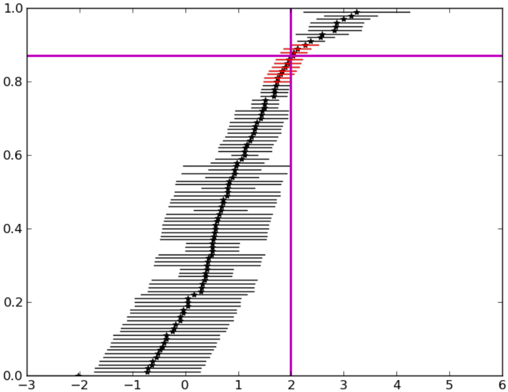


Selective algorithm

For each sample $\omega_\ell^i \in \Omega$ we have

$$|X(\omega_\ell^i) - X_\ell^\delta(\omega_\ell^i)| \leq \delta_\ell^i \quad \text{and} \quad \mathcal{C}(X_\ell^\delta(\omega_\ell^i)) = (\delta_\ell^i)^{-q}.$$

- $N = 100$
- $y = 2$
- $\delta_\ell^i = 0.25$
- $\#I_3 = 11$

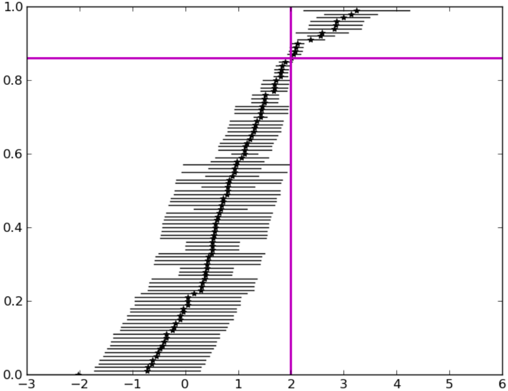


Selective algorithm

For each sample $\omega_\ell^i \in \Omega$ we have

$$|X(\omega_\ell^i) - X_\ell^\delta(\omega_\ell^i)| \leq \delta_\ell^i \quad \text{and} \quad \mathcal{C}(X_\ell^\delta(\omega_\ell^i)) = (\delta_\ell^i)^{-q}.$$

- $N = 100$
- $y = 2$
- $\delta_\ell^i = 0.0125$
- $\# I_3 = 11$

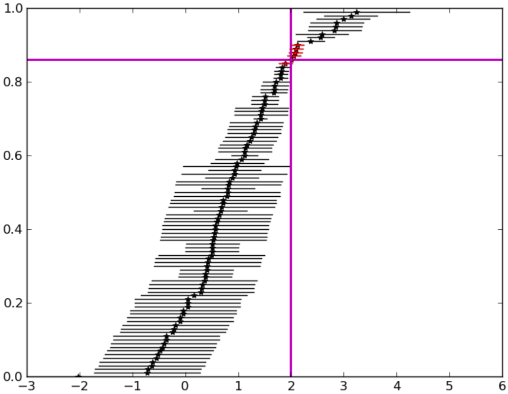


Selective algorithm

For each sample $\omega_\ell^i \in \Omega$ we have

$$|X(\omega_\ell^i) - X_\ell^\delta(\omega_\ell^i)| \le \delta_\ell^i \quad \text{and} \quad \mathcal{C}(X_\ell^\delta(\omega_\ell^i)) = (\delta_\ell^i)^{-q}.$$

- $N = 100$
- $y = 2$
- $\delta_\ell^i = 0.0125$
- $\# I_4 = 6$

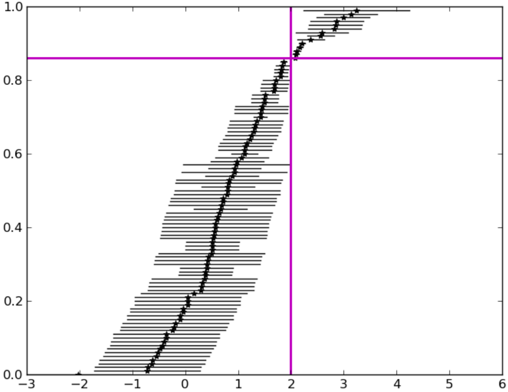


Selective algorithm

For each sample $\omega_\ell^i \in \Omega$ we have

$$|X(\omega_\ell^i) - X_\ell^\delta(\omega_\ell^i)| \le \delta_\ell^i \quad \text{and} \quad \mathcal{C}(X_\ell^\delta(\omega_\ell^i)) = (\delta_\ell^i)^{-q}.$$

- $N = 100$
- $y = 2$
- $\delta_4 = 0.00625$
- $\#I_4 = 6$



Theorem (Computable complexity for the Multilevel Monte Carlo method with selective refinement)

There exist a constant L and a sequence $\{N_\ell\}$ such that the RMSE is less then ε, with the required work in terms of ε,

$$\mathbb{E}\left[\mathcal{C}_q\left(\widehat{Q}^{MLS}_{\{N_\ell\},\delta}\right)\right] \lesssim \begin{cases} \varepsilon^{-2} & q < 2 \\ \varepsilon^{-2}(\log \varepsilon)^2 & q = 2 \\ \varepsilon^{-q} & q > 2 \end{cases},$$

The method is optimal in the sense:

- $(q < 2)$ same as the standard MC method on level $= 0$
- $(q > 2)$ same complexity as one sample on the finest level L

$$\mathbb{E}\left[\mathcal{C}_q\left(\widehat{Q}^{MLS}_{\{N_\ell\},\delta}\right)\right] \lesssim \begin{cases} N & q < 2 \\ \mathcal{C}_q\left(Q^\delta_L(\omega)\right) & q > 2 \end{cases}$$

Example 1:

Solving a PDE in $2D$ to accuracy ε, on a uniform mesh, using a numerical method with convergence rate $p = 1$, and using multigrid to solve the linear system. The computational cost is $\sim \delta^{-2}$.

Example 2:

Solving a PDE in $3D$ to accuracy ε, on a uniform mesh, using a numerical method with convergence rate $p = 1$, and using multigrid to solve the linear system. The computational cost is $\sim \delta^{-3}$.

Numerical verification: **Demonstrational problem**

- Estimate $p = F(y)$ for $q = \{1, 2, 3\}$