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Multilevel Subset Simulation

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Outline

Introduction: **Rare Event Probabilities**

Definition: Rare event

- G – The limit state functiona, stochastic variable
- The rare event probability P is:

$$P = \Pr(G \leq 0)$$

where P is unknown but **small**.

Introduction: **Problem formulation**

Model problem

- $\mathcal{M}$ – model
- Ω – sample space

We assume that there exists a unique solution u given any $\omega \in \Omega$:

$$\mathcal{M}(\omega, u) = 0.$$

- $G(u) : V \to \mathbb{R}$ – Limit state function (functional) of the solution u
- The solution u is uniquely determined by the data, $G(\omega) := G(u(\omega))$

Introduction: **Numerical errors**

Assumption: Numerical error for samples

- For each sample $\omega_i \in \Omega$ the numerical approximation $G_\ell^\epsilon(\omega_i)$ of $G(\omega_i)$ satisfies
$$|G(\omega_i) - G_\ell^\epsilon(\omega_i)| \leq \epsilon_\ell,$$
for any $\epsilon_\ell > 0$
- The work W for computing $G_\ell^\epsilon(\omega_i)$ depends on the error tolerances and satisfies
$$C\epsilon_\ell^{-q} \leq W(G^\epsilon(\omega_i)) \leq \epsilon_\ell^{-q},$$
where $C \leq 1$ and $q > 0$ are independent of ω_i

Introduction: **Standard Monte Carlo for rare events estimation**

- Large stochastic dimension ⇒ Monte Carlo based method
- Standard Monte Carlo (MC) methods need a large number of samples:

$$\frac{\sqrt{V[\widehat{P}_{MC}]}}{E[\widehat{P}_{MC}]} \leq \epsilon \Rightarrow N \gtrsim p^{-1}$$

Subset simulation: **Idea**

- Subset simulation – A variance reduction techniques
- Idea: Uses Bayes theorem

$$\Pr(B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(A|B)}$$

if $B \subset A \Rightarrow \Pr(A|B) = 1$ and we have

$$\Pr(B) = \Pr(B|A)\Pr(A).$$

- Define $\widetilde{F}_\ell = G_L^\gamma(\omega) \leq c_\ell$, where $c_0 > c_1 > \cdots > c_L = 0$
- We have that $\widetilde{F}_\ell \subset \widetilde{F}_{\ell-1}$ are subsets
- Applying Bayes theorem on the sequence of subsets, $\widetilde{F}_L \subset \widetilde{F}_{L-1} \subset \cdots \subset \widetilde{F}_0$, yields

$$\Pr(G_L^\epsilon(\omega) \leq 0) = \Pr(\widetilde{F}_L) = \Pr(\widetilde{F}_0) \prod_{\ell=1}^{L} \Pr(\widetilde{F}_\ell | \widetilde{F}_{\ell-1})$$

Subset simulation: **Work estimate**

- Choosing $\Pr(\widetilde{F}_\ell|\widetilde{F}_{\ell-1}) \sim 0.1$ we get

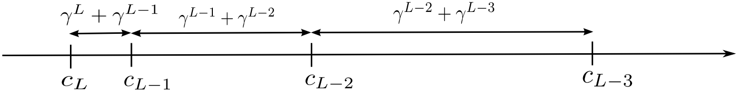
$$p \sim 0.1^L$$

  and $L \sim \log(p^{-1})$.
- Number of samples for each failure event $\Pr(\widetilde{F}_\ell|\widetilde{F}_{\ell-1})$ independent of P
- Hence, total number of samples

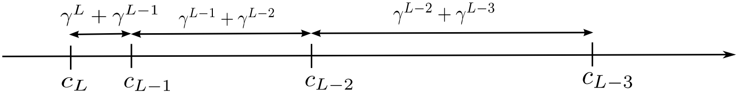
$$N \sim \log(p^{-1})$$

Multilevel subset simulation: **Subset property**

- Multilevel approach has been proven successful in other application, e.g., multigrid and multilevel Monte Carlo
- Define $F_\ell = G_\ell^\gamma(\omega) \leq c_\ell$, where $c_0 > c_1 > \cdots > c_L = 0$
- Different model resolution: $\Pr(F_{\ell-1} \mid F_\ell) \neq 1$



- Fix: $\Pr(F_{\ell-1} \mid F_\ell) = 1$ if $c_{\ell-1} - c_\ell > \gamma^{\ell-1} + \gamma^\ell$,



Multilevel subset simulation: **Estimate numerical bias**

- Recall: $\Pr(F_L) = \Pr(F_L) = \Pr(F_0)\prod_{\ell=1}^{L}\Pr(F_\ell | F_{\ell-1})$
- Lets define: $F_\infty = G_\infty^\gamma < c_0$, then

$$P = \Pr(F_\infty) = \Pr(F_0)\Pr(F_\infty | F_{L-1})\prod_{\ell=1}^{L-1}\Pr(F_\ell | F_{\ell-1})$$

  since $\Pr(F_{L-1} | F_\infty) = 1$
- The numerical bias can be estimated

$$|P - \Pr(F_L)| = \Pr(F_0)\prod_{\ell=1}^{L-1}\Pr(F_\ell | F_{\ell-1})|Pr(F_\infty | F_{L-1}) - Pr(F_L | F_{L-1})|$$

$$\approx \frac{\widehat{P}}{\widehat{P}_{F_L | F_{L-1}}}|\widehat{P}_{F_{L+1} | F_{L-1}} - \widehat{P}_{F_L | F_{L-1}}|$$

Multilevel subset simulation: **Root mean square error**

- $|\mathbb{E}(\widehat{P}) - \Pr(F_L)|$ i dominated by $\mathbb{V}(\widehat{P})$
- $|P - \Pr(F_L)| \approx \widehat{P}/\widehat{P}_{F_L|F_{L-1}}|\widehat{P}_{F_{L+1}|F_{L-1}} - \widehat{P}_{F_L|F_{L-1}}|$

Relative root mean square error

$$e^2(\widehat{P}) = \frac{\mathbb{V}(\widehat{P}) + (\mathbb{E}(\widehat{P}) - \Pr(F_L) + \Pr(F_L) - P)^2}{\mathbb{E}(\widehat{P})^2}$$

$$\approx \frac{\mathbb{V}(\widehat{P})}{\mathbb{E}(\widehat{P})^2} + \frac{(\Pr(F_L) - P)^2}{\mathbb{E}(\widehat{P})^2}$$

$$\lesssim \frac{\mathbb{V}(\widehat{P})}{\mathbb{E}(\widehat{P})^2} + \frac{(\widehat{P}_{F_{L+1}|F_{L-1}} - \widehat{P}_{F_L|F_{L-1}})^2}{\widehat{P}^2_{F_L|F_{L-1}}} \leq \mathrm{TOL}^2$$

Multilevel subset simulation: **Algorithm**

Input: L (number of layers) and TOL (desired tolerance)
Determine intermediate subsets F_ℓ
Set $\epsilon = \mathrm{TOL}/(2L^{1/2})$
for $\ell = 1, \ldots, L$ **do**
    Estimate $\widehat{P}_\ell \approx \Pr(F_\ell \mid F_{\ell-1})$ such that $e(\widehat{P}_\ell) \leq \epsilon$
end for
Set the subset estimator to $\widehat{P} = \prod_{i=1}^{L} \widehat{P}_i$
Estimate error $e(\widehat{P})$

- Estimate $\widehat{P}_1$ using i.i.d samples or standard subset simulation
- Estimate $\widehat{P}_\ell$ for $\ell > 1$ using conditional samples (Metropolis-Hasting)

Multilevel subset simulation: **Adaptive algorithm**

Input: $L = 0$ (number of layers), TOL (desired tolerance)
while $e(\widehat{P}) \leq \mathrm{TOL}$ **do**
  Set $L = L + 1$
  Run multilevel(L, TOL)
end while

Numerical experiment: **Model problem**

- One phase flow model on a unit square:

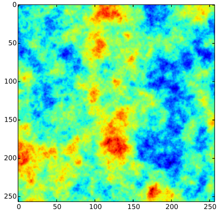
$$-\nabla \cdot A(\omega)\nabla u = 0,$$

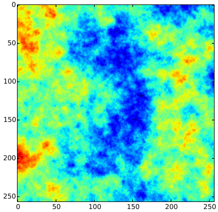
with Boundary condition No-flow up and down, $u = 1$ left boundary Γ_L, and $u = 0$ right

- The limit state function

$$G(\omega) = 3.1 - \int_{\Gamma_L} \nu \cdot A(\omega)\nabla u \, \mathrm{d}x$$

- Permeability: $A(\omega)$ is a Gaussian random field with exponential covariance with correlation length $\lambda = 0.1$





- Set $L = 4$ and $\mathrm{TOL} = 0.5$, and 5 Markov chains
- Run algorithm `multilevel`(L, TOL)
- Numerical tolerance on the finest level 0.03125
- Repeat 100 times to get statistial data

Result:

- Sample rare event probability $\approx 5.50 \cdot 10^{-6}$
- Sample relative standard deviation ≈ 0.419