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Multilevel Subset Simulation for rare event probabilities

Daniel Elfverson
daniel.elfverson@it.uu.se

Division of Scientific Computing
Uppsala University
Sweden

Joint work with: Robert Scheichl, University of Bath

Introduction: **Rare Event Probabilities**

Definition: Rare event

- $G : \Omega \to \mathbb{R}$ – The limit state functiona, stochastic variable
- The rare event probability P is:

$$P = \Pr(G \leq 0)$$

where $P \ll 1$ is unknown but **small**.

Introduction: **Problem formulation**

Model problem

- $\mathcal{M}$ – model
- Ω – sample space

We assume that there exists a unique solution u given any $\omega \in \Omega$:

$$\mathcal{M}(\omega, u(\omega)) = 0.$$

- $G(u) : V \to \mathbb{R}$ – Limit state function (functional) of the solution u
- The solution u is uniquely determined by the data, $G(\omega) := G(u(\omega))$

Introduction: **Numerical errors**

Assumption: Numerical error for samples

- For $\omega \in \Omega$ the numerical approximation $G_\ell(\omega)$ of $G(\omega)$ satisfies

$$|G(\omega) - G_\ell(\omega)| \leq \epsilon_\ell,$$

  for $\epsilon_\ell > 0$

- The work W for computing $G_\ell(\omega_i)$ satisfy

$$W(G(\omega_i)) = \mathcal{O}(\epsilon_\ell^{-q}),$$

  where $q > 0$ are independent of ω_i

Introduction: **Monte Carlo for rare events estimation**

- Large stochastic dimension $\Rightarrow$ Monte Carlo based method
- Standard Monte Carlo (MC) methods need a large number of samples:

$$\frac{\sqrt{V[\widehat{P}_{MC}]}}{E[\widehat{P}_{MC}]} \leq \epsilon \quad \Longrightarrow \quad N \geq \frac{(1-P)}{P\epsilon^2} \sim P^{-1}$$

Subset simulation: **Idea**

- Subset simulation – A variance reduction techniques (Au and Beck, 2001)
- Idea: Uses Bayes theorem

$$\Pr(B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(A|B)}$$

if $B \subset A \Rightarrow \Pr(A|B) = 1$ and we have

$$\Pr(B) = \Pr(B|A)\Pr(A).$$

- Define $\widetilde{F}_\ell = G_L(\omega) \leq c_\ell$, where $c_0 > c_1 > \cdots > c_L = 0$
- We have that $\widetilde{F}_{\ell+1} \subset \widetilde{F}_\ell$ are subsets
- Applying Bayes theorem on the sequence of subsets, $\widetilde{F}_L \subset \widetilde{F}_{L-1} \subset \cdots \subset \widetilde{F}_0$, yields

$$\Pr(G_L(\omega) \leq 0) = \Pr(\widetilde{F}_L) = \Pr(\widetilde{F}_0) \prod_{\ell=1}^{L} \Pr(\widetilde{F}_\ell | \widetilde{F}_{\ell-1})$$

Subset simulation: **Work estimate**

- Choosing $\Pr(\widetilde{F}_\ell|\widetilde{F}_{\ell-1}) \sim 0.1$ we get

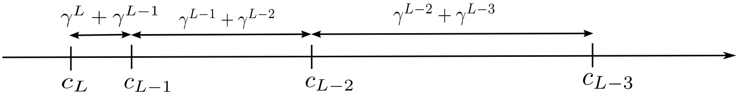
$$P \sim 0.1^L$$

  and $L \sim \log(P^{-1})$.
- Number of samples for each failure event $\Pr(\widetilde{F}_\ell|\widetilde{F}_{\ell-1})$ is independent of P
- Hence, total number of samples

$$N \sim \log(P^{-1})$$

Multilevel subset simulation: **Subset property**

- Multilevel approach has been proven successful in other application, e.g., multigrid and multilevel Monte Carlo
- Define $F_\ell = G_\ell(\omega) \leq c_\ell$, where $c_0 > c_1 > \cdots > c_L = 0$
- Different model resolution: $\Pr(F_\ell \mid F_{\ell+1}) \neq 1$
- Fix: $\Pr(F_\ell \mid F_{\ell+1}) = 1$ if $|c_\ell - c_{\ell+1}| > \gamma^\ell + \gamma^{\ell+1}$



Multilevel subset simulation: **Estimate numerical bias**

- Recall: $\Pr(F_L) = \Pr(F_1)\prod_{\ell=2}^{L}\Pr(F_\ell|F_{\ell-1})$
- Lets define: $F_\infty = G_\infty < c_0$, then

$$P = \Pr(F_\infty) = \Pr(F_1)\Pr(F_\infty|F_{L-1})\prod_{\ell=2}^{L-1}\Pr(F_\ell|F_{\ell-1})$$

since $\Pr(F_{L-1}|F_\infty) = 1$

- The numerical bias can be estimated

$$|P - \Pr(F_L)| = \Pr(F_1)\prod_{\ell=2}^{L-1}\Pr(F_\ell|F_{\ell-1})\,|Pr(F_\infty|F_{L-1}) - Pr(F_L|F_{L-1})|$$

$$\approx \frac{\widehat{P}}{\widehat{P}_{F_L|F_{L-1}}}\left|\widehat{P}_{F_{L+1}|F_{L-1}} - \widehat{P}_{F_L|F_{L-1}}\right|$$

Multilevel subset simulation: **Algorithm**

Input: L (number of layers) and TOL (desired tolerance)
Determine intermediate subsets F_ℓ
Set $\epsilon = \mathrm{TOL}/(2L^{1/2})$
for $\ell = 1, \ldots, L$ **do**
  Estimate $\widehat{P}_\ell \approx \Pr(F_\ell \mid F_{\ell-1})$ such that $e(\widehat{P}_\ell) \le \epsilon$
end for
Set the subset estimator to $\widehat{P} = \prod_{i=1}^{L} \widehat{P}_i$
Estimate error $e(\widehat{P})$

- Estimate $\widehat{P}_1$ using i.i.d samples or Standard subset simulation
- Estimate $\widehat{P}_\ell$ for $\ell > 1$ using conditional samples (Metropolis-Hasting)

Numerical experiment: **Model problem**

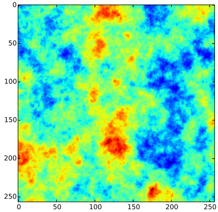
- One phase flow model on a unit square:

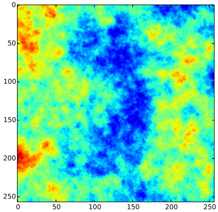
$$-\nabla \cdot A(\omega)\nabla u = 0,$$

  with Boundary condition No-flow up and down, $u = 1$ on Γ left boundary and $u = 0$ right
- Limit state function

$$G(\omega) = 3.1 - \int_{\Gamma} \nu \cdot A(\omega)\nabla u \, \mathrm{d}x$$

- Permeability – $A(\omega)$ is a Gaussian random field with exponential covariance with correlation length $\lambda = 0.1$





Numerical experiment: **Setup**

Settings:

- Set $L = 4$, $\gamma = 1/2$, $\mathrm{TOL} = 0.5$, and 5 Markov chains

Estimators:

- $\mathrm{SS}(\mathrm{TOL})$ (Standard subset simulation)
- $\mathrm{ML}_1(L, \mathrm{TOL})$ (Multilevel with i.i.d samples on level 1)
- $\mathrm{ML}_2(L, \mathrm{TOL})$ (Multilevel with subset simulation on level 1)

Numerical experiment: **ML_1 estimator**

- Mean number of samples and mean cost on each level from 10 runs of the multilevel subset simulation estimator ML_1

ℓ	1	2	3	4
$\widehat{P}_\ell$	0.00028	0.064	0.18	0.38
$N_{\ell-1}$	0	486	297	260
N_ℓ	41857	87	37	24
Cost	41857	834	1780	5696

Numerical experiment: $\mathbf{ML}_2$ estimator

- Mean number of samples and mean cost on each level from 10 runs of the multilevel subset simulation estimator ML_2

ℓ	1	2	3	4
$\widehat{P}_\ell$	0.00035	0.13	0.31	0.31
$N_{\ell-1}$	0	193	150	159
N_ℓ	912	64	31	29
Cost	912	448	1096	4400

Numerical experiment: **Comparison**

- Repeat all estimators 10 times to get statistial data
- Standard Monte Carlo cost is estimated

	ML_2	ML_1	SS	MC
Cost	1	7.3	13	3011

Numerical experiment: **Propertie of the ML_2 estimator**

- Repeat estimator 100 times to get statistial data
- The estimate mean rare event probability and relative variance are:

$$\mathbb{E}\left[\widehat{P}^{\mathrm{ML}_2}\right] \approx 2.29 \cdot 10^{-6} \quad \text{and} \quad \frac{\left(\mathbb{V}\left[\widehat{P}^{\mathrm{ML}_2}\right]\right)^{1/2}}{\mathbb{E}\left[\widehat{P}^{\mathrm{ML}_2}\right]} \approx 0.71 \leq 1$$

- The estimated relative variance of the estimator is less than the tolerance $\mathrm{TOL} = 1$.