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# Underestimation of the Radius in the Radon Transform for Circles and Spheres

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## Abstract

In this technical report we compute the underestimation of the radius in the Radon transform for circles and spheres. Our implementation of the Radon transform uses spheres with a Gaussian profile, and normalises the grey-value of each of the spheres so that a very large sphere matching only a couple of segments will not get a higher confidence (value of the peak in the parameter space) than a very small sphere completely matched in the image. This normalisation causes an underestimation of the radius.

## Contents

## 1 Introduction

The Radon transform for an $N$-dimensional hyper-sphere is defined as

$$P(\mathbf{x}, r) = \int C_b(\mathbf{p}, r) I(\mathbf{x} - \mathbf{p}) d\mathbf{p} \quad , \tag{1}$$

with

$$\begin{aligned} C_b(\mathbf{x}, r) &= \frac{1}{S_N(r)} \delta\big(\frac{1}{2}\sqrt{2}(\|\mathbf{x}\| - r)\big) * G(\mathbf{x}; \sigma_p) \\ &= \frac{1}{S_N(r)} G(\|\mathbf{x}\| - r; \sigma_p) \end{aligned} \tag{2}$$

the convolution kernel that defines the shape of the sphere, and $I(\mathbf{x})$ the input image. In these equations, $S_N(r)$ is the surface area of the sphere of radius $r$. $G(\mathbf{x}; \sigma)$ denotes the Gaussian function:

$$G(\mathbf{x}; \sigma) = \frac{1}{\left(\sigma\sqrt{2\pi}\right)^N} \, e^{-\frac{1}{2}\left(\frac{\|\mathbf{x}\|}{\sigma}\right)^2} \quad . \tag{3}$$

Let us assume that the input image has a single sphere with a Gaussian profile centered at $\mathbf{x} = 0$, having a radius of $R$ and a Gaussian parameter $\sigma_i$:

$$I(\mathbf{x}) = G(\|\mathbf{x}\| - R; \sigma_i) \quad . \tag{4}$$

$P(0, r)$, has a shape given by the integral of the product of two Gaussian curves (that of $C_b(\mathbf{x}, r)$ and that of $I(\mathbf{x})$),

$$\begin{aligned} P(0, r) &= \int C_b(\mathbf{p}, r) \; I(0 - \mathbf{p}) \; d\mathbf{p} \\ &= \int \frac{1}{S_N(r)} \; G(\|\mathbf{p}\| - r; \sigma_p) \; G(\|\mathbf{p}\| - R; \sigma_i) \; d\mathbf{p} \\ &= S_N(1) \int_0^\infty \frac{1}{S_N(r)} \; G(\rho - r; \sigma_p) \; G(\rho - R; \sigma_i) \; d\rho \quad . \end{aligned}$$

The location of the maximum along $r$ is then used as an estimate of the radius of the sphere in the input image.

Using the substitutions $\sigma_s^2 = \sigma_p^2 + \sigma_i^2$, $\frac{1}{\sigma_e^2} = \frac{1}{\sigma_p^2} + \frac{1}{\sigma_i^2}$ and $s = \sigma_e^2 \left( \frac{r}{\sigma_p^2} + \frac{R}{\sigma_i^2} \right)$, The product of the Gaussians can be re-written as

$$
\begin{aligned}
&G(\rho - r; \sigma_p)\, G(\rho - R; \sigma_i) \\
&= \frac{1}{\sqrt{2\pi}\sigma_p} \exp\left( -\frac{1}{2} \frac{(\rho - r)^2}{\sigma_p^2} \right) \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left( -\frac{1}{2} \frac{(\rho - R)^2}{\sigma_i^2} \right) \\
&= \frac{1}{2\pi\sigma_p\sigma_i} \exp\left( -\frac{1}{2} \Big[ \frac{\rho^2 - 2\rho r + r^2}{\sigma_p^2} + \frac{\rho^2 - 2\rho R + R^2}{\sigma_i^2} \Big] \right) \\
&= \frac{1}{2\pi\sigma_p\sigma_i} \exp\left( -\frac{1}{2\sigma_e^2} \left[ \rho^2 - 2\rho\sigma_e^2 (\frac{r}{\sigma_p^2} + \frac{R}{\sigma_i^2}) \right] - \frac{1}{2} (\frac{r^2}{\sigma_p^2} + \frac{R^2}{\sigma_i^2}) \right) \\
&= \frac{1}{2\pi\sigma_p\sigma_i} \exp\left( -\frac{1}{2\sigma_e^2} \left[ \rho - \sigma_e^2 (\frac{r}{\sigma_p^2} + \frac{R}{\sigma_i^2}) \right]^2 - \frac{(r - R)^2}{2\sigma_s^2} \right) \\
&= \frac{\sigma_e\sigma_s}{\sigma_p\sigma_i}\, G(\rho - s; \sigma_e)\, G(r - R; \sigma_s) \\
&= G(\rho - s; \sigma_e)\, G(r - R; \sigma_s) \quad .
\end{aligned}
$$

Using this, one obtains

$$
\begin{aligned}
P(0, r) &= \frac{S_N(1)}{S_N(r)}\, G(r - R; \sigma_s) \int_0^\infty \rho^{N-1} G(\rho - s; \sigma_e)\, d\rho \\
&= \frac{1}{r^{N-1}}\, G(r - R; \sigma_s) \int_{-s}^\infty (x + s)^{N-1} G(x; \sigma_e)\, dx \\
&= \frac{1}{r^{N-1}}\, G(r - R; \sigma_s) \int_{-s}^\infty \sum_{k=0}^{N-1} \binom{N-1}{k} x^k s^{N-k-1} G(x; \sigma_e)\, dx \\
&= \frac{1}{r^{N-1}}\, G(r - R; \sigma_s) \sum_{k=0}^{N-1} \binom{N-1}{k} s^{N-k-1} \int_{-s}^\infty x^k G(x; \sigma_e)\, dx \quad .
\end{aligned}
\tag{5}
$$

The integral can be split

$$
\int_{-s}^\infty x^k G(x; \sigma_e)\, dx = \int_0^\infty x^k G(x; \sigma_e)\, dx + \int_0^{-s} x^k G(x; \sigma_e)\, dx \quad ,
$$

but this does not lead us anywhere. The first term is the Gamma function, and the second term is called the incomplete Gamma function, and does not have a closed solution. Thus we need to choose a dimensionality at this stage.

## 2 The 2D case

$$
P(0, r) = \frac{1}{r}\, G(r - R; \sigma_s) \int_{-s}^\infty (x + s) G(x; \sigma_e)\, dx \tag{6}
$$

The solution to the integral is given by

$$\begin{aligned} s\int_{-s}^{\infty} G(x;\sigma_e)\,dx + \int_{-s}^{\infty} xG(x;\sigma_e)\,dx \\ = s\left[\int_{0}^{\infty} G(x;\sigma_e)\,dx - \int_{0}^{-s} G(x;\sigma_e)\,dx\right] + \int_{-s}^{\infty} xG(x;\sigma_e)\,dx \\ = s\left[1 - \operatorname{erf}(\frac{-s}{\sqrt{2}\sigma_e})\right] + \sigma_e^2\,G(-s;\sigma_e) \end{aligned}$$

Assuming $R > 3\sigma_i$ or so (if this is not true, the input image will not have recognizable circles), and knowing that the position of the peak is close to its expected location, $r \approx R$, it can be assumed that $R > 3\sigma_e$. This means that the error function takes a value of approximately $-1$, and the Gaussian of $0$.

$$P(0,r) \approx \frac{2s}{r}\,G(r-R;\sigma_s) = 2\sigma_e^2\left(\frac{1}{\sigma_p^2} + \frac{R}{\sigma_i^2 r}\right)\,G(r-R;\sigma_s)$$

To find the position of the maximum, we equate the derivative to zero.

$$\frac{dP}{dr} \approx 2\sigma_e^2\left[\frac{-R}{\sigma_i^2 r^2} + \left(\frac{1}{\sigma_p^2} + \frac{R}{\sigma_i^2 r}\right)\frac{R-r}{\sigma_s^2}\right] G(r-R;\sigma_s) = 0$$

Solving for $x = r - R$ yields

$$\begin{aligned} 0 &= \frac{-R}{\sigma_i^2 r^2} + \frac{R-r}{\sigma_p^2\sigma_s^2} + \frac{R^2 - Rr}{\sigma_i^2\sigma_s^2 r} \\ \Longrightarrow \quad 0 &= \sigma_s^2\sigma_p^2 R - \sigma_i^2 R r^2 + \sigma_i^2 r^3 - \sigma_p^2 R^2 r + \sigma_p^2 R r^2 \\ &= \sigma_i^2 x^3 + (2\sigma_i^2 + \sigma_p^2) R x^2 + \sigma_s^2 R^2 x + \sigma_s^2\sigma_p^2 R \quad , \end{aligned}$$

a third-degree equation. Again assuming $R > \sigma_i$ and $x$ is close to $0$,

$$0 \approx (2\sigma_i^2 + \sigma_p^2) R x^2 + \sigma_s^2 R^2 x + \sigma_s^2\sigma_p^2 R$$

has a simpler solution. We select the root closest to $0$.

$$x \approx \frac{-\sigma_s^2 R}{2(2\sigma_i^2 + \sigma_p^2)} + \sqrt{\frac{\sigma_s^4 R^2}{4(2\sigma_i^2 + \sigma_p^2)^2} - \frac{\sigma_s^2\sigma_p^2}{(2\sigma_i^2 + \sigma_p^2)}} \tag{7}$$

A fifth-order approximation around $1/R = 0$ of this is given by

$$\begin{aligned} x &\approx -\frac{\sigma_p^2}{R} - \frac{\sigma_p^4}{R^3}\frac{\sigma_p^2 + 2\sigma_i^2}{\sigma_s^2} + O\left(R^{-5}\right) \\ &= -\frac{\sigma_p^2}{R} - \frac{\sigma_p^2}{R^3}\left(\sigma_p^2 + \sigma_e^2\right) + O\left(R^{-5}\right) \end{aligned} \tag{8}$$

Note that both $\sigma_s$ and $\sigma_e$ depend on the input image, but the first term (which is good for an approximation of the third order), depends only on $\sigma_p$, defined by the algorithm.

## 3 The 3D case

$$P(0,r) = \frac{1}{r^2}\, G(r-R;\sigma_s) \int_{-s}^{\infty} (x+s)^2 G(x;\sigma_e)\; dx \tag{9}$$

The solution to the integral is given by

$$\begin{aligned}
& s^2 \int_{-s}^{\infty} G(x;\sigma_e)\; dx + 2s \int_{-s}^{\infty} xG(x;\sigma_e)\; dx + \int_{-s}^{\infty} x^2 G(x;\sigma_e)\; dx \\
& \qquad \{\text{partial integration of last term}\} \\
&= s^2 \int_{-s}^{\infty} G(x;\sigma_e)\; dx + 2s \int_{-s}^{\infty} xG(x;\sigma_e)\; dx - \sigma_e^2 x G(x;\sigma_e)\Big|_{-s}^{\infty} \\
&\qquad + \sigma_e^2 \int_{-s}^{\infty} G(x;\sigma_e)\; dx \\
&= (s^2+\sigma_e^2)\left[\int_0^{\infty} G(x;\sigma_e)\; dx - \int_0^{-s} G(x;\sigma_e)\; dx\right] + 2s \int_{-s}^{\infty} xG(x;\sigma_e)\; dx \\
&\qquad - \sigma_e^2 x G(x;\sigma_e)\Big|_{-s}^{\infty} \\
&= (s^2+\sigma_e^2)\left[1 - \operatorname{erf}(\frac{-s}{\sqrt{2}\sigma_e})\right] + \sigma_e^2 s\, G(-s;\sigma_e)
\end{aligned}$$

Again the error function takes a value of approximately $-1$, and the Gaussian of $0$.

$$\begin{aligned}
P(0,r) &\approx \frac{2}{r^2}\left(s^2+\sigma_e^2\right)\; G(r-R;\sigma_s) \\
&= \frac{2}{r^2}\left(\sigma_e^4\left(\frac{r}{\sigma_p^2}+\frac{R}{\sigma_i^2}\right)^2 + \sigma_e^2\right) G(r-R;\sigma_s) \\
&= \frac{2}{\sigma_s^4}\frac{\left(\sigma_i^2 r + \sigma_p^2 R\right)^2 + \sigma_i^2\sigma_p^2\sigma_s^2}{r^2}\; G(r-R;\sigma_s)
\end{aligned}$$

To find the position of the maximum, we equate the derivative to zero.

$$\begin{aligned}
\frac{dP}{dr} \approx \frac{2}{\sigma_s^4}\Bigg[ & \frac{2\left(\sigma_i^2 r + \sigma_p^2 R\right)\sigma_i^2}{r^2} \\
& + \frac{\left(\sigma_i^2 r + \sigma_p^2 R\right)^2 + \sigma_i^2\sigma_p^2\sigma_s^2}{r^2}\left(\frac{R-r}{\sigma_s^2} - \frac{2}{r}\right)\Bigg] G(r-R;\sigma_s) = 0
\end{aligned}$$

Solving for $x = r - R$ yields

$$\begin{aligned}
& 0 = \frac{2\left(\sigma_i^2 r + \sigma_p^2 R\right)\sigma_i^2}{r^2} + \frac{\left(\sigma_i^2 r + \sigma_p^2 R\right)^2 + \sigma_i^2\sigma_p^2\sigma_s^2}{r^2}\left(\frac{R-r}{\sigma_s^2} - \frac{2}{r}\right) \\
\Longrightarrow\quad & 0 = 2\left(\sigma_i^2 r + \sigma_p^2 R\right)\sigma_i^2\sigma_s^2 r + \left(\left(\sigma_i^2 r + \sigma_p^2 R\right)^2 + \sigma_i^2\sigma_p^2\sigma_s^2\right)\left(Rr - r^2 - 2\sigma_s^2\right) \\
& \quad = \sigma_i^4 x^4 + (\sigma_i^2 + 2\sigma_s^2)\sigma_i^2 R x^3 + \left((2\sigma_i^2+\sigma_s^2)R^2 + \sigma_p^2\sigma_i^2\right)\sigma_s^2 x^2 \\
& \qquad + \left(\sigma_s^2 R^2 + 3\sigma_p^2\sigma_i^2\right)\sigma_s^2 R x + 2(R^2+\sigma_i^2)\sigma_p^2\sigma_s^4
\end{aligned}$$

a fourth-degree equation. Again assuming $R > \sigma_i$ and $x$ is close to $0$,

$$0 \approx \left((2\sigma_i^2 + \sigma_s^2)R^2 + \sigma_p^2\sigma_i^2\right) x^2 + \left(\sigma_s^2 R^2 + 3\sigma_p^2\sigma_i^2\right) Rx + 2(R^2 + \sigma_i^2)\sigma_p^2\sigma_s^2$$

has a simpler solution. We select the root closest to $0$.

$$x \approx \frac{-\left(\sigma_s^2 R^2 + 3\sigma_p^2\sigma_i^2\right) R}{2(2\sigma_i^2 + \sigma_s^2)R^2 + 2\sigma_p^2\sigma_i^2} + \sqrt{\frac{\left(\sigma_s^2 R^2 + 3\sigma_p^2\sigma_i^2\right)^2 R^2}{\left(2(2\sigma_i^2 + \sigma_s^2)R^2 + 2\sigma_p^2\sigma_i^2\right)^2} - \frac{2(R^2 + \sigma_i^2)\sigma_p^2\sigma_s^2}{(2\sigma_i^2 + \sigma_s^2)R^2 + \sigma_p^2\sigma_i^2}}$$

A fifth-order approximation around $1/R = 0$ of this is given by

$$\begin{aligned} x &\approx -\frac{2\sigma_p^2}{R} - \frac{2\sigma_p^2}{R^3}\frac{2\sigma_p^4 + 4\sigma_p^2\sigma_i^2 + \sigma_i^4}{\sigma_s^2} + O\left(R^{-5}\right) \\ &= -\frac{2\sigma_p^2}{R} - \frac{2\sigma_p^2}{R^3}\left(\sigma_p^2 + \sigma_s^2 + \sigma_e^2\right) + O\left(R^{-5}\right) \end{aligned} \quad (10)$$

As in the 2D case, the first term (which is good for an approximation of the third order), depends only on $\sigma_p$, defined by the algorithm.

## 4 Kernel normalization

To show that the most important part of the under-estimation of the radius is caused by the normalisation, we compute the position of the maximum along the $r$-axis of $C_b(\mathbf{x}, r)$. We expect this maximum to be close to $\|\mathbf{x}\| = R$. We recall the definition of $C_b(\mathbf{x}, r)$,

$$C_b(\mathbf{x}, r) = \frac{1}{S_N(r)} G(\|\mathbf{x}\| - r; \sigma_p) = \frac{K}{r^{N-1}} G(R - r; \sigma_p) \quad ,$$

with $K$ some constant that depends on the dimensionality $N$. The derivative along $r$ is given by

$$\frac{dC_b}{dr} = K\left[\frac{-(N-1)}{r^N} + \frac{R - r}{\sigma_p^2\, r^{N-1}}\right] G(R - r; \sigma_p)$$

Equating it to zero and solving for $x = r - R$ yields

$$\begin{aligned} 0 = -\frac{N-1}{r^N} + \frac{R - r}{\sigma_p^2\, r^{N-1}} &= -(N-1)\sigma_p^2 + (R - r)r \\ &= -(N-1)\sigma_p^2 - x(x + R) \quad , \end{aligned}$$

$$x \approx -\frac{(N-1)\sigma_p^2}{R} - \frac{(N-1)^2\sigma_p^4}{R^3} + O\left(R^{-5}\right) \quad . \quad (11)$$

This explains the first term of equations (8) and (10), as well as a portion of the second term. The rest of those equations is due to the asymmetry of the peak resulting from the convolution. This asymmetry is caused by the curvature of the two interacting shapes.

To correct for this bias, we need to draw the kernel with an alternate radius $R'$. This should be selected such that the maximum in the $r$-direction lies exactly at $r$. For larger $R$ this is:

$$
\begin{aligned}
& r = R' + \frac{(N-1)\sigma_p^2}{R'} \\
\Longrightarrow \quad & 0 = R'^2 - rR' + (N-1)\sigma_p^2 \\
\Longrightarrow \quad & R' = \frac{1}{2}r + \sqrt{\frac{1}{4}r^2 - (N-1)\sigma_p^2} \quad .
\end{aligned}
\tag{12}
$$