Volumes, areas, and masses

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1. The Euclidean unit ball and unit sphere

Let $V_n = \operatorname{vol}(B^n)$ denote the volume of the Euclidean unit ball B^n in $\mathbf{R}^n$, and let $A_{n-1} = \operatorname{area}(S^{n-1})$ be the area of its boundary, the unit sphere S^{n-1} in $\mathbf{R}^n$. It is then easy to prove the two relations

$$V_n = \frac{A_{n-1}}{n} \qquad \text{and} \qquad A_n = 2\pi V_{n-1}.$$

The first just says that the volume of B^n is the integral of the areas of all spheres rS^{n-1} when r varies from 0 to 1. The second expresses the fact that over each point x in B^{n-1} there is a circle in S^n, the inverse image of x under the projection $p\colon S^n \to B^{n-1}$ defined by $p(x_1, ..., x_{n+1}) = (x_1, ..., x_{n-1})$, and that the n-dimensional measure of the set $p^{-1}(M)$ over a set M in B^{n-1} is just 2π times the $(n-1)$-dimensional measure of M. These two formulas give rise to 2-step induction formulas:

$$V_n = \frac{2\pi}{n} V_{n-2} \qquad \text{and} \qquad A_n = \frac{2\pi}{n-1} A_{n-2},$$

from which all volumes and areas can be calculated. We need only start with $V_0 = 1$, $V_1 = 2$ and $A_0 = 2$, $A_1 = 2\pi$ to get generally in $\mathbf{R}^n$:

In $\mathbf{R}^0 = \{0\}$	$V_0 = 1$	$A_{-1} = 0$
In $\mathbf{R}^1$	$V_1 = 2$	$A_0 = 2$
In $\mathbf{R}^2 = \mathbf{C}$	$V_2 = \pi$	$A_1 = 2\pi$
In $\mathbf{R}^3$	$V_3 = \dfrac{4\pi}{3}$	$A_2 = 4\pi$
In $\mathbf{R}^4 = \mathbf{C}^2$	$V_4 = \dfrac{\pi^2}{2}$	$A_3 = 2\pi^2$
In $\mathbf{R}^5$	$V_5 = \dfrac{8\pi^2}{15}$	$A_4 = \dfrac{8\pi^2}{3}$
In $\mathbf{R}^6 = \mathbf{C}^3$	$V_6 = \dfrac{\pi^3}{6}$	$A_5 = \pi^3$
In $\mathbf{R}^7$	$V_7 = \dfrac{16\pi^3}{105}$	$A_6 = \dfrac{16\pi^3}{15}$
In $\mathbf{R}^8 = \mathbf{C}^4$	$V_8 = \dfrac{\pi^4}{24}$	$A_7 = \dfrac{\pi^4}{3}$

The general formulas are

$$V_n = \frac{\pi^{n/2}}{(n/2)!}, \qquad A_{n-1} = \frac{2\pi^{n/2}}{(n/2-1)!}.$$

To prove this it is enough to check that the formula for V_n satisfies the 2-step induction formula above and gives the right value for $n=0$ and $n=1$. And for the latter is is good to remember that $\frac{1}{2}! = \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$. In general, $\Gamma(z) = (z-1)! = \int_0^\infty e^{-t}t^{z-1}dt$ for $\operatorname{Re} z > 0$.

The formulas look a bit simpler in $\mathbf{R}^{2k} = \mathbf{C}^k$:

$$V_{2k} = \frac{\pi^k}{k!}, \qquad A_{2k-1} = \frac{2\pi^k}{(k-1)!}.$$

Therefore the sum of the volumes of all unit balls in $\mathbf{C}^k$, $k \in \mathbf{N}$, is e^π. (But is it allowed to add volumes of different dimensions?) In $\mathbf{R}^n$ for odd $n = 2k+1$ they are

$$V_{2k+1} = \frac{2^{k+1}\pi^k}{(2k+1)!!}, \qquad A_{2k} = \frac{2^{k+1}\pi^k}{(2k-1)!!}.$$

Here $(2k+1)!! = 1 \cdot 3 \cdot 5 \cdots (2k+1)$.

Let r_n be the radius of a ball with the same volume as the cube $[-1,1]^n$, that is, with volume 2^n. Of course this radius is smaller than the radius of the circumscribed sphere, which is $\sqrt{n}$, so it is reasonable to set $r_n = \theta_n\sqrt{n}$. In fact r_n should be somewhere between this radius and the radius of the inscribed ball in the cube, which is 1. Now $r_n = 2V_n^{-1/n}$, so $\theta_n = 2V_n^{-1/n}n^{-1/2} \to \sqrt{2/\pi e}$, which is approximately 0.48394. We have for example $\theta_6 \approx 0.620971$; $\theta_{12} \approx 0.5636$; $\theta_{138} \approx 0.4947104$.

Another way to calculate A_n is to study the integrals

$$I_n = \int_{\mathbf{R}^n} e^{-\|x\|_2^2}dx = A_{n-1}\int_0^\infty e^{-r^2}r^{n-1}dr = \frac{1}{2}A_{n-1}\Gamma(n/2),$$

where $\|x\|_2$ is the Euclidean norm, $\|x\|_2 = \sqrt{\sum x_j^2}$. Using iterated integrals we see that $I_n = I_1^n$. On the other hand it is easy to calculate I_2 using polar coordinates. One finds $I_2 = \pi$, thus $I_n = \pi^{n/2}$. This gives the value for A_{n-1} already found.

2. Other norms

The l^1 norm $\|x\|_1 = \sum |x_j|$ and the l^∞ norm $\|x\|_\infty = \max |x_j|$ are sometimes used. For these norms, the volumes of the unit balls in $\mathbf{R}^n$ are

$$V_n^1 = \frac{2^n}{n!}, \qquad V_n^\infty = 2^n.$$

However, we can also use complex l^p norms: $\|z\|_1 = \sum |z_j|$ and $\|z\|_\infty = \max |z_j|$. Then the volumes of the unit balls in $\mathbf{C}^k$ are $(2\pi)^k/(2k)!$ and π^k respectively.

A little table is useful for comparison:

	$p=1$	$p=2$	$p=\infty$
$\mathbf{R}^n$	$\frac{2^n}{n!}$	$\frac{\pi^{n/2}}{(n/2)!}$	2^n
$\mathbf{R}^{2k}$	$\frac{4^k}{(2k)!}$	$\frac{\pi^k}{k!}$	4^k
$\mathbf{C}^k$	$\frac{(2\pi)^k}{(2k)!}$	$\frac{\pi^k}{k!}$	π^k

3. The mass defined by a subharmonic function

Let E be a radial fundamental solution of $\Delta E = \delta$ in $\mathbf{R}^n$; this implies that we have

$$\frac{\partial E}{\partial r} = \frac{1}{A_{n-1}} r^{1-n}, \qquad n = 1, 2, 3, ...,$$

in all dimensions. Therefore, after having made an unimportant choice of a constant,

$$E(x) = \frac{\|x\|_2^{2-n}}{(2-n)A_{n-1}}, \quad n \neq 2, \quad \text{and} \quad E(x) = \frac{1}{2\pi}\log\|x\|_2, \quad n = 2.$$

It will be convenient in the sequel to use a notation for the average of a function f over a set A:

$$⨍_A f(x)dx = \int_A f(x)dx \Big/ \int_A dx,$$

provided $0 < \int_A dx < +\infty$.

Proposition. *Let $u \in C^2(\Omega)$, where Ω is an open set containing $R\overline{B}$, the closed ball of radius R and center at the origin. Let $\mu = \Delta u$ be its Laplacian. Then*

$$\int_{RB} (E(R) - E(x))\mu = ⨍_{RS} (u - u(0)). \tag{3.1}$$

From this it follows that if $\mu \geqslant 0$, $\theta > 1$,

$$⨍_{rB} \mu \leqslant \frac{n(n-2)}{1-\theta^{-n+2}} r^{-2} ⨍_{\theta r S} (u - u(0)), \qquad n \neq 2.$$

In particular if $n = 1$:

$$⨍_{rB} \mu \leqslant \frac{1}{\theta - 1} r^{-2} ⨍_{\theta r S} (u - u(0)) = \frac{1}{(\theta-1)r^2}\left(\frac{1}{2}u(\theta r) - u(0) + \frac{1}{2}u(-\theta r)\right)$$

and if $n = 2$:

$$⨍_{rB} \mu \leqslant \frac{2}{r^2 \log\theta} ⨍_{\theta r S} (u - u(0)).$$

Choosing $u(x) = (E(x) - E(r))^+$ we see that these inequalities cannot be improved.

Proof. We start from Green's[1] formula:

$$\int_\Omega (u\Delta v - v\Delta u) = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial N} - v\frac{\partial u}{\partial N}\right).$$

[1]George Green (1793–1841).

Let $\mu = \Delta u$, $v(x) = E(x) - E(R) \leqslant 0$ in $\Omega = RB$. Then $v = 0$ on $\partial\Omega = RS$, and $\Delta v = \delta$. We get

$$\int_{RB} (u\delta - v\mu) = \int_{RS} u\frac{\partial E}{\partial r} = \fint_{RS} u, \tag{3.2}$$

for $\partial E/\partial r$ is a radial function whose mean value over RS is 1. We can write (3.2) as

$$\int_{RB} (-v)\mu = \fint_{RS} (u - u(0)),$$

i.e., we have proved (3.1). Now $E(\theta r) - E(r) = (\theta^{-n+2} - 1)E(r) \geqslant 0$ if $n \neq 2$ and $\theta > 1$, and $E(\theta r) - E(r) = \dfrac{1}{2\pi}\log\theta > 0$ when $n = 2$, so we get

$$\fint_{rB} \mu = \frac{1}{r^n V_n}\int_{rB} \mu \leqslant \frac{1}{r^n V_n(E(\theta r) - E(r))}\fint_{RS} (u - u(0));$$

the last inequality follows since $-v \geqslant E(R) - E(r) > 0$ in $|x| \leqslant r < \theta r = R$. Continuing we see that

$$\fint_{rB} \mu = \frac{1}{r^n V_n(\theta^{-n+2} - 1)E(r)}\fint (u - u(0)) = \frac{n(n-2)}{1 - \theta^{-n+2}} r^{-2} \fint (u - u(0))$$

for $n \neq 2$, and

$$\fint_{rB} \mu \leqslant \frac{1}{r^2 V_2 \frac{1}{2\pi}\log\theta}\fint (u - u(0)) = \frac{2}{r^2\log\theta}\fint (u - u(0))$$

when $n = 2$.