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Characterizing digital straightness and digital convexity by means of difference operators

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Abstract. We characterize straightness of digital curves in the integer plane by means of difference operators. Earlier definitions of digital rectilinear segments have used, respectively, Rosenfeld's chord property, word combinatorics, Reveilles' double Diophantine inequalities, and the author's refined hyperplanes. We prove that all these definitions are equivalent.

We also characterize convexity of integer-valued functions on the integers with the help of difference operators.

1. Introduction

The problems considered here originate in geometry—digital geometry. They will be treated using methods from Cartesian geometry as well as from word combinatorics, Diophantine inequalities, and from the calculus of difference operators. While the first three methods are not new, the use of difference operators in this context seems to be so. It is hoped that the combination of all these different methods and aspects can contribute to enriching the theory and the available methods, and to helping our understanding.

It might be helpful to start with an analogy from the calculus of real variables. If $F\colon \mathbf{R} \to \mathbf{R}$ is a twice differentiable function on the real line satisfying the equation $F'' = 0$, then, by an elementary result for ordinary differential equations, F represents a straight line, i.e., it is an affine function $F(x) = Ax + B$, $x \in \mathbf{R}$, for some constants A and B. And if $F'' \geqslant 0$, then F is convex, i.e., it satisfies Jensen's inequality

$$F((1-\lambda)x + \lambda y) \leqslant (1-\lambda)F(x) + \lambda F(y), \qquad x, y \in \mathbf{R},\ 0 \leqslant \lambda \leqslant 1. \tag{1.1}$$

The purpose of this paper is to establish analogues of these results in the case of functions of an integer variable, replacing the differential operator $F \mapsto F''$ by difference operators. We shall see that there is a crucial difference between functions $f\colon \mathbf{Z} \to \mathbf{R}$ with real values and functions $f\colon \mathbf{Z} \to \mathbf{Z}$ with integer values: the function spaces $\mathbf{R}^{\mathbf{Z}}$ (partially discretized) and $\mathbf{Z}^{\mathbf{Z}}$ (totally discretized) are very different in nature. The first case is completely elementary; the second very far from it, since it is ripe with combinatorial problems.

We shall see that we can characterize refined digital lines (equivalently: balanced binary words) with the help of difference operators, but not lines in the sense of Reveillès; the latter form a narrower class of digital lines, the chain codes of which do not include the so called skew Sturmian words in the sense of Morse and Hedlund (1940:8); cf. Theorem 10.1. However, for digital rectilinear segments, thus for finite sets, all definitions are equivalent (Theorem 10.2). The chord property of Rosenfeld (1974) will also be studied and it is shown that it can be characterized using difference operators.

The study of these discrete analogues of the differential equation $F'' = 0$ in the function space $\mathbf{Z}^{\mathbf{Z}}$ is equivalent to the study of straight lines in the digital plane $\mathbf{Z}^2$, and therefore also to the theory of balanced words from an alphabet of two letters. This theory is highly developed, and much research is going on; see, e.g., Morse & Hedlund (1940), Hung & Kasvand (1984), Bruckstein (1991), Rosenfeld & Klette (2001), Lothaire (2002), Pytheas Fogg (2002), Vuillon (2003), Klette & Rosenfeld (2004), Samieinia (2007), Uscka-Wehlou (2009a, 2009b), Berthé (2009; with 94 references), Samieinia (2010a, 2010b), and Bédaride et al. (2010). Nevertheless, the analogy with $F'' = 0$ may lead to a new, more numerical aspect of the theory, and certain results, like Theorem 9.3 on the extension of rectilinear segments, receive easy proofs. Viewed as a problem in combinatorics, this theorem says that a balanced finite binary word can be extended to a periodic balanced infinite word, moreover to infinitely many words with different periods, and with control over the periods obtained—and also to infinitely many nonperiodic balanced infinite words.

There is also a relation between continued fractions and the digitizations of a straight line in the plane. To describe it, let us first recall some basic concepts.

Any real number α can be written as a continued fraction,

$$\alpha = s_0 + \cfrac{1}{s_1 + \cfrac{1}{s_2 + \cfrac{1}{s_3 + \cdots}}} = [s_0; s_1, s_2, s_3, \dots],$$

where the s_n, defined for all $n \in \mathbf{N}$ when α is irrational and for $0 \leqslant n \leqslant N$ for some $N \geqslant 0$ when α is rational, are integers defined as follows. We first define a sequence $(\alpha_n)_{n\in\mathbf{N}}$ by

$$\alpha_0 = \alpha, \quad \alpha_{n+1} = (\alpha_n - \lfloor \alpha_n \rfloor)^{-1}, \qquad n \in \mathbf{N},$$

provided α_n is not an integer—otherwise, the induction stops there. Then we define $s_n = \lfloor \alpha_n \rfloor$. We have

$$s_0 \in \mathbf{Z}, \quad s_n \in \dot{\mathbf{N}} = \mathbf{N} \smallsetminus \{0\} \text{ for } n \geqslant 1.$$

The rational numbers

$$\frac{p_n}{q_n} = [s_0; s_1, s_2, \dots, s_n], \qquad n \in \mathbf{N},$$

defined by truncation of the continued fraction, are called *convergents* of α, and are, in a precise sense, best possible rational approximants to α. We have

$$\frac{p_n}{q_n} \leqslant \alpha \leqslant \frac{p_m}{q_m}$$

when n is even and m is odd.

The relation between digital straight lines and continued fractions is given by a theorem of Felix Klein (1895). He studied the set of points with positive integer coordinates below and above the line $y = \alpha x$, thus the sets

$$M_-(\alpha) = \{(x, y) \in \dot{\mathbf{N}} \times \dot{\mathbf{N}}; y \leqslant \alpha x\} \text{ and } M_+(\alpha) = \{(x, y) \in \dot{\mathbf{N}} \times \dot{\mathbf{N}}; y \geqslant \alpha x\}.$$

(Note that we exclude the point $(0, 0)$, which lies on the line.) The boundary of the convex hull of $M_-(\alpha)$ is a polygon, and the theorem of Klein says that its vertices (finitely many if α is rational, infinitely many otherwise) are given by the convergents of α with even index, i.e.,

$$(q_0, p_0), (q_2, p_2), (q_4, p_4), \ldots .$$

Similarly the vertices of the convex hull of $M_+(\alpha)$ are given by the convergents of α with odd index,

$$(q_1, p_1), (q_3, p_3), (q_5, p_5), \ldots .$$

For later developments, see Hübler et al. (1981; with an algorithm for the calculation of convex hulls), Bruckstein (1991; where continued fractions are briefly mentioned), Voss (1991), Debled (1995), and Uscka-Wehlou (2009b; for a survey of chain codes and continued fractions).

So much for the many studies of discrete analogues of the equation $F'' = 0$. Discrete analogues of the inequality $F'' \geqslant 0$, on the other hand, are not so well known. We will develop integer analogues of this inequality, which then yield a new way of looking at convexity for functions $f \in \mathbf{Z}^{\mathbf{Z}}$; cf. Eckhart (2001), Murota (2003) and Kiselman (2004). (Admittedly, Murota's book is mainly about functions with extended real values, but since he considers convexity of sets as well, it also concerns integer-valued functions via their finite epigraphs; cf. Section 3 below.)

The focus of interest in this article is the space $\mathbf{Z}^{\mathbf{Z}}$ of functions with discrete values. We sum up our results on straightness as well as some already known results in Section 10. The much easier space $\mathbf{R}^{\mathbf{Z}}$ is mentioned briefly for comparison in Section 4. However, as a preparation for future work, the basic definitions in Section 3 are given for $\mathbf{Z}^{(\mathbf{Z}^n)}$ and even more generally—that does not cost more.

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2. Difference operators

Definition 2.1. Given any $a \in \mathbf{R}$ we define a difference operator $D_a\colon \mathbf{R}^{\mathbf{R}} \to \mathbf{R}^{\mathbf{R}}$ by

$$(D_aF)(x) = F(x+a) - F(x), \qquad x \in \mathbf{R},\ a \in \mathbf{R},\ F \in \mathbf{R}^{\mathbf{R}}. \tag{2.1}$$

If $a \in \mathbf{N}$, D_a operates also from $\mathbf{R}^{\mathbf{Z}}$ to $\mathbf{R}^{\mathbf{Z}}$ and from $\mathbf{Z}^{\mathbf{Z}}$ to $\mathbf{Z}^{\mathbf{Z}}$; we shall use the same symbol for its restrictions to $\mathbf{R}^{\mathbf{Z}}$ and $\mathbf{Z}^{\mathbf{Z}}$.

We combine two of these operators to obtain the *Jensen operator* $J_{a,b}$,

$$\begin{aligned}(J_{a,b}F)(x) &= \tfrac{a}{a+b}D_bF(x+a) - \tfrac{b}{a+b}D_aF(x)\\ &= \tfrac{b}{a+b}F(x) - F(x+a) + \tfrac{a}{a+b}F(x+a+b), \quad x \in \mathbf{R},\ a,b > 0.\end{aligned} \tag{2.2}$$

A function $F \in \mathbf{R}^{\mathbf{R}}$ is convex if and only if $J_{a,b}F \geqslant 0$ for all positive real numbers a, b. When $a, b \in \dot{\mathbf{N}}$, the Jensen operator maps $\mathbf{Z}^{\mathbf{Z}}$ into $\mathbf{Q}^{\mathbf{Z}}$.

We shall use the fact that $(J_{a,b}F)(x) = H(x+a) - F(x+a)$, where H is the affine function which takes the same values as F at the points x and $x+a+b$, and thus measures the deviation from being affine.

Another second-order difference operator is D_bD_a, given by

$$(D_bD_aF)(x) = F(x+a+b) - F(x+a) - F(x+b) + F(x). \tag{2.3}$$

It is well known that a continuous function $F\colon \mathbf{R} \to \mathbf{R}$ is convex if and only if $D_aD_aF \geqslant 0$ for all real $a > 0$; equivalently $D_bD_aF \geqslant 0$ for all $a, b > 0$. We note that

$$D_bD_a = J_{a,b} + J_{b,a}, \tag{2.4}$$

an operator with integer coefficients. In particular $D_aD_a = 2J_{a,a}$.

For functions in $\mathbf{R}^{\mathbf{Z}}$, the conditions $D_1D_1f = 0$ and $D_1D_1f \geqslant 0$ give easy and satisfying results (Section 4). For functions in $\mathbf{Z}^{\mathbf{Z}}$, on the other hand, these conditions yield very narrow classes of functions. But if we relax them to $|D_1D_1f| \leqslant 1$ and $D_1D_1f \geqslant -1$, we get classes of functions which are much too wide to be of interest. It turns out, perhaps surprisingly, that a simple compromise, intermediate between the two conditions, viz.

$$|D_bD_af| \leqslant 1 \text{ and } D_bD_af \geqslant -1, \qquad a, b \in \dot{\mathbf{N}},$$

respectively, yields classes with good properties. These inequalities are equivalent to $|J_{a,b}f| < 1$ and $J_{a,b}f > -1$ for all $a, b \in \dot{\mathbf{N}}$, respectively.

Defining T as the translation operator $(Tf)(x) = f(x+1)$ and I as the identity operator, we can write $D_1 = T - I$ and

$$D_a = T^a - I = P_a(T - I) = P_aD_1, \qquad a \in \dot{\mathbf{N}},$$

a telescoping series, where

$$P_a = T^{a-1} + T^{a-2} + \cdots + T^2 + T + I, \qquad a \in \dot{\mathbf{N}}.$$

The operator $D_b D_a$, $a, b \in \dot{\mathbf{N}}$, can be factorized as $D_b D_a = P_b P_a D_1 D_1$. Therefore the condition $D_b D_a f \geqslant -1$ can be expressed using instead $g = D_1 f$ or $h = D_1 D_1 f$: it is equivalent to $P_b P_a D_1 g \geqslant -1$ as well as to $P_b P_a h \geqslant -1$.

Since P_a and P_b have positive coefficients, the condition $h = D_1 D_1 f \geqslant 0$ implies that $D_b D_a f = P_b P_a h \geqslant 0$ for all $a, b \in \dot{\mathbf{N}}$. But if $h \geqslant -1$, we can conclude only that $D_b D_a f \geqslant -ab$. (The function $u(x) = -\frac{1}{2}x(x-1)$ with $D_1 D_1 u = -1$ and $D_b D_a u = -ab$ shows that the conclusion cannot be improved.) This indicates that the condition $D_b D_a f \geqslant -1$ for all $a, b \in \dot{\mathbf{N}}$ is much stronger than requiring it just for $a = b = 1$.

For a calculus of the difference operators $D_b D_a$ in several variables, see Kiselman & Samieinia (2010).

3. Defining convexity

It is most convenient to define convex functions with the help of convex sets. This also has the advantage that we can treat functions with infinite values without difficulty.

3.1. Basic definitions

A subset A of $\mathbf{R}^n$ is said to be *convex* if

(3.1) $$\{a, b\} \subset A \text{ implies } [a, b] \subset A,$$

where

$$[a, b] = \{(1-t)a + tb; t \in \mathbf{R},\ 0 \leqslant t \leqslant 1\}$$

is the *segment* with a and b as endpoints. A segment $[a, b]$ with endpoints a, b in a given set will be called a *chord* of that set, and we define the *chord set* of any set A as

$$\operatorname{chord}(A) = \bigcup_{a,b \in A} [a, b] \subset \mathbf{R}^n, \qquad A \subset \mathbf{R}^n.$$

Thus a set A is convex if and only if

(3.2) $$\operatorname{chord}(A) \subset A.$$

It is justified, I think, to call this property the *chord property in the sense of Euclid*. Indeed, Definition 4 in his first book of the *Stoikheía* ‘The Elements’ reads according to Heath: “A straight line is a line which lies evenly with the points on itself.” (Euclid 1956:165). This can arguably be interpreted as $\operatorname{chord}(A) \subset A$, which together with the property of a line being a “breadthless length” (Definition 2; Euclid 1956:158) implies that the set is an *eutheía*, a rectilinear segment or an infinite straight line.

The smallest convex set containing a set A is called its *convex hull* and will be denoted by $\operatorname{cvxh}(A)$; it is well defined since any intersection of convex sets is convex.

The operation cvxh is increasing, idempotent, and extensive, in other words, a cleistomorphism (closure operator) in $\mathscr{P}(\mathbf{R}^n)$. If we instead regard it as a mapping from the complete lattice $\mathscr{P}(\mathbf{R}^n)$ to the complete lattice of all convex subsets of $\mathbf{R}^n$, it is a dilation. The operation chord, on the other hand, is increasing and extensive, but not idempotent in dimension $n \geqslant 2$.

In one dimension we have chord = cvxh; in $\mathbf{R}^2$ we have, in view of Carathéodory’s theorem,

$$A \subset \operatorname{chord}(A) \subset \operatorname{chord}(\operatorname{chord}(A)) = \operatorname{cvxh}(A).$$

(In $\mathbf{R}^n$ we need to take the operation chord n times to arrive at $\operatorname{cvxh}(A)$.)

We also note that

$$\operatorname{chord}(A) \subset A \;\Leftrightarrow\; \operatorname{cvxh}(A) \subset A.$$

Since the chord property of Euclid (3.2) is unreasonable in a digital setting, it has been weakened by Azriel Rosenfeld in a sense which turned out to be successful: We shall say that a set $A \subset \mathbf{R}^2$ has the *chord property in the sense of Rosenfeld* (1974) if

$$\operatorname{chord}(A) \subset A + U, \tag{3.3}$$

where U is the open unit ball in $\mathbf{R}^2$ for the l^∞ norm $\|x\|_\infty = \max(|x_1|, |x_2|)$,

$$U = \{x \in \mathbf{R}^2, \|x\|_\infty < 1\}.$$

We note that

$$(c + U) \cap \mathbf{Z}^2 = \{c\}, \qquad c \in \mathbf{Z}^2,$$

which implies that, for any set $A \subset \mathbf{Z}^2$ having the chord property, we have

$$\operatorname{chord}(A) \cap \mathbf{Z}^2 = A; \tag{3.4}$$

cf. (3.7) below. In particular, $\{p, q\} \subset A$ implies $[p, q] \cap \mathbf{Z}^2 \subset A$ if A has the chord property and $p_1 = q_1$ or $p_2 = q_2$ (we say that A is vertically and horizontally convex).

3.2. Counting with infinities

To any subset Y of $\mathbf{R}$ we add two elements $-\infty$, $+\infty$: we define

$$Y_! = Y \cup \{-\infty, +\infty\}.$$

In particular, we have

$$\mathbf{R}_! = [-\infty, +\infty] = \mathbf{R} \cup \{-\infty, +\infty\},$$

the set of *extended real numbers*, and

$$\mathbf{Z}_! = [-\infty, +\infty]_{\mathbf{Z}} = \mathbf{Z} \cup \{-\infty, +\infty\},$$

the set of *extended integers*.

We extend the ceiling function $\mathbf{R} \ni y \mapsto \lceil y \rceil \in \mathbf{Z}$ to a function $\mathbf{R}_! \to \mathbf{Z}_!$, keeping the notation. It is then a dilation between the two complete lattices $\mathbf{R}_!$ and $\mathbf{Z}_!$:

$$\sup_j \lceil y_j \rceil = \lceil \sup_j y_j \rceil,$$

but it is not an erosion, since it may happen that

$$\inf_j \lceil y_j \rceil > \lceil \inf_j y_j \rceil.$$

Similarly, the floor function $y \mapsto \lfloor y \rfloor$ is an erosion but not a dilation.

We also extend the ceiling and floor functions to functions $F\colon X \to \mathbf{R}_!$ defined on any set X and with values in the set of extended reals $\mathbf{R}_!$. Then $\lceil F \rceil \in {\mathbf{Z}_!}^X$.

3.3. Graphs and epigraphs

To every mapping $f\colon X \to Y$ of a set X into a set Y we associate its *graph*,

$$\operatorname{graph}(f) = \{(x, y) \in X \times Y; y = f(x)\}.$$

The relation between functions and sets is provided by the notion of finite epigraph. To every function $f\colon X \to Y_!$, where $Y \subset \mathbf{R}$ and $Y_! = Y \cup \{-\infty, +\infty\}$, we associate its *epigraph*

$$\operatorname{epi}(f) = \{(x, y) \in X \times Y_!; f(x) \leqslant y\} \subset X \times \mathbf{R}_!,$$

and its *finite epigraph*

$$\operatorname{epi}^{\mathrm{F}}(f) = \{(x, y) \in X \times Y; f(x) \leqslant y\} = \operatorname{epi}(f) \cap (X \times Y) \subset X \times \mathbf{R}.$$

Note that $-\infty, +\infty$ are never elements of a finite epigraph. (The finite epigraph of the constant function $+\infty$ is empty.) If the codomain of f is a subset of $\mathbf{R}$, then of course $\operatorname{epi}^{\mathrm{F}}(f) = \operatorname{epi}(f)$; the superscript $^{\mathrm{F}}$ is not necessary.

We shall also need the *strict finite epigraph*:

$$\operatorname{epi}^{\mathrm{F}}_{\mathrm{s}}(f) = \{(x, y) \in X \times Y; f(x) < y\}.$$

3.4. Convex functions

A function $f\colon \mathbf{R}^n \to \mathbf{R}_!$ is said to be *convex* if its finite epigraph is convex as a subset of $\mathbf{R}^n \times \mathbf{R}$. Given a function $f\colon X \to \mathbf{R}_!$, where $X \subset \mathbf{R}^n$, the largest convex function $F\colon \mathbf{R}^n \to \mathbf{R}_!$ such that $F|_X \leqslant f$ is called the *convex envelope* of f and will be denoted by $\operatorname{cvxe}(f)$. In general we have

$$\operatorname{cvxe}(f)(x) = \inf_{y \in \mathbf{R}} \Big(y; (x, y) \in \operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))\Big) \tag{3.5}$$

and

$$\operatorname{epi}^{\mathrm{F}}_{\mathrm{s}}(\operatorname{cvxe}(f)) = \operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}_{\mathrm{s}}(f)) \subset \operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f)) \subset \operatorname{epi}^{\mathrm{F}}(\operatorname{cvxe}(f)). \tag{3.6}$$

3.5. Discrete convexity

We shall now generalize the notion of convexity as follows.

Definition 3.1. Given a subset W of $\mathbf{R}^n$ we shall say that a subset A of W is *W-convex* if there exists a convex subset C of $\mathbf{R}^n$ such that $A = C \cap W$.

When $W = \mathbf{R}^n$ we get usual convexity; when $W = \emptyset$, only the empty set is W-convex. Of interest in this paper are the cases $W = \mathbf{Z}^n$ and $W = \mathbf{Z}^{n-1} \times \mathbf{R}$.

The convex set C is not uniquely determined by A, and it is often convenient to take the smallest convex set that can serve in the definition; this set is $\operatorname{cvxh}(A)$, the convex hull of A taken in $\mathbf{R}^n$. Since we always have $A \subset \operatorname{cvxh}(A) \cap W$, W-convexity of A is equivalent to the inclusion

$$\operatorname{cvxh}(A) \cap W \subset A. \tag{3.7}$$

Kim & Rosenfeld (1982) established a perfect digital analogue in $\mathbf{Z}^2$ of the Euclidean definition of convexity (3.1): they proved that a subset A of $\mathbf{Z}^2$ is $\mathbf{Z}^2$-convex if and only if any two of its points can be connected by a digital straight line segment in the sense of Rosenfeld contained in A.

Proposition 3.2. *For subsets of $\mathbf{Z}^2$, the chord property* (3.3) *in the sense of Rosenfeld implies $\mathbf{Z}^2$-convexity. The converse implication does not hold.*

Proof. Assume that $A \subset \mathbf{Z}^2$ has the chord property, and let $t \in \operatorname{cvxh}(A) \cap \mathbf{Z}^2$. We have to prove that $t \in A$. By Carathéodory's theorem, there are three not necessarily distinct points $a, b, c \in A$ such that $t \in \operatorname{cvxh}(\{a, b, c\})$. The vertical line $\{x; x_1 = t_1\}$ cuts two of the segments $[a, b]$, $[b, c]$, $[c, a]$. Denote these two points by $p = (p_1, p_2) = (t_1, p_2)$ and $q = (q_1, q_2) = (t_1, q_2)$. By the chord property there exists a point $r \in A$ with $\|r - p\|_\infty < 1$. Since t_1 is an integer, we must have $r_1 = p_1 = t_1$. Similarly there is a point $s \in A$ with $s_1 = q_1 = t_1$ and $|s_2 - q_2| < 1$.

We thus have two point $r, s \in A$ with the same first coordinate, and the vertical segment $[r, s]$ contains the given point t. It follows from (3.4) that $[r, s]_{\mathbf{Z}^2}$ must be a subset of A. In particular $t \in [r, s]_{\mathbf{Z}^2} \subset A$, and we are done.

That the converse implication does not hold is shown by the set $\{(0, 0), (2, 1)\}$, which is $\mathbf{Z}^2$-convex but does not have the chord property. (There is a lack of connectivity here.) However, the graph of a function $f\colon \{0, 1, 2\} \to \mathbf{Z}$ with $f(0) = 0$, $f(2) = 1$ is $\mathbf{Z}^2$-convex if and only if it has the chord property (this happens if and only if $f(1) \in \{0, 1\}$). For a more general result, see Theorem 5.2 below. $\square$

Definition 3.3. Given a subset X of $\mathbf{R}^n$, a subset Y of $\mathbf{R}$, and a subset W of $X \times Y$, we shall say that a function $f\colon X \to Y_!$ is *W-convex* if its finite epigraph is a W-convex set in the sense of Definition 3.1.

Thus f is W-convex if and only if $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f)) \cap W \subset \operatorname{epi}^{\mathrm{F}}(f)$; cf. (3.7). We remark here that for $W = \mathbf{Z}^2$, the condition $\operatorname{epi}^{\mathrm{F}}_{\mathrm{s}}(\operatorname{cvxe}(f)) \cap \mathbf{Z}^2 \subset \operatorname{epi}^{\mathrm{F}}(f)$ is too weak to give reasonable results, whereas the condition $\mathrm{epi}^{\mathrm{F}}(\mathrm{cvxe}(f)) \cap \mathbf{Z}^2 \subset \mathrm{epi}^{\mathrm{F}}(f)$ is too strong; cf. (3.6).

When X is all of $\mathbf{R}^n$, Y is all of $\mathbf{R}$, and $W = \mathbf{R}^n \times \mathbf{R}$, thus for $(\mathbf{R}^n \times \mathbf{R})$-convexity, we get usual convexity for functions $F \in \mathbf{R}_!{}^{(\mathbf{R}^n)}$.

For functions defined in $\mathbf{Z}^n$ and with values in $\mathbf{R}_!$ there is a simple characterization of $(\mathbf{Z}^n \times \mathbf{R})$-convexity in terms of extensions:

Proposition 3.4. *A function $f\colon \mathbf{Z}^n \to \mathbf{R}_!$ is $(\mathbf{Z}^n \times \mathbf{R})$-convex if and only if it admits an $(\mathbf{R}^n \times \mathbf{R})$-convex extension, thus an extension $F\colon \mathbf{R}^n \to \mathbf{R}_!$ which is convex in the usual sense.*

Proof. If $F\colon \mathbf{R}^n \to \mathbf{R}_!$ is convex, we shall prove that its restriction $f = F|_{\mathbf{Z}^n}$ is $(\mathbf{Z}^n \times \mathbf{R})$-convex, i.e., that $(x, y) \in \mathrm{cvxh}(\mathrm{epi}^{\mathrm{F}}(f)) \cap (\mathbf{Z}^n \times \mathbf{R})$ implies $(x, y) \in \mathrm{epi}^{\mathrm{F}}(f)$. Since $\mathrm{epi}^{\mathrm{F}}(F)$ is now convex, the convex hull of $\mathrm{epi}^{\mathrm{F}}(f) = \mathrm{epi}^{\mathrm{F}}(F) \cap (\mathbf{Z}^n \times \mathbf{R})$ is contained in $\mathrm{epi}^{\mathrm{F}}(F)$. So if (x, y) belongs to $\mathrm{cvxh}(\mathrm{epi}^{\mathrm{F}}(f)) \cap (\mathbf{Z}^n \times \mathbf{R})$, then it belongs also to $\mathrm{epi}^{\mathrm{F}}(F) \cap (\mathbf{Z}^n \times \mathbf{R}) = \mathrm{epi}^{\mathrm{F}}(f)$.

Conversely, assume that f is $(\mathbf{Z}^n \times \mathbf{R})$-convex and denote by $F = \mathrm{cvxe}(f)$ its convex envelope. When $x \in \mathbf{Z}^n$, (3.5) shows that

$$F(x) = \inf\Big(y; (x, y) \in \mathrm{cvxh}(\mathrm{epi}^{\mathrm{F}}(f))\Big) = \inf\Big(y; (x, y) \in \mathrm{epi}^{\mathrm{F}}(f)\Big) = f(x). \qquad \square$$

For $(\mathbf{Z} \times \mathbf{Z})$-convexity there is no simple characterization like Proposition 3.4.

In view of Proposition 3.4, a $(\mathbf{Z}^n \times \mathbf{R})$-convex function may also be called *convex extensible*, cf. Kiselman & Samieinia (2010). However, we should be aware of the fact that Murota (2003:93) used this term in another, narrower sense as shown by the following example.

Example 3.5. Define $f\colon \mathbf{Z}^2 \to \mathbf{Z} \cup \{+\infty\}$ by $f(x_1, 0) = 0$ for all $x_1 \in \mathbf{Z}$, $f(0, 1) = 0$, $f(1, 1) = 1$, and $f(x) = +\infty$ for all other points $x \in \mathbf{Z}^2$. This function has a convex extension $\mathrm{cvxe}(f)\colon \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$; it is thus $(\mathbf{Z}^2 \times \mathbf{R})$-convex. But it is not convex extensible in the sense of Murota (2003:93), for the function $\overline{f}$ constructed in definition (3.56) there satisfies

$$\overline{f}(1, 1) = 0 < 1 = f(1, 1) = (\mathrm{cvxe}(f))(1, 1).$$

We must thus take care and not believe that being convex extensible in Murota's sense is the same thing as having a convex extension. We always have $\overline{f} \leqslant \mathrm{cvxe}(f)$ in $\mathbf{R}^n$, and the inequality may be strict even at some integer points as we have seen. The function $\overline{f}$ is always lower semicontinuous, whereas the convex envelope $\mathrm{cvxe}(f)$ need not be. In fact, $\overline{f}$ is the second Fenchel transform of f.

Note that $f > -\infty$. If we allow $-\infty$ as a value, there are simple examples even in one variable: define $g\colon \mathbf{Z} \to \mathbf{Z}_!$ by $g(0) = -\infty$, $g = +\infty$ in $\mathbf{Z} \smallsetminus \{0\}$. Then $\overline{g} = -\infty < +\infty = \mathrm{cvxe}(g)$ in $\mathbf{Z} \smallsetminus \{0\}$. $\square$

To any function $f \in \mathbf{Z}_!{}^{\mathbf{Z}}$ we associate the function $g \in \mathbf{R}_!{}^{\mathbf{Z}}$ taking the same values. Then $\mathrm{epi}^{\mathrm{F}}(f) = \mathrm{epi}^{\mathrm{F}}(g) \cap (\mathbf{Z} \times \mathbf{Z}) \subset \mathrm{epi}^{\mathrm{F}}(g)$ with a strict inclusion except when both finite epigraphs are empty. However, their convex hulls are the same. This is because, for every $(p, p') \in \operatorname{epi}^{\mathrm{F}}(f)$, the whole ray $(p, p') + L$, where

$$L = \{(0, z') \in \mathbf{R}^2; z' \geqslant 0\},$$

is contained in $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$, so that both convex hulls can be described as the convex hull of the union of all sets $(p, p') + L$ with (p, p') varying in $\operatorname{epi}^{\mathrm{F}}(f)$. (When $f(p)$ is finite, we can take $p' = f(p)$.)

Proposition 3.6. *Let $f \in \mathbf{Z}_!^{\mathbf{Z}}$. Every point in $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$ is of the form*

$$(x_1, y_2 + z_2) \in \mathbf{R} \times \mathbf{R},$$

where $z_2 \geqslant 0$ and (x_1, y_2) is on a segment $[p, q]$ with $p, q \in \operatorname{epi}^{\mathrm{F}}(f)$.

If $f > -\infty$, it is enough to take segments $[(p_1, f(p_1)), (q_1, f(q_1))]$, $p_1 \leqslant q_1$, such that for all points s_1 with $p_1 < s_1 < q_1$, the point $(s_1, f(s_1))$ lies strictly above the segment $[(p_1, f(p_1)), (q_1, f(q_1))]$.

Proof. In view of Carathéodory's theorem every point in $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$ is in the convex hull of three points in $\operatorname{epi}^{\mathrm{F}}(f)$, but in view of the special form of a finite epigraph, we can simplify the description as follows.

Let x be a point in the convex hull of three points p, q, r in $\operatorname{epi}^{\mathrm{F}}(f)$. Now, any point inside a triangle in $\mathbf{R}^2$ must be on or above one of its three sides. This means that x is on or above one of the three segments $[p, q]$, $[q, r]$, $[r, p]$; let us say the first one. If $p_1 = q_1$, we may assume that $p_2 \leqslant q_2$, and then $x_2 = p_2 + z_2$ with $z_2 \geqslant 0$. If on the other hand $p_1 \neq q_1$, then we define y_2 by letting (x_1, y_2) be the point on the segment $[p, q]$, x_1 being given. So (x_1, y_2) belongs to a segment with endpoints in $\operatorname{epi}^{\mathrm{F}}(f)$, and $x_2 \geqslant y_2$, so that $x_2 = y_2 + z_2$ with $z_2 \geqslant 0$.

If $f > -\infty$ we need only points on the graph, thus we may take $p = (p_1, f(p_1))$ etc. If there is a point $(s_1, f(s_1))$ with $p_1 \neq s_1 \neq q_1$ on or below the segment $[(p_1, f(p_1)), (q_1, f(q_1))]$, we can use instead one of the segments

$$[(p_1, f(p_1)), (s_1, f(s_1))],\ [(s_1, f(s_1)), (q_1, f(q_1))]$$

to get a new representation, and then go on until there are no more points $(s_1, f(s_1))$ of the graph with $p_1 \neq s_1 \neq q_1$ on or below the segments used. ☐

4. Real-valued convex extensible functions

For function in $\mathbf{R}^{\mathbf{Z}}$ the questions can be resolved easily:

Theorem 4.1. *A function $f\colon \mathbf{Z} \to \mathbf{R}$ satisfies the equation*

$$D_1 D_1 f = 0, \tag{4.1}$$

equivalently

$$D_b D_a f = 0, \qquad a, b \in \dot{\mathbf{N}}, \tag{4.2}$$

if and only if there are real constants A and B such that $f(x) = Ax + B$. It satisfies the inequality

$$D_1 D_1 f \geqslant 0, \tag{4.3}$$

equivalently

$$D_b D_a f \geqslant 0, \qquad a, b \in \dot{\mathbf{N}}, \tag{4.4}$$

if and only if it is $(\mathbf{Z} \times \mathbf{R})$-convex. Equivalent conditions are $J_{1,1} f = 0$ and $J_{1,1} f \geqslant 0$, respectively.

Proof. The equivalence of (4.1) and the seemingly stronger condition (4.2) follows from the factorization $D_b D_a = P_b P_a D_1 D_1$. Similarly, (4.3) is equivalent to (4.4). The latter inequality should be compared with inequality (1.39) in Murota (2003:25), which is his starting point for the introduction of M-convex functions. □

5. Characterizations of straightness

5.1. Rosenfeld: the chord property

In order to characterize straightness of finite subsets of $\mathbf{Z}^2$, Azriel Rosenfeld (1974) introduced the chord property already mentioned in (3.3).

We may define the P-digitization of a subset M of $\mathbf{R}^n$ as the set

$$\operatorname{dig}_P(M) = (M + P) \cap \mathbf{Z}^n, \qquad M \in \mathscr{P}(\mathbf{R}^n).$$

Here P is a pixel or voxel located at the origin—it may in fact be any subset of $\mathbf{R}^n$. We may take $P = \{0\}$, but then many sets will have empty digitization; the role of P is to fatten M before intersecting it with the grid $\mathbf{Z}^n$.

We note that dig_P is a dilation $\mathscr{P}(\mathbf{R}^n) \to \mathscr{P}(\mathbf{Z}^n)$ for any P, i.e.,

$$\operatorname{dig}_P(\textstyle\bigcup M_j) = \bigcup \operatorname{dig}_P(M_j)$$

for any family (M_j) of subsets of $\mathbf{Z}^n$. We even have

$$\operatorname{dig}_{\bigcup P_k}(\textstyle\bigcup M_j) = \bigcup_k \bigcup_j \operatorname{dig}_{P_k}(M_j).$$

We also note that the operation commutes with translations by an integer vector:

$$\operatorname{dig}_P(c + M) = c + \operatorname{dig}_P(M), \qquad c \in \mathbf{Z}^n,\ M \in \mathscr{P}(\mathbf{R}^n),$$

as well as the symmetry $\operatorname{dig}_P(M) = \operatorname{dig}_M(P)$.

Rosenfeld took P as the cross

$$R = \left(\left[-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \{0\}\right) \cup \left(\{0\} \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right[\right) \subset \mathbf{R}^2.$$

Then the straight line L in $\mathbf{R}^2$ defined by an equation $x_2 = F(x_1) = \alpha x_1 + \beta$ with $|\alpha| < 1$ gives rise to a function $f\colon \mathbf{Z} \to \mathbf{Z}$. Indeed, given $z_1 \in \mathbf{Z}$, there is one and only one z_2 such that (z_1, z_2) belongs to $L + R$. Actually $z_2 = f(z_1) = \lceil \alpha z_1 + \beta - \frac{1}{2} \rceil$, so that this digitization of the real line with equation $x_2 = F(x_1)$ has the equation $z_2 = \lceil \alpha z_1 + \beta - \frac{1}{2} \rceil$. For each z_1 one chooses the integer closest to $\alpha z_1 + \beta$ if there is a unique closest integer, and, by convention $\alpha z_1 + \beta - \frac{1}{2}$ if $\alpha z_1 + \beta$ is a half-integer (the choice between $\alpha z_1 + \beta - \frac{1}{2}$ and $\alpha z_1 + \beta + \frac{1}{2}$, made to obtain uniqueness, introduces of course a certain asymmetry). The differences $(D_1 f)(z) = f(z_1 + 1) - f(z_1)$ form an upper mechanical word, also called a β-sequence (see, e.g., Uscka-Wehlou 2009b), to be compared with the constant $F(x_1 + 1) - F(x_1) = \alpha$ for the original function.

Rosenfeld (1974) proved that a finite digital arc A, in particular the graph of a function $f\colon [c, d]_{\mathbf{Z}} \to \mathbf{Z}$ with $|D_1 f| \leqslant 1$, has the chord property if and only if $A = \operatorname{dig}_R(L)$ for some rectilinear segment $L = [p, q]$ in $\mathbf{R}^2$.

When A is the graph of a function $f \in \mathbf{Z}^{\mathbf{Z}}$ the chord property can be formulated as follows. Given $p < t < q$ with integers p and q and a real number t, we let $H\colon \mathbf{R} \to \mathbf{R}$ be the affine function which takes the values of f at p and q. Then the chord property says that

$$|H(t) - f(\lfloor t \rfloor)| < 1 \text{ or } |H(t) - f(\lceil t \rceil)| < 1. \tag{5.1}$$

If t happens to be an integer, this simplifes to

$$|H(t) - f(t)| < 1. \tag{5.2}$$

Theorem 5.1. *Let $f\colon \mathbf{Z} \to \mathbf{Z}$ be a function with integer values. Then its graph has the chord property if and only if $|D_1 f(x)| \leqslant 1$ and $|J_{a,b} f(x)| < 1$ for all $(x, a, b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}}$. The corresponding result holds also for a function defined on an interval $[c, d]_{\mathbf{Z}}$ or $[c, +\infty[_{\mathbf{Z}}$ or $]-\infty, d]_{\mathbf{Z}}$ of $\mathbf{Z}$.*

This result is equivalent to Theorem 3.1 in Samieinia (2010a). In fact, the vertical distance between the vertex $(s, f(s))$ of a boomerang and a chord $[(p, f(p)), (q, f(q))]$, $p < s < q$, is equal to $|J_{s-p,q-s} f(p)|$.

Proof. Assume first that graph(f) has the chord property. Then $|D_1 f(x)| \leqslant 1$, for otherwise the midpoint $(x + \frac{1}{2}, \frac{1}{2} f(x) + \frac{1}{2} f(x+1))$ of the segment

$$[(x, f(x)), (x + 1, f(x + 1))]$$

would not belong to graph$(f) + U$.

Let $x < x + a < x + a + b$ be given with $x \in \mathbf{Z}$, $a, b \in \dot{\mathbf{N}}$. Then $(J_{a,b} f)(x) = H(x + a) - f(x + a)$, where H is the affine function which takes the values $f(x)$ at x and $f(x + a + b)$ at $x + a + b$. In the chord property we take $p = x$, $t = x + a$,

$q = x + a + b$. It follows that $(x + a, H(x + a))$ belongs to $\operatorname{graph}(f) + U$, and since $t = x + a$ is now an integer, (5.2) says that $|(J_{a,b}f)(x)| = |H(x + a) - f(x + a)| < 1$.

Conversely, assume that $|(J_{a,b}f)(x)| = |H(x+a)-f(x+a)| < 1$ for all $(x, a, b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}}$. We have to prove that (5.1) holds for any real t with $p < t < q$. If t is an integer we choose x, a, b so that $p = x$, $t = x + a$, $q = x + a + b$ and get the inequality. If t is not an integer, we define $t_0 = \lfloor t \rfloor$ and $t_1 = t_0 + 1 = \lceil t \rceil$. Then we know that $|H(t_j) - f(t_j)| < 1$, $j = 0, 1$. This implies that, for $j = 0, 1$,

$$\begin{aligned} v_0 = \min(f(t_0), f(t_1)) - 1 \leqslant f(t_j) - 1 < H(t_j) \\ < f(t_j) + 1 \leqslant \max(f(t_0), f(t_1)) + 1 = v_1, \end{aligned}$$

where v_0 and v_1 are defined by the equations. Thus $H(t_0)$ and $H(t_1)$ both belong to the open interval $]v_0, v_1[$, which implies that $H(t)$, which is obtained by interpolation between $H(t_0)$ and $H(t_1)$, is also in this interval, in other words that the point $(t, H(t))$ belongs to the open rectangle $\Omega =]t_0, t_1[\times]v_0, v_1[$.

We now invoke the other hypothesis, viz. that $|D_1 f| \leqslant 1$, which implies that $v_1 - v_0 \leqslant 3$ and hence that Ω is a subset of the dilation $\operatorname{graph}(f) + U$. Thus finally $(t, H(t)) \in \operatorname{graph}(f) + U$. We are done. ☐

We shall also establish a partial converse to Proposition 3.2:

Theorem 5.2. *The graph of an integer-valued function defined on an interval of* $\mathbf{Z}$ *and satisfying* $|D_1 f| \leqslant 1$ *is* $\mathbf{Z}^2$*-convex if and only if it possesses the chord property.*

Proof. One direction has already been proved in Proposition 3.2. Let $f\colon [c, d]_{\mathbf{Z}} \to \mathbf{Z}$ be such that its graph A is $\mathbf{Z}^2$-convex. We claim that $|J_{a,b}f| < 1$. To reach a contradiction, we assume that $J_{a,b}f(x) \leqslant -1$; then $H(x+a)-f(x+a) \leqslant -1$, where H is the affine function which takes the same values as f at x and $x + a + b$. Since $(x + a, f(x + a))$ belongs to A and

$$H(x + a) \leqslant f(x + a) - 1 < f(x + a)$$

and since also $(x+a, H(x+a))$ belongs to $\operatorname{cvxh}(A)$, it follows that $(x+a, f(x+a)-1)$ belongs to $\operatorname{cvxh}(A)$, hence to A in view of its $\mathbf{Z}^2$-convexity. But this contradicts the fact that A is a graph. The conclusion now follows from Theorem 5.1 above. ☐

5.2. Characterizations by means of balanced words

Theorem 5.3. *A function* $f \in \mathbf{Z}^{\mathbf{Z}}$ *with* $0 \leqslant D_1 f \leqslant 1$ *satisfies*

$$|D_b D_a f(x)| \leqslant 1, \qquad x \in \mathbf{Z}, \quad a, b \in \dot{\mathbf{N}}, \tag{5.3}$$

if and only if the binary word $D_1 f$ *is balanced.*

For the proof we recall some notions from word theory. By a *word* we understand here a doubly infinite sequence $(w_j)_{j\in\mathbf{Z}}$ of letters w_j; it is *binary* if there are only two letters; we shall then take them as 0 and 1. (Often one studies words that are infinite in only one direction, $(w_j)_{j\in\mathbf{N}}$.)

A factor $w' = (w_j)_{j=p}^{q}$ of a word w is said to have *length* $q-p+1$:

$$\operatorname{length}(w') = q - p + 1.$$

The empty word $\varepsilon = (w_j)_{j=p}^{p-1}$ has length 0.

If w is binary, the number of ones in a factor $w' = (w_j)_{j=p}^{q}$ is called its *height*:

$$\operatorname{height}(w') = \sum_{j=p}^{q} w_j.$$

A function $f \in \mathbf{Z}^{\mathbf{Z}}$ is said to have *chain code* $c = c(f) = (c_j)_{j\in\mathbf{Z}}$, where

$$c_j = f(j+1) - f(j) = D_1 f(j), \qquad j \in \mathbf{Z}.$$

Conversely, every sequence $(c_j)_{j\in\mathbf{Z}}$ determines a family of functions having this chain code; we take

$$f(x) = C + \sum_{j=0}^{x-1} c_j \text{ for } x \geqslant 0 \text{ and } f(x) = C - \sum_{j=x}^{-1} c_j \text{ for } x < 0,$$

where C is an arbitrary constant equal to $f(0)$.

A binary word w is said to be *balanced* if for any two factors w' and w'' of w we have

$$\text{(5.4)} \qquad \operatorname{length}(w') = \operatorname{length}(w'') \text{ implies } |\operatorname{height}(w') - \operatorname{height}(w'')| \leqslant 1.$$

Let now $w' = (w_j)_{j=p'}^{q'}$, $w'' = (w_j)_{j=p''}^{q''}$ be two factors of the same binary word w. That they have the same length means that $q' - p' + 1 = q'' - p'' + 1$. Their heights are

$$\operatorname{height}(w') = \sum_{j=p'}^{q'} w_j, \quad \operatorname{height}(w'') = \sum_{j=p''}^{q''} w_j.$$

Now, writing $w_j = D_1 f(j)$, we obtain

$$\operatorname{height}(w') = \sum_{j=p'}^{q'} D_1 f(j) = D_a f(p'), \qquad \text{where } a = q' - p' + 1.$$

Proof of Theorem 5.3. Given f, let $w' = (w_j)_{j=p'}^{q'}$ and $w'' = (w_j)_{j=p''}^{q''}$ be two factors of the same length of the binary word $w = D_1 f$. For reasons of symmetry we may assume that $p' \leqslant p''$. Define $x = p'$, $a = q' - p' + 1 = q'' - p'' + 1$, the common length of the intervals, and $b = p'' - p' = q'' - q'$, the distance between their left endpoints. Then $x + a = q' + 1$, $x + b = p''$, and $x + a + b = q'' + 1$, so that

$$\operatorname{height}(w'') - \operatorname{height}(w') = D_a f(p'') - D_a f(p') = D_b D_a f(p') = D_a D_b f(p').$$

We see that condition (5.3) translates directly to condition (5.4). □

Thus the equality (4.1) or (4.2) for functions in $\mathbf{R}^{\mathbf{Z}}$ is replaced by the inequality (5.3) for functions in $\mathbf{Z}^{\mathbf{Z}}$, which we can understand as a kind of approximate equality. Note that we require this inequality for $(x, a, b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}}$; it seems that this requirement cannot be considerably weakened (except of course that we may restrict attention to $0 < a \leqslant b$).

5.3. Hyperplanes in the sense of Reveillès

Jean-Pierre Reveillès (1991:45) introduced digital lines in the digital plane as solutions to a double Diophantine inequality: he considered sets of the form

$$\{x \in \mathbf{Z}^2; \beta \leqslant \alpha_1 x_1 + \alpha_2 x_2 < \gamma\}, \tag{5.5}$$

where α_1 and α_2 are real numbers, not both of them zero, and β and γ are real numbers. We shall refer to such a set as a *digital straight line in the sense of Reveillès*. He considers in particular the case when α_1 and α_2 are integers; then he calls the digital line *rational*—indeed, if $\alpha_2 \neq 0$, its slope $-\alpha_1/\alpha_2$ is a rational number. It is obvious how to generalize this definition to higher dimensions: we then speak about *digital hyperplanes in the sense of Reveillès*.

5.4. Refined digital hyperplanes

Let us first define slabs in $\mathbf{R}^n$:

$$\begin{aligned} T &= T(\alpha, \beta, \gamma) = \{x \in \mathbf{R}^n; \beta \leqslant \alpha \cdot x \leqslant \gamma\}, \\ T^* &= T^*(\alpha, \beta, \gamma) = \{x \in \mathbf{R}^n; \beta \leqslant \alpha \cdot x < \gamma\}, \\ T_* &= T_*(\alpha, \beta, \gamma) = \{x \in \mathbf{R}^n; \beta < \alpha \cdot x \leqslant \gamma\}, \\ T_*^* &= T_*^*(\alpha, \beta, \gamma) = \{x \in \mathbf{R}^n; \beta < \alpha \cdot x < \gamma\}. \end{aligned} \tag{5.6}$$

We shall also need to talk about the real hyperplanes

$$T^0 = \{x \in \mathbf{R}^n; \alpha \cdot x = \beta\}, \qquad T^1 = \{x \in \mathbf{R}^n; \alpha \cdot x = \gamma\}.$$

If $\beta \leqslant \gamma$, we have $T = T_*^* \cup T^0 \cup T^1$.

A digital hyperplane D in the sense of Reveillès is of the form $T^*(\alpha, \beta, \gamma) \cap \mathbf{Z}^n$ and therefore satisfies

$$(T \cap \mathbf{Z}^n) \smallsetminus D \subset T^0,$$

i.e., the points in $T \cap \mathbf{Z}^n$ not in D all belong to a single real hyperplane in $\mathbf{R}^n$.

In Kiselman (2004:456) we generalized this to the following. Let us denote by $\pi_k\colon \mathbf{Z}^n \to \mathbf{Z}^{n-1}$ the projection which forgets the coordinate x_k, $k = 1, \dots, n$. A set D is a *refined digital hyperplane* if D is $\mathbf{Z}^n$-convex, if

$$T_*^* \cap \mathbf{Z}^n \subset D \subset T \cap \mathbf{Z}^n$$

for some choice of $\alpha \in \mathbf{R}^n \smallsetminus \{0\}$ and $\beta, \gamma \in \mathbf{R}$, and if in addition, for some k, the sets $\pi_k(D \cap T^0)$ and $\pi_k(D \cap T^1)$ are disjoint and together fill all of $\pi_k((T^0 \cup T^1) \cap \mathbf{Z}^2)$.

In two dimensions this definition can be expressed in a simple way. We take $n = 2$, $(\alpha_1, \alpha_2) = (-\alpha, 1)$ and define strips in $\mathbf{R}^2$ as follows.

$$\begin{aligned} S(\alpha,\beta,\gamma) &= T = \{x \in \mathbf{R}^2; \beta \leqslant x_2 - \alpha x_1 \leqslant \gamma\}, \\ S^*(\alpha,\beta,\gamma) &= T^* = \{x \in \mathbf{R}^2; \beta \leqslant x_2 - \alpha x_1 < \gamma\}, \\ S_*(\alpha,\beta,\gamma) &= T_* = \{x \in \mathbf{R}^2; \beta < x_2 - \alpha x_1 \leqslant \gamma\}, \\ S_*^*(\alpha,\beta,\gamma) &= T_*^* = \{x \in \mathbf{R}^2; \beta < x_2 - \alpha x_1 < \gamma\}. \end{aligned} \tag{5.7}$$

Then a straight line in $\mathbf{Z}^2$ in the sense of Reveillès is, possibly after a permutation and a sign change of the coordinates, equal to the intersection $S^*(\alpha,\beta,\gamma) \cap \mathbf{Z}^2$, for some α, β, γ, $|\alpha| \leqslant 1$. (We could as well have used $S_*(\alpha,\beta,\gamma)$ here, for $S_*(\alpha,\beta,\gamma) = -S^*(\alpha,-\gamma,-\beta)$.)

A refined digital line with $|\alpha| \leqslant 1$ and $\gamma = \beta + 1$ is either a digital line in the sense of Reveillès or, possibly after a reflection, of the form

$$\begin{aligned} D(\alpha,\beta,p) = \{x \in \mathbf{Z}^2 \cap S^*(\alpha,\beta,\beta+1); x_1 < p\} \\ \cup \{x \in \mathbf{Z}^2 \cap S_*(\alpha,\beta,\beta+1); x_1 \geqslant p\} \end{aligned} \tag{5.8}$$

for some $p \in \mathbf{Z}$. This is because the only pairs of $\mathbf{Z}$-convex complementary subsets of the digital line are $(\mathbf{Z}, \varnothing)$ and $(]-\infty, p[_{\mathbf{Z}}\,, [p, +\infty[_{\mathbf{Z}})$, $p \in \mathbf{Z}$.

Theorem 5.4. *Every digital line in the sense of Reveillès is a refined digital line.*

Conversely, given $|\alpha| \leqslant 1$ and β real, we consider four cases for the set

$$D = S(\alpha,\beta,\beta+1) \cap \mathbf{Z}^2,$$

defining

$$D^j = \{x \in D; x_2 - \alpha x_1 = \beta + j\}, \qquad j = 0, 1:$$

(A). *The slope α is rational and $\beta \in \mathbf{Z} + \alpha\mathbf{Z}$. Then D^0 and D^1 contain infinitely many points and D is not a refined digital line. For any integer p, the set $D(\alpha,\beta,p)$, obtained by removing from D certain points in $D^0 \cup D^1$ (see* (5.8)*), is a refined digital line. The sets $D \smallsetminus D^0$ and $D \smallsetminus D^1$ are digital lines in the sense of Reveillès.*

(B). *The slope α is rational and $\beta \notin \mathbf{Z} + \alpha\mathbf{Z}$ (for instance when β is irrational). Then D^0 and D^1 are empty, so that $D = S_*^*(\alpha,\beta,\beta+1) \cap \mathbf{Z}^2$ and D is a digital straight line in the sense of Reveillès.*

(C). *The slope α is irrational and D^0 is empty. Then $D = S(\alpha, \beta, \beta+1) \cap \mathbf{Z}^2 = S_*^*(\alpha, \beta, \beta+1) \cap \mathbf{Z}^2$ is a digital straight line in the sense of Reveillès.*

(D). *The slope α is irrational and D^0 is a singleton set. Then D^1 is also a singleton set, and D is not a refined digital line. But $D \smallsetminus D^0$ and $D \smallsetminus D^1$ are digital straight lines in the sense of Reveillès.*

Thus in cases (B), (C) and (D) the two notions coincide; in case (A) they are different. In cases (A) and (D) we have to remove certain points in the bounding lines D^0, D^1 to obtain what we want, while this is not necessary in cases (B) and (C).

For case (A), cf. Example 7.4; for case (C) take $\alpha = 1/\sqrt{2}$ and $\beta = \frac{1}{2}$; for case (D), cf. Example 7.5.

Proof. We note that generally $D^1 = D^0 + (0, 1)$, which means that, in order to get a digital line in the sense of Reveillès, we always have to remove one of D^0 and D^1 unless they are empty. Cases (A) and (B) are then straightforward.

For cases (C) and (D) we note that, since α is irrational, we cannot have two points p and q with $p_1 \neq q_1$ in D^0; otherwise $\alpha = (q_2 - p_2)/(q_1 - p_1)$ would be rational. Thus D^0 and D^1 are either empty or singleton sets. $\square$

6. Jensen's inequality in the discrete case

The difference operator $D_b D_a$ has the advantage that it is symmetric in a and b and that it has entire coefficents. A drawback is that it involves four points if $0 \neq a \neq b \neq 0$. The Jensen operator, on the other hand, involves only three points but has the drawback that it does not have integer coefficients. In the proofs of this paper either one can be used—it is mostly a matter of taste which to choose. Actually the paper was first written using $D_b D_a$, and only later was the Jensen operator introduced as an alternative. To pass from one to the other, we note the following result.

Theorem 6.1. *A function $f \colon \mathbf{Z} \to \mathbf{Z}$ satisfies the condition*

$$D_b D_a f \geqslant -1 \quad \textit{for all } a, b \in \dot{\mathbf{N}}$$

if and only it satisfies

$$J_{a,b} f > -1 \quad \textit{for all } a, b \in \dot{\mathbf{N}}.$$

Similarly, $|D_b D_a f| \leqslant 1$ for all $a, b \in \dot{\mathbf{N}}$ is equivalent to $|J_{a,b} f| < 1$ for all $a, b \in \dot{\mathbf{N}}$. The corresponding results hold for functions which are defined on an interval $[c, d]_{\mathbf{Z}} = [c, d] \cap \mathbf{Z}$; in this case $(D_b D_a f)(x)$ and $(J_{a,b} f)(x)$ can only be defined for $c \leqslant x < x + a < x + a + b \leqslant d$.

Proof. The equality $D_b D_a f = J_{a,b} + J_{b,a}$ shows that $J_{a,b} f, J_{b,a} f > -1$ implies $D_b D_a f > -2$; hence, since $D_b D_a f$ has integer values, that $D_b D_a f \geqslant -1$.

Conversely, we shall prove that if there exists points x, a, b such that $J_{a,b}f(x) \geqslant 1$, then there exists points x', a', b' such that $D_{b'}D_{a'}f(x') > 1$. (Actually these points may be chosen so that $x' + a' = x + a$.) So suppose that $J_{a,b}f(x) \geqslant 1$. We may then assume that a and b are minimal with this property, for otherwise we can either replace x by a larger value and a by a smaller value or b by a smaller value, in both cases keeping $x + a$ fixed and keeping $J_{a,b}f(x)$ or making it even larger. If $a = b$, then $D_bD_af(x) = 2J_{a,b} \geqslant 2 > 1$ and we have obtained what we want. If $a \neq b$, say $a < b$, then the minimality implies that $J_{b,a}f(x) > 0$, for otherwise the points x, $x + a$ and $x + b = x + a + b'$ would be new points with $0 < b' = b - a < b$ such that $(J_{a,b'}f)(x) \geqslant (J_{a,b}f)(x) \geqslant 1$. If on the other hand $a > b$, then similarly $J_{b,a}f(x) > 0$, for otherwise the points $x' = x + b$, $x' + a' = x + a$ and $x + a + b$ would be new points with $a' = a - b < a$ such that $(J_{a',b}f)(x') \geqslant (J_{a,b}f)(x) \geqslant 1$. Hence in all cases $D_bD_af(x) = J_{a,b}f(x) + J_{b,a}f(x) > J_{a,b}f(x) \geqslant 1$. We are done. ☐

We extend the Jensen operator to functions with infinite values as

$$((J_{a,b})_!)F(x) = \frac{b}{a+b}F(x) \dotplus (-F(x+a)) \dotplus \frac{a}{a+b}F(x+a+b). \tag{6.1}$$

Here $\dotplus$ denotes upper addition, an extension of usual addition to an operation $\mathbf{R}_! \times \mathbf{R}_! \to \mathbf{R}_!$ satisfying, e.g., $(+\infty) \dotplus (-\infty) = +\infty$.

Theorem 6.2. *A function $f\colon \mathbf{Z} \to \mathbf{Z}_!$ is $(\mathbf{Z} \times \mathbf{Z})$-convex if and only if*

$$((J_{a,b})_!f)(x) > -1 \qquad \text{for all } \ (x, a, b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}},$$

equivalently

$$\lceil((J_{a,b})_!f)(x)\rceil \geqslant 0 \qquad \text{for all } \ (x, a, b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}}.$$

Explicitly, this is the case if and only if, for all points $p, s, q \in \mathbf{Z}$ with $p < s < q$, we have

$$f(s) \leqslant \lceil(1-\lambda)f(p) \dotplus \lambda f(q)\rceil, \quad \text{where } \lambda = (s-p)/(q-p). \tag{6.2}$$

Thus a suitable weakening of Jensen's inequality (1.1) gives the right condition.

Proof. First assume that f is $(\mathbf{Z} \times \mathbf{Z})$-convex, and take three points $p < s < q$. If one of $f(p)$, $f(q)$ is equal to $+\infty$, then (6.2) obviously holds. If one of them, say $f(p)$, is equal to $-\infty$ while $f(q) < +\infty$, then (p, p') belongs to $\operatorname{epi}^{\mathrm{F}}(f)$ for every $p' \in \mathbf{Z}$, which implies that (s, s') belongs to $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$ for negative numbers s' with arbitrarily large absolute values. Hence $f(s) = -\infty$; the inequality holds.

The case when both $f(p)$ and $f(q)$ are finite remains to be considered. Then $(p, f(p))$ and $(q, f(q))$ belong to $\operatorname{epi}^{\mathrm{F}}(f)$, so the point

$$(s, s') = (1-\lambda)(p, f(p)) + \lambda(q, f(q))$$

belongs to its convex hull, hence also $(s, \lceil s' \rceil)$. Since f is convex extensible and $(s, \lceil s' \rceil)$ has integer coordinates, this point must belong to $\mathrm{epi}^{\mathrm{F}}(f)$. We are done.

Conversely, if (6.2) holds, then we shall prove that every point in

$$\mathrm{cvxh}(\mathrm{epi}^{\mathrm{F}}(f)) \cap \mathbf{Z}^2$$

belongs to $\mathrm{epi}^{\mathrm{F}}(f)$. In view of Proposition 3.6, every point in $\mathrm{cvxh}(\mathrm{epi}^{\mathrm{F}}(f))$ is of the form $(x, y' + z')$, where $z' \geqslant 0$ and (x, y') is on some segment $[(p, p'), (q, q')]$ with $(p, p'), (q, q') \in \mathrm{epi}^{\mathrm{F}}(f)$. But (6.2) says that such a point $(x, y' + z')$ with $x = s \in \mathbf{Z}$ and $y' + z' \in \mathbf{Z}$ belongs to $\mathrm{epi}^{\mathrm{F}}(f)$: $f(s) \leqslant \lceil y' \rceil \leqslant y' + z'$. □

Corollary 6.3. *The graph of a function $f \colon \mathbf{Z} \to \mathbf{Z}$ satisfies the chord property in the sense of Rosenfeld if and only if it satisfies*

$$(\lfloor (J_{a,b}f)(x) \rfloor, \lceil (J_{a,b}f)(x) \rceil) = (0,1) \ \textit{or}\ (0,0) \ \textit{or}\ (-1,0) \quad \textit{for all } x \in \mathbf{Z}, a, b \in \dot{\mathbf{N}}.$$

Proof. In view of Theorem 5.1 this follows on applying Theorem 6.2 to f and $-f$. □

7. Discretization

Definition 7.1. Given any function $F \colon \mathbf{R}^n \to \mathbf{R}_!$ we introduce its *lower discretization* $\mathrm{discr}_*(F)$ and its *upper discretization* $\mathrm{discr}^*(F)$ by

$$\mathrm{discr}_*(F) = \lfloor F|_{\mathbf{Z}^n} \rfloor \colon \mathbf{Z}^n \to \mathbf{Z}_!, \quad \mathrm{discr}^*(F) = \lceil F|_{\mathbf{Z}^n} \rceil \colon \mathbf{Z}^n \to \mathbf{Z}_!.$$

Proposition 7.2. *For any $(\mathbf{Z}^n \times \mathbf{Z})$-convex function $f \colon \mathbf{Z}^n \to \mathbf{Z}_!$ and any $x \in \mathbf{Z}^n$ we have one of the following cases for $F = \mathrm{cvxe}(f)$, $\lfloor F(x) \rfloor = (\mathrm{discr}_*(F))(x)$ and $\lceil F(x) \rceil = (\mathrm{discr}^*(F))(x)$.*
(A). *$F(x) = f(x)$. Then $\lfloor F(x) \rfloor = \lceil F(x) \rceil = f(x)$;*
(B). *$f(x) - 1 < F(x) < f(x)$. Then $f(x) - 1 = \lfloor F(x) \rfloor < \lceil F(x) \rceil = f(x)$;*
(C). *$F(x) = f(x) - 1$. Then $\lfloor F(x) \rfloor = \lceil F(x) \rceil = f(x) - 1$.*
All three cases can occur as we shall see in Example 7.4 (cases (A) *and* (C)*), and Example 7.5 (case* (B)*).*

Proof. We note that we always have

$$\lfloor F|_{\mathbf{Z}^n} \rfloor \leqslant F|_{\mathbf{Z}^n} \leqslant \lceil F|_{\mathbf{Z}^n} \rceil \leqslant f.$$

If f is convex extensible we also know that $\mathrm{epi}^{\mathrm{F}}_{\mathrm{s}}(F) \subset \mathrm{cvxh}(\mathrm{epi}^{\mathrm{F}}(f))$, which leads to $f - 1 \leqslant F|_{\mathbf{Z}^n}$. Hence, for $(\mathbf{Z}^n \times \mathbf{Z})$-convex functions we have

$$f - 1 \leqslant \lfloor F|_{\mathbf{Z}^n} \rfloor \leqslant F|_{\mathbf{Z}^n} \leqslant \lceil F|_{\mathbf{Z}^n} \rceil \leqslant f.$$

From this we easily deduce the conclusion in the proposition. □

Theorem 7.3. *Assume that* $F\colon \mathbf{R} \to \mathbf{R}_!$ *is convex. Then* $\operatorname{discr}^*(F)$ *and* $\operatorname{discr}_*(F)$ *are both* $(\mathbf{Z} \times \mathbf{Z})$*-convex.*

Proof. That the upper discretization is $(\mathbf{Z}\times\mathbf{Z})$-convex is an immediate consequence of the definitions; we just have to observe that

$$\operatorname{epi}^{\mathrm{F}}(\operatorname{discr}^*(F)) = \operatorname{epi}^{\mathrm{F}}(F) \cap \mathbf{Z}^2.$$

It is perhaps surprising that also the lower discretization is $(\mathbf{Z} \times \mathbf{Z})$-convex. In general we have, writing $g = \operatorname{discr}^*(F)$ and $h = \operatorname{discr}_*(F)$, that $h(x) = g(x)$ when $F(x) \in \mathbf{Z}_!$ and $h(x) = g(x) - 1$ when $F(x) \in \mathbf{R} \smallsetminus \mathbf{Z}$. Since both cases can occur, it is not obvious that (6.2) for g implies the same inequality for h.

However, we always have $h \leqslant F < h + 1$ at points where F is finite, so that

$$h(s) \leqslant F(s) \leqslant (1-\lambda)F(p) + \lambda F(q) < (1-\lambda)h(p) + \lambda h(q) + 1,$$

assuming $h(p)$ and $h(q)$ to be finite. Since for any integer m, the inequality $m < t+1$ is equivalent to $m \leqslant \lceil t \rceil$, we see that (6.2) holds for h.

If one of $h(p)$, $h(q)$ is equal to $+\infty$, the inequality certainly holds; if one is equal to $-\infty$ while the other is less than $+\infty$, then also $h(s) = -\infty$ and the inequality holds as well. □

Example 7.4. Define $f(x) = 0$ for $x < 0$, $f(x) = 1$ for $x \geqslant 0$, $x \in \mathbf{Z}$ and $F = \operatorname{cvxe}(f)$. Then $F = 0$ everywhere, and $\operatorname{discr}^*(F) = \operatorname{discr}_*(F) = 0$. We have $F = f - 1$ on the natural numbers. The word $D_1 f$ was called a skew Sturmian word by Morse & Hedlund (1940:8)—it is not periodic but ultimately periodic. This function satisfies $|D_b D_a f| \leqslant 1$ as well as $|J_{a,b} f| < 1$ for all $a, b \in \dot{\mathbf{N}}$. In fact, $J_{a,b}f(x)$ can be any rational number in $]-1, 1[$, and $D_b D_a f(x)$ can assume any of the values $-1, 0, 1$. But the graph of f is not a discrete straight line in the sense of Reveillès (1991:45). It is, however, a refined digital hyperplane in the sense of Kiselman (2004:456, Definition 6.2).

For more general examples, see case (A) in Theorem 5.4. Also in this case the convex envelope of the corresponding function f is affine: $F(x) = \alpha x + \beta$, $x \in \mathbf{R}$. □

Example 7.5. Define $F(x) = \alpha x$, $x \in \mathbf{R}$, for an irrational number α, and let $f = \operatorname{discr}^*(F)$, $g = \operatorname{discr}_*(F)$. Then f and g are convex extensible and we note that $g = f - 1$ except at the origin, where $f(0) = g(0) = 0$. The convex hull of the finite epigraph of g is the open half plane $\{x \in \mathbf{R}^2; y > \alpha x - 1\}$; that of f is the open half plane $\{x \in \mathbf{R}^2; y > \alpha x\}$ with the point $(0, 0)$ added. Neither is closed, so both f and g have irregular points as defined in Section 8. In fact, all points are irregular for g, all but the origin are irregular for f. The convex envelope of f is F, that of g is $F - 1$, so that $\operatorname{cvxe}(f) = \operatorname{cvxe}(g) + 1 = F$.

We note that in this example, $\operatorname{discr}^*(\operatorname{cvxe}(f)) = f$ but

$$\operatorname{discr}^*(\operatorname{cvxe}(g)) = \operatorname{discr}^*(F - 1) = f - 1,$$

which takes the value -1 at the origin. Thus we have $\operatorname{discr}^*(\operatorname{cvxe}(g)) = g - 1$ at the origin.

Both the graph of f and that of g can be described as discrete straight lines in the sense of Reveillès (1991:45): the graph of f is defined by $0 \leqslant -\alpha x + y < 1$; that of g by $0 \leqslant \alpha x - y < 1$.

This example is related to the functions appearing in the theorem of Klein mentioned in the introduction, but there is an important difference: here we define f and g at all integers, whereas in Klein's theorem we used only their values for $x \geqslant 1$. $\square$

The discretization operators may be used for rescaling of convex extensible functions. Let us define, given two positive numbers α, β, and a function $F \in \mathbf{R}_!{}^{\mathbf{R}^n}$, its rescaling $F_\alpha^\beta(x) = \beta F(x/\alpha)$, $x \in \mathbf{R}^n$. Then for $f \in \mathbf{Z}_!{}^{\mathbf{Z}^n}$ we define its rescalings

$$f_\alpha^{\beta,*}(x) = \operatorname{discr}^*\Big((\operatorname{cvxe}(f))_\alpha^\beta\Big) \text{ and } f_{\alpha,*}^\beta(x) = \operatorname{discr}_*\Big((\operatorname{cvxe}(f))_\alpha^\beta\Big), \qquad x \in \mathbf{Z}^n.$$

Both functions are convex extensible. If f is $(\mathbf{Z}^n \times \mathbf{Z})$-convex, they are reasonable candidates for rescaled functions of f.

8. Regular and irregular points

Given $f \colon \mathbf{Z} \to \mathbf{R}_!$, $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$ is either empty or the convex hull of a denumerably infinite set in the plane. It may or may not be closed.

We define $C(s) = \{(s, y) \in \operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))\}$, $s \in \mathbf{R}$. This set may be empty or equal to a straight line; if not, it is a vertical ray with endpoint $(s, F(s))$, where $F = \operatorname{cvxe}(f)$. We shall say that $s \in \mathbf{R}$ is a *regular point for f* if $C(s)$ is closed, and that s is an *irregular point for f* if $C(s)$ is not closed.

Proposition 8.1. *Let $f \in \mathbf{Z}_!{}^{\mathbf{Z}}$ and write F for* $\operatorname{cvxe}(f)$*. If $F(s) \geqslant f(s)$ for a point $s \in \mathbf{Z}$ (case* (A) *in Proposition 7.2), then s is regular. The converse does not hold. For a convex extensible function f, $s \in \mathbf{Z}$ regular implies $F(s) > f(s) - 1$ (case* (A) *or* (B) *in Proposition 7.2). The converse does not hold. Thus for convex extensible functions we have for all integer points s,*

$$F(s) \geqslant f(s) \;\Rightarrow\; s \text{ is regular } \;\Rightarrow\; F(s) > f(s) - 1 \;\Leftrightarrow\; \lceil F(s) \rceil = f(s). \quad \square$$

Proof. For the second assertion, cf. Proposition 7.2. That the converse does not hold is clear from Example 7.5. $\square$

Proposition 8.2. *Let f be a function in $\mathbf{R}_!{}^{\mathbf{Z}}$. If* $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$ *is closed, then all points are regular. Conversely, if $F =$* $\operatorname{cvxe}(f) > -\infty$ *and all points are regular, then*

$$\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f)) = \operatorname{epi}^{\mathrm{F}}(F) = \overline{\operatorname{epi}_{\mathrm{s}}^{\mathrm{F}}(F)},$$

a closed set. $\square$

If f is allowed to take the value $-\infty$, regularity does not imply that $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$ is closed:

Example 8.3. Define $f\colon \mathbf{Z} \to \mathbf{Z}_!$ by $f(x) = -\infty$ for $x < 0$, $f(0) = 0$, and $f(x) = +\infty$ for $x > 0$. Then f is $(\mathbf{Z} \times \mathbf{Z})$-convex and every point is regular, but $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$ is not closed: the point $(0, -1)$ belongs to its closure but not to the set itself. □

Proposition 8.4. *Assume that $f \in {\mathbf{Z}_!}^{\mathbf{Z}}$ is $(\mathbf{Z} \times \mathbf{Z})$-convex and that $a, b \in \mathbf{R}$, $a \leqslant b$, are regular points for f. Then all points in $[a, b]$ are regular for f. In particular we have $\lceil \operatorname{cvxe}(f) \rceil = f$ on $[a, b]_{\mathbf{Z}}$.*

Proof. Assume that a and b are regular, and that s is irregular for a $(\mathbf{Z} \times \mathbf{Z})$-convex function $f \in {\mathbf{Z}_!}^{\mathbf{Z}}$, $a, b, s \in \mathbf{R}$, $a < s < b$. We shall reach a contradiction. Let H be the affine function which agrees with $F = \operatorname{cvxe}(f)$ at a and b. Then $H(s) > F(s)$, but on the other hand there must exist points $r_j, t_j \in \mathbf{R}$ with $r_j < s < t_j$, such that the affine function H_j which agrees with F at r_j and t_j has the property that $H_j(s)$ tends to $F(s)$. However, we can take r_j and t_j as integers. This is because F is not an arbitrary convex function but the convex envelope of a function which is $+\infty$ on $\mathbf{R} \smallsetminus \mathbf{Z}$.

Indices j such that $r_j \leqslant a < b \leqslant t_j$ cannot contribute to this convergence. Indeed, for these indices we must have $H_j(a) \geqslant H(a)$ and $H_j(b) \geqslant H(b)$ so that also $H_j(s) \geqslant H(s) > F(s)$, which prevents convergence to $F(s)$.

Also indices such that $a < r_j \leqslant t_j < b$ cannot contribute to the convergence. Indeed, since the r_j and t_j are integers, there are only finitely many different functions H_j for such indices, and for all of them we have $H_j(s) > F(s)$.

Finally we need to look at indices j such that $a < r_j < b \leqslant t_j$ or $r_j \leqslant a < t_j < b$. For an index such that $a < r_j < b \leqslant t_j$ we must have $H_j(b) \geqslant H(b) = F(b)$. Let K_j be the affine function which agrees with H_j at r_j and with H at b. Then there are only finitely many different values $K_j(s)$, and they are all strictly larger than $F(s)$, so, since $H_j(s) \geqslant K_j(s)$, these indices cannot contribute to the convergence. The case $r_j \leqslant a < t_j < b$ is symmetric.

Thus in all cases we have found a contradiction.

For the last assertion, see Proposition 8.1. □

Proposition 8.5. *Suppose that $s \in \mathbf{Z}$ is an irregular point for a function $f\colon \mathbf{Z} \to \mathbf{Z}_!$. Then the boundary of the finite epigraph of $F = \operatorname{cvxe}(f)$ is a polygon with finitely or infinitely many vertices at integer points, and either all points in $[s, +\infty[$ are irregular and F agrees with an affine function on that interval, or all points in $]-\infty, s]$ are irregular and F equals the restriction of an affine function there.*

Proof. The previous proposition shows that there cannot be regular points both to the left and to the right of an irregular point. That F agrees with an affine function on an interval consisting of irregular points follows easily. □

For $(\mathbf{Z} \times \mathbf{Z})$-convex functions $f\colon \mathbf{Z} \to \mathbf{Z}_!$ we have a priori six cases when comparing f and $F = \operatorname{cvxe}(f)$ at a point $s \in \mathbf{Z}$:

1. $C(s) = \emptyset$, a closed set. Then $F(s) = f(s) = +\infty$.

2. $C(s) = \{s\} \times \mathbf{R}$, a closed set. Then $F(s) = f(s) = -\infty$.

3. $C(s)$ is closed and $F(s)$ is an integer. Then the endpoint $(s, F(s))$ of $C(s)$ belongs to $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$ and $F(s) = \lceil F(s) \rceil = f(s)$. Example: $f(x) = \lceil \frac{1}{2}x \rceil$, s any even integer.

4. $C(s)$ is closed and $F(s) \in \mathbf{R} \smallsetminus \mathbf{Z}$. Then the endpoint $(s, F(s))$ belongs to $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$ and $F(s) < \lceil F(s) \rceil = f(s)$. Example: $f(x) = \lceil \frac{1}{2}x \rceil$, s any odd integer.

5. $C(s)$ is not closed and $F(s)$ is an integer. Then the endpoint $(s, F(s))$ of $C(s)$ does not belong to $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$ and $F(s) = \lceil F(s) \rceil = f(s) - 1 < f(s)$. Example: $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geqslant 0$, s any natural number. See Example 7.4.

6. $C(s)$ is not closed and $F(s) \in \mathbf{R} \smallsetminus \mathbf{Z}$. Then the endpoint $(s, F(s))$ of $C(s)$ does not belong to $\operatorname{cvxh}(\operatorname{epi}^{\mathrm{F}}(f))$ and $F(s) < \lceil F(s) \rceil = f(s)$. Example: $f(x) = \lceil x/\sqrt{2}\, \rceil$, $F(x) = x/\sqrt{2}$, s any nonzero integer. See Example 7.5.

For such functions, s is regular in cases 1, 2, 3, and 4 and irregular in cases 5 and 6.

Note that $\lceil F(s) \rceil = f(s)$ in all cases except 5, and that $\lceil F(s) \rceil = f(s) - 1 < f(s)$ in case 5.

9. Extending rectilinear segments

Let us consider functions defined on an interval: let c and d be two integers and consider functions $f\colon [c, d]_{\mathbf{Z}} \to \mathbf{Z}$. We can then form $D_b D_a f(x)$ only for $c \leqslant x \leqslant d - a - b$, $a, b \in \dot{\mathbf{N}}$. A natural question is whether the conditions $|D_b D_a f(x)| \leqslant 1$ for these finitely many a, b, x are sufficient to ensure that f represents a straight line segment; in other words, whether we can find an extension g to all of $\mathbf{Z}$ of the function f which satisfies the conditions everywhere. The answer is in the affirmative, but the extension is never unique.

We recall the definitions in (5.7) of various strips $S(\alpha, \beta, \gamma)$ etc. in $\mathbf{R}^2$. All these strips have height $\gamma - \beta$.

Theorem 9.1. *If $f\colon [c, d]_{\mathbf{Z}} \to \mathbf{Z}$ satisfies $|D_b D_a f(x)| \leqslant 1$ for all x, a, b for which the expression is defined, then its graph is contained in an open strip $S^*_*(\alpha, \beta, \gamma)$ with rational α and of height $\gamma - \beta < 1$. If a function $f\colon \mathbf{Z} \to \mathbf{Z}$ defined on the whole integer axis satisfies $|D_b D_a f| \leqslant 1$, its graph is contained in a closed strip $S(\alpha, \beta, \beta + 1)$ of height* 1.

Proof. Given $f\colon [c, d]_{\mathbf{Z}} \to \mathbf{Z}$ with $c \leqslant d$ and a real number α, there exist real numbers β and γ such that the strip $S(\alpha, \beta, \gamma)$ contains the graph of f. For every

α we choose $\beta = \beta(\alpha)$ maximal and $\gamma = \gamma(\alpha)$ minimal. Then there is at least one p and one q such that $f(p) = \alpha p + \beta$ and $f(q) = \alpha q + \gamma$.

Next we vary α to minimize the height $\gamma(\alpha) - \beta(\alpha)$: we obtain a strip with smallest height which contains the graph of f. If $d - c \geqslant 1$ it is unique. Unless $d - c \leqslant 1$ (a trivial case), the graph has at least three points and it is clear that there must be either at least two points on the lower boundary and one on the upper boundary of the strip, or vice versa. For reasons of symmetry, we may assume that we have $p < s < q$ with $(p, f(p))$ and $(q, f(q)$ on the lower boundary, $(s, f(s))$ on the upper boundary, i.e., $f(p) = \alpha p + \beta$ and $f(q) = \alpha q + \beta$, while $f(s) = \alpha s + \gamma$. If for instance $(p, f(p))$ and $(s, f(s))$ are on the lower boundary and $(q, f(q))$ on the upper boundary, and there is no point on the upper boundary to the left of q and no point on the lower boundary to the right of s, it is easy to see that there are strips with a smaller height.

The condition $|J_{a,b}f(x)| < 1$ can now be written, taking $x = p$, $a = s - p$, and $b = q - s$,

$$-1 < \frac{b}{a+b} f(p) - f(s) + \frac{a}{a+b} f(q) \leqslant 0,$$

which, if we insert the values of a, b, $f(p)$, $f(s)$, and $f(q)$, just says that $\beta \leqslant \gamma < \beta + 1$.

Therefore all points in the graph of f lie in the closed strip $S(\alpha, \beta, \gamma)$, which is contained in the half-open strip $S^*(\alpha, \beta, \beta + 1)$. It is of course also contained in the open strip $S^*_*(\alpha, \beta - \varepsilon, \gamma + \varepsilon)$ of height $\gamma + \varepsilon - (\beta - \varepsilon) < 1$ for small ε.

This means that $f(x) = \lceil \alpha x + \beta \rceil$ for all x in the domain of definition of f.

We can choose $\alpha = (f(q) - f(p))/(q - p)$, a rational number. However, since $\gamma - \beta < 1$, we can also vary α in some interval and choose infinitely many rational or irrational values for the slope of the line.

Finally, if $f\colon \mathbf{Z} \to \mathbf{Z}$, we consider its restriction f_k to the interval $[-k, k]_{\mathbf{Z}}$, $k \in \mathbf{N}$, and apply what we know about f_k: there is a strip $S(\alpha_k, \beta_k, \gamma_k)$ of height $\gamma_k - \beta_k < 1$. It is not difficult to show that, as $k \to +\infty$, the sequences (α_k), (β_k) and (γ_k) tend to limits α, β and γ and that the graph of f is contained in the closed strip $S(\alpha, \beta, \gamma)$. But we can only say that $\gamma - \beta \leqslant 1$; cf. Example 7.4. $\square$

Theorem 9.2. *If the graph of a function $f\colon \mathbf{Z} \to \mathbf{Z}$ or $f\colon [c, d]_{\mathbf{Z}} \to \mathbf{Z}$ is contained in a half-open strip $S^*(\alpha, \beta, \beta + 1)$ or $S_*(\alpha, \beta, \beta + 1)$, then $|(D_b D_a f)(x)| \leqslant 1$ for all x and $a, b \in \dot{\mathbf{N}}$ for which the expression is defined.*

Proof. If $\alpha x + \beta \leqslant f(x) < \alpha x + \beta + 1$ or $\alpha x + \beta < f(x) \leqslant \alpha x + \beta + 1$ we get

$$\begin{aligned}(D_b D_a f)(x) &= f(x + a + b) - f(x + a) - f(x + b) + f(x) \\ &< \alpha(x + a + b) + \beta + 1 - \alpha(x + a) - \beta - \alpha(x + b) - \beta + \alpha x + \beta + 1 \\ &= 2.\end{aligned}$$

Since $(D_b D_a f)(x)$ is an integer for functions with integer values, we must have $(D_b D_a f)(x) \leqslant 1$. By symmetry, $(D_b D_a f)(x) \geqslant -1$. $\square$

Theorem 9.3. *Let* $f\colon [c,d]_{\mathbf{Z}} \to \mathbf{Z}$ *be given such that* $|D_bD_af(x)| \leqslant 1$ *for all* a, b, x *for which the expression is defined, i.e., for* $c \leqslant x \leqslant d-a-b$, $a,b \in \dot{\mathbf{N}}$. *Then* f *can be extended to a function* $g\colon \mathbf{Z} \to \mathbf{Z}$ *such that* $|D_bD_ag(x)| \leqslant 1$ *for all* $x \in \mathbf{Z}$ *and all* $a,b \in \dot{\mathbf{N}}$.

If we look at this as a combinatorial problem for chain codes, i.e., for binary words, the theorem says, in case $0 \leqslant D_1f \leqslant 1$, that a balanced finite binary word can be extended to a periodic balanced infinite word, moreover to infinitely many words with different periods—and also to infinitely many balanced nonperiodic infinite words.

Proof. Applying Theorem 9.1, we find a half-open strip $S^*(\alpha,\beta,\beta+1)$ or $S_*(\alpha,\beta,\beta+1)$ which contains the graph of f. In the first case $f(x) = \lceil \alpha x + \beta \rceil$, in the second $f(x) = \lfloor \alpha x + \beta + 1 \rfloor$ for all $x \in [c,d]_{\mathbf{Z}}$. We can now define the extension $g(x) = \lceil \alpha x + \beta \rceil$ or $g(x) = \lfloor \alpha x + \beta + 1 \rfloor$ for all $x \in \mathbf{Z}$. According to Theorem 9.2 we now have $|D_bD_ag| \leqslant 1$. □

Example 9.4. Let f_k be the restriction to $[-k,k]_{\mathbf{Z}}$ of the function in Example 7.4 with $k \in \dot{\mathbf{N}}$. Then the construction gives a slope $\alpha_k = 1/(k+1) > 0$ and height $\gamma_k - \beta_k = k/(k+1) < 1$. Actually we may choose any α_k with $0 < \alpha_k \leqslant 1/(k+1)$ and still get $\gamma_k - \beta_k < 1$. But it is not possible to choose $\alpha_k = 0$, for then $\gamma_k - \beta_k = 1$.

This means that the straight line constructed from a restriction of f to a finite interval $[c,d]$ containing -1 and 0 must always have a positive slope, although f itself represents a line with slope zero. If we choose a rational slope, the chain code D_1g of the extension will be periodic, while D_1f is not. The function in Example 7.4 can never appear as an extension in the construction used in the proof of Theorem 9.3. □

10. Digital straightness

Combining what we have learned about digital straightness so far we obtain the following result.

Theorem 10.1. *Let* $f \in \mathbf{Z}^{\mathbf{Z}}$, *assume that* $0 \leqslant D_1f \leqslant 1$, *and consider the following properties.*

(A). *The graph of* f *has the chord property;*

(B). *Both* f *and* $-f$ *are convex extensible;*

(C). *The graph of* f *is a* $\mathbf{Z}^2$*-convex set;*

(D). *The inequality* $|(D_bD_af)(x)| \leqslant 1$ *holds for all* $(x,a,b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}}$*;*

(E). *The inequality* $|(J_{a,b}f)(x)| < 1$ *holds for all* $(x,a,b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}}$*;*

(F). *The binary word* $D_1f\colon \mathbf{Z} \to \mathbf{Z}$ *is balanced;*

(G). *The function* f *defines a refined digital hyperplane in* $\mathbf{Z}^2$ *in the sense of Kiselman (2004);*

(H). *The function f defines a digital straight line in the sense of Reveillès (1991).*

All conditions (A), (B), (C), (D), (E), (F) *and* (G) *are equivalent, and they are implied by* (H).

Proof. The equivalences follow on combining Theorems 5.1, 5.3 and 6.1.

That (G) $\not\Rightarrow$ (H) follows from Example 7.4. □

Some of the equivalences in this theorem have a long history. Morse & Hedlund (1940) proved that Sturmian words (aperiodic words of minimal complexity) are balanced, and conversely. That balance of a binary word is equivalent to the property of being a mechanical word is proved in the case of irrational slope in Lothaire (2002: Theorem 2.1.13).

Theorem 10.2. *Let $f\colon [c,d]_{\mathbf{Z}} \to \mathbf{Z}$ be defined on a finite interval $[c,d]_{\mathbf{Z}}$ and assume that $|D_1 f(x)| \leqslant 1$ for all x such that $c \leqslant x \leqslant d-1$. Then the following properties are all equivalent.*

(A). *The graph of f has the chord property;*

(B). *Both f and $-f$ are convex extensible;*

(C). *The graph of f is a $\mathbf{Z}^2$-convex set;*

(D). *The inequalty $|(D_b D_a f)(x)| \leqslant 1$ holds for all $(x,a,b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}}$ such that $c \leqslant x < x+a+b \leqslant d$;*

(E). *The inequality $|(J_{a,b} f)(x)| < 1$ holds for all $(x,a,b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}}$ such that $c \leqslant x < x+a+b \leqslant d$;*

(F). *The binary word $D_1 f\colon [c,d-1]_{\mathbf{Z}} \to \mathbf{Z}$ is balanced;*

(G). *The function f defines a subset of a refined digital hyperplane in $\mathbf{Z}^2$ in the sense of Kiselman (2004);*

(H). *The function f defines a subset of a digital straight line in the sense of Reveillès (1991).*

Proof. In view of Theorem 10.1 it is enough to prove that (H) is implied by any of the other conditions. From Theorem 9.1 it follows that the graph of a function satisfying (D) is contained in an open strip $S(\alpha,\beta,\gamma)$ of height $\gamma-\beta<1$. So then it is contained in the digital straight line in the sense of Reveillès $S^*(\alpha,\beta,\beta+1)$ (as well as in others). Hence (H) holds. □

We also note the following result on locality of the various properties. Let us say that a property of functions $f \in \mathbf{Z}_!{}^A$, where A is an arbitrary subinterval of $\mathbf{Z}$, is *local* if it is true that it has the property if and only if all its restrictions $f|_{[c,d]_{\mathbf{Z}}}$ to finite intervals $[c,d]_{\mathbf{Z}} \subset A$ have the property.

Proposition 10.3. *The properties* (A), (B), (C), (D), (E), (F) *and* (G) *of Theorems 10.1 and 10.2, understood respectively for functions defined on* $\mathbf{Z}$ *and on subintervals of* $\mathbf{Z}$*, are local properties. The property* (H) *is not local.*

Proof. For properties (A)–(F) this is obvious. For property (G) we just return to (D) for example. That (H) is not local follows on comparing Example 7.4 with (H) of Theorem 10.2. ☐

11. Extending convex extensible functions

We may also extend a function defined and finite on a finite interval $[c,d]_{\mathbf{Z}}$ if it satisfies the conditions $D_bD_af \geqslant -1$ whenever the expression has a sense. The function considered thus has a convex extension with values in $\mathbf{R}_!$; the problem is to find an extension with finite values. Since the conditions are now onesided, we have even more freedom in the choice of extension.

Theorem 11.1. *Let $f\colon [c,d]_{\mathbf{Z}} \to \mathbf{Z}$ be given such that $D_bD_af(x) \geqslant -1$ for all a,b,x for which the expression is defined, i.e., for $c \leqslant x \leqslant d-a-b$, $a,b \in \dot{\mathbf{N}}$. Then f can be extended to a function $g\colon \mathbf{Z} \to \mathbf{Z}$ such that $D_bD_ag(x) \geqslant -1$ for all $x \in \mathbf{Z}$ and all $a,b \in \dot{\mathbf{N}}$.*

Proof. We shall define g as follows.

$$g(x) = \begin{cases} f(c) + \alpha(x-c), & x < c; \\ f(x), & c \leqslant x \leqslant d; \\ f(d) + \beta(x-d), & x > d. \end{cases}$$

Here

$$\alpha = \inf_{t\in\mathbf{Z}}(D_1f(t); c \leqslant t \leqslant d-1) \in \mathbf{Z}$$

and

$$\beta = \sup_{t\in\mathbf{Z}}(D_1f(t); c \leqslant t \leqslant d-1) \in \mathbf{Z}.$$

This means that we define g by affine functions outside the given interval $[c,d]_{\mathbf{Z}}$.

We shall prove by induction that g satisfies $D_bD_ag \geqslant -1$. This is true by hypothesis if $c \leqslant x < x+a+b \leqslant d$. Assume that it is true for $p \leqslant x < x+a+b \leqslant q$ for a certain p and a certain q with $p \leqslant c$ and $q \geqslant d$, and let us prove that it is true if we augment q by one unit as well as if we decrease p by one unit. It is enough to consider $p \leqslant x < x+a+b = q+1$, i.e., to pass from $[p,q]_{\mathbf{Z}}$ to $[p,q+1]_{\mathbf{Z}}$. We shall then compare $D_bD_ag(x)$ with $D_{b-1}D_ag(x)$, i.e., we move $x+b$ and $x+a+b$ one step to the left to get the rightmost point inside the interval $[p,q]_{\mathbf{Z}}$. Then

$$\begin{aligned} D_bD_ag(x) &= g(x) - g(x+a) - g(x+b) + g(q+1) \\ &= g(x) - g(x+a) - g(x+b) + g(q) + \beta \\ &\geqslant g(x) - g(x+a) - g(x+b-1) + g(q) = D_{b-1}D_ag(x) \geqslant -1 \end{aligned}$$

in view of our choice of β, which guarantees that $g(q+1) = g(q) + \beta$ while $g(x+b) \leqslant g(x+b-1) + \beta$. ☐

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