# Vyacheslav Zakharyuta's Complex Analysis

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ABSTRACT. The paper gives an account of the work of Vyacheslav Pavlovich Zakharyuta in the domain of complex analysis, in particular pluripotential theory, showing the influence of his research during several decades.

## 1. Introduction

Professor Vyacheslav Pavlovich Zakharyuta has, over many years, made outstanding contributions to mathematics. He has very early found important phenomena in complex analysis, thereby initiating new roads of research. I shall try here to outline some of the most significant of his contributions to analysis in several complex variables.

I cannot limit myself to an account of his results only; I find it important to put them into the framework of a more general development of the field of several complex variables, especially pluripotential theory and the theory of bases of topological vector spaces of holomorphic functions.

Professor Zakharyuta has also been a very successful advisor of graduate students. More than 30 Master Degree students have finished their degree with him as an advisor, and eleven doctoral theses were successfully defended with him as an advisor or co-advisor: he has been the principal advisor of S. N. Kadampatta, N. I. Skiba, P. A. Chalov, N. S. Manzhikova (Nadbitova), Alexander P. Goncharov, L. V. Runov, V. A. Znamenskiĭ, M. A. Shubarin, and B. A. Derzhavets. He has been assistant advisor of Thabet Abdeljavad and Erdal Karapınar.

## 2. The global extremal function

Józef Siciak introduced (1961, 1962) an extremal function of several complex variables analogous to the Green function for the unbounded component of the complement of a compact set in the complex plane and with pole at infinity. He emphasized that the Green function plays a fundamental role in the theory of interpolation and approximation of holomorphic functions of one variable by polynomials. Indeed his

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function was to play a similar role in several variables and his article became the starting point for a rich development.

The extremal function $z \mapsto \Phi(z, E, b)$ depends on a given subset $E$ of $\mathbb{C}^n$ and a given function $b$ defined on $E$. Siciak's original definition used Lagrange interpolation of the values $\exp b(p^\nu)$ to define a polynomial taking those values at certain points $p^\nu$ in $E$, then choosing the points in an extremal way (in analogy with the Fekete points in one variable) and finally passing to the limit. A consequence was the *Bernstein–Walsh inequality* for polynomials $P$ of degree at most $j$, viz.

$$|P(z)| \leqslant \|P\|_E \, \Phi(z, E, 0)^j, \qquad z \in \mathbb{C}^n, \tag{2.1}$$

where the norm is the supremum norm on $E$.

Siciak proved that the strict sublevel sets of the extremal function, i.e., the sets

$$E_R = \{z;\ \Phi(z, E, 0) < R\}, \qquad R > 1,$$

determine the possible holomorphic extensions of a given function $f$ on a compact set $E$. More precisely, assuming $\Phi(\,\cdot\,, E, 0)$ to be continuous, $f$ was shown to admit a holomorphic extension to the open set $E_R$ if and only if

$$\limsup_{j\to\infty} \|f - \pi_j\|_E^{1/j} \leqslant 1/R,$$

where $\pi_j$ is a polynomial of degree at most $j$ which best approximates $f$ on $E$ (1962:346). This was a striking generalization of the corresponding one-dimensional result, due to Bernstein (in the case of an interval), and Walsh and Russell; cf. J. L. Walsh (1935:79).

Later (2.1) was taken as the definition, i.e., one usually defined

$$\Phi(z, E, 0) = \sup_{j\geqslant 1} \sup_P \big(|P(z)|^{1/j};\ \|P\|_E \leqslant 1\big), \qquad z \in \mathbb{C}^n, \tag{2.2}$$

where $P$ varies in the space of polynomials of degree at most $j$.

In a talk at an All-Union Conference in Kharkov in 1971, Zakharyuta introduced an extremal function defined in terms of plurisubharmonic functions

$$V_E(z) = \sup_u \big(u(z);\ u \in \mathscr{L},\ u \leqslant 0 \text{ on } E\big), \qquad z \in \mathbb{C},$$

where $\mathscr{L}$ denotes the class of plurisubharmonic functions with logarithmic growth, i.e.,

$$\mathscr{L} = \left\{u \in PSH(\mathbb{C}^n);\ \sup_{z\in\mathbb{C}^n} \big(u(z) - \log(1 + \|z\|)\big) < +\infty\right\}.$$

Here and in the sequel, $\log = \log_e = \ln$ is the natural logarithm.

This definition was published in his Sbornik paper (1975:382) in connection with a study of multidimensional analogues of classical characteristics of compacta such as the transfinite diameter, Chebyshev constants, and capacity. The main result of that paper is that, in all dimensions, the limit in Franciszek Leja's definition of the transfinite diameter exists, and that the transfinite diameter is equal to the Chebyshev constant.

The methods in the Sbornik paper have been widely used, for instance in arithmetic geometry by Robert Rumely and Chi Fong Lai (1994) and by these two authors joint with Robert Varley (2000). In December, 2007, Thomas Bloom and Norman Levenberg deposited a paper (2007) in the ArXiv, where they discuss a general framework for various types of transfinite diameter in the spirit of Zakharyuta (1975).

Zakharyuta's definition of the extremal function was introduced in connection with his new proof of the Bernstein–Walsh theorem based on the use of orthogonal bases of polynomials published in (1976/77).

Liouville's theorem for plurisubharmonic functions says that a nonconstant plurisubharmonic function cannot grow slower than a positive constant times the function $z \mapsto \log \|z\|$. Therefore $\mathscr{L}$ is called the class of plurisubharmonic functions of *minimal growth.* It is a subclass of $PSH(\mathbb{C}^n)$ of great interest. The upper semicontinuous envelope $V_E^*$ of $V_E$ is either plurisubharmonic (when $E$ is not pluripolar) or identically $+\infty$ (when $E$ is pluripolar).

If we use (2.2) to define $\Phi(z, E, 0)$, it is obvious that $\log \Phi(\,\cdot\,, E, 0) \leqslant V_E$ for any set $E$. Zakharyuta proved that $V_K^* = \log \Phi(\,\cdot\,, K, 0) = V_K$ if $K$ is a compact set such that $V_K^*$ is zero on $K$ (1976/77:146). Siciak (1976, 1981, 1982:23) proved that $V_K = \log \Phi(\,\cdot\,, K, 0)$ for general compact sets $K$. A fourth proof, using Hörmander's $L^2$ methods, was given by Jean-Pierre Demailly in his notes (1989). Thus a definition that had its origin in interpolation problems in one complex variable came to be directly expressed using plurisubharmonic functions.

A striking characterization of algebraic varieties in terms of the global extremal function was established by Azimbay Sadullaev (1982). Let a connected analytic variety $A$ in an open subset of $\mathbb{C}^n$ be given as well as a compact subset $K$ of $A$, and assume that $K$ is not pluripolar in $A$. Then $V_K$ is locally bounded on $A$ if and only if $A$ is a piece of an algebraic variety.

It is not easy to calculate $V_E$. Sadullaev (1985) determined $V_K$ when $K$ is a ball in $\mathbb{R}^n \subset \mathbb{C}^n$ and noted that it is not a smooth function. More generally, Magnus Lundin (1985) determined $V_K$ when $K$ is a convex, symmetric, compact subset of $\mathbb{R}^n \subset \mathbb{C}^n$. From the special form of $V_K$ in Lundin's case, one can see easily that the sublevel sets $\big\{z \in \mathbb{C}^n;\ V_K^*(z) < c\big\}$, $c \in \mathbb{R}$, are convex. It is a general result of László Lempert that these sublevel sets are convex if $K$ is any convex compact subset of $\mathbb{C}^n$. Lempert's result relies on a beautiful description of $V_K$ (published by Siegfried Momm 1996:160) when $K$ is strongly convex and has real analytic boundary, viz.

$$V_K(z) = \inf_{r,f} \big(\log r;\ r > 1,\ f(r) = z\big), \qquad z \in \mathbb{C}^n \smallsetminus K,$$

where $f$ varies in the class of all holomorphic mappings of the complement of the closed unit disk into $\mathbb{C}^n$ such that $f(t)/|t|$ is bounded and $f$ has a continuous extension to the unit circle, mapping it into $K$.

This description has been rendered even more beautiful by the use of the method of disk functionals developed by Finnur Lárusson and Ragnar Sigurdsson; see their paper on the Siciak–Zakharyuta extremal function (2005). It works even for weighted functions as shown by Magnússon and Sigurdsson (2007). The disk envelope formulas for $V_K$ can also be used to characterize polynomial convexity (Lárusson and Sigurdsson 2007).

Bloom (1997) presented a survey of several results from pluripotential theory, in particular those of Zakharyuta. An essential role is played by the global extremal function.

Bedford and Taylor (1986) gave precise estimates for the measure $(dd^cV_K)^n$ when $K$ is compact and contained in $\mathbb{R}^n$ and gave an exact expression for it when $K$ is convex and symmetric.

Zeriahi (1996) investigated the global extremal function on nonsingular algebraic varieties and extended results in $\mathbb{C}^n$ to that case. To treat the more general case of analytic spaces, he introduced an axiomatic approach (2000) in that he replaced the class $\mathscr{L}$ by a class of functions satisfying certain axioms.

The global Siciak–Zakharyuta extremal function has had a great significance in many results on approximation and the problems of isomorphisms between spaces of holomorphic functions, and even in real analysis; see, e.g., Pawłucki and Pleśniak (1986) and the surveys by Klimek (1991) and Zakharyuta (1994).

## 3. Capacities defined by the global extremal function

The notion of capacity appeared in classical potential theory as a measure of the size of sets in $\mathbb{R}^n$, and was a model for the capacity of a metal conductor to hold electric charges: how many coulombs can you put into the conductor while not letting the tension exceed one volt? An early attempt to generalize this notion to several complex variables was the Γ-capacity of Ronkin (1971). It is built up from the logarithmic capacity in $\mathbb{C}$ using induction over the dimension, and is not invariant under biholomorphic mappings. Zakharyuta (1975) and Siciak (1981) studied the functionals

$$\gamma(E) = \limsup_{\|z\|\to+\infty} \big(V_E(z) - \log\|z\|\big) \text{ and } c(E) = \exp(-\gamma(E)), \qquad E \subset \mathbb{C}^n. \tag{3.1}$$

In fact, for $n = 1$, $c(E)$ is the classical logarithmic capacity of $E$, so it was natural to expect that the behavior of the extremal function at infinity would reflect important properties of the set. The functional $c$ was called a capacity by analogy (e.g., by Zakharyuta 1975:383), without claiming that it is actually a capacity in Choquet's sense.

Gustave Choquet (1915–2006) introduced an axiomatic approach to capacities in his immensely influential paper (1955). He defined a *capacity* as a functional

$$\varphi\colon \mathscr{E} \to [-\infty, +\infty]$$

which is defined on an arbitrary family $\mathscr{E}$ of subsets of a topological space $X$ and which is increasing and continuous on the right (1955:174). He then defined the interior capacity related to $\varphi$ as

$$\varphi_*(A) = \sup_E \big(\varphi(E);\ E \in \mathscr{E}, E \subset A\big), \qquad A \subset X,$$

with the modification that $\varphi_*(A) = \inf_E(\varphi(E);\ E \in \mathscr{E})$ when there is no element of $\mathscr{E}$ contained in $A$ (this is to define a zero level for $\varphi$), and the exterior capacity as

$$\varphi^*(A) = \inf_\omega \big(\varphi_*(\omega);\ \omega \text{ open},\ \omega \supset A\big), \qquad A \subset X.$$

Choquet called a set *capacitable* if the interior and exterior capacities agree on it. The continuity on the right means precisely that $\varphi(E) = \varphi^*(E)$ for all $E \in \mathscr{E}$, and clearly $\varphi_*(E) = \varphi(E)$ when $E \in \mathscr{E}$, so all elements of $\mathscr{E}$ are capacitable. For which other sets $A$ does the equality $\varphi_*(A) = \varphi^*(A)$ hold?

Before Choquet it was not known whether all Borel sets are capacitable for the classical Newtonian capacity (Cartan 1945:94). Choquet solved the problem affirmatively. His famous theorem of capacitability (1955:223) says that every $K$-analytic set is capacitable for every capacity in a very large class. The class of

$K$-analytic sets contains all Borel sets in $\mathbb{R}^n$ and in particular the sets

$$\{x \in \mathbb{R}^n;\ u(x) < u^*(x)\},$$

where $u = \limsup u_j$, $(u_j)$ being a sequence of subharmonic functions which is locally bounded from above.

Soon afterwards Choquet streamlined his definition. Specialized to the case of the family of all compact subsets of a Hausdorff space $X$, his new definition read as follows (1959:84): an *abstract capacity* (later to become known as a *Choquet capacity*) is an increasing functional $f$ defined on all subsets of $X$ with values in $[-\infty, +\infty]$ and satisfying

$$f\big(\bigcap K_j\big) = \lim f(K_j) \quad \text{and} \quad f\big(\bigcup A_j\big) = \lim f(A_j) \tag{3.2}$$

for every decreasing sequence $(K_j)_{j\in\mathbb{N}}$ of compact sets and every increasing sequence $(A_j)_{j\in\mathbb{N}}$ of arbitrary subsets of $X$.

In his new theory, he called a set $A$ *$f$-capacitable* if $f(A) = \sup_K f(K)$, the supremum being taken over all compact sets $K$ contained in $A$. All $K$-Suslin sets (in many cases the same as the $K$-analytic sets) are capacitable for all abstract capacities. Links between the two systems of axioms are provided by two facts:

(i) The exterior capacity associated to a capacity in Choquet's theory (1955) is always an abstract capacity (Brelot 1959:59); and

(ii) An abstract capacity in the sense of Choquet (1959) is a capacity in the sense of Choquet (1955) when $\mathscr{E}$ is the family of compact sets, provided that the underlying space is locally compact.

For a full account of the history of potential theory, see Brelot (1954, 1972) and Choquet (1986), who presented his personal reflections on the birth of capacity theory.

Sławomir Kołodziej (1988) proved the remarkable result that the functional $c$ defined in (3.1) actually satisfies Choquet's axioms (3.2)—the difficult point being the first condition on decreasing sequences of compact sets. Therefore all theorems on abstract capacities can be applied to this functional: Borel sets can be approximated from the inside by compact sets and from the outside by open sets.

Kołodziej later discovered new fundamental properties of extremal functions (1989) and showed his result in (1988) to be an easy consequence of them.

## 4. The relative extremal function and common bases

An extremal function which has become known as the *relative extremal function* was introduced by Siciak (1969:154). Given an open set $\Omega$ in $\mathbb{C}^n$ and a compact subset $E$ of $\Omega$ he defined a function $(U_{E,\Omega})^*$, written $U^*_{E,\Omega}$, where the star denotes the upper semicontinuous envelope, and where

$$U_{E,\Omega}(z) = \sup_u \big(u(z);\ u \in PSH(\Omega),\ u \leqslant 0 \text{ on } E,\ u \leqslant 1 \text{ in } \Omega\big), \qquad z \in \Omega. \tag{4.1}$$

The definition makes sense of course for any subset $E$ of $\Omega$. Siciak noted that $U^*_{E,\Omega}$ is *extremal* in the sense that any plurisubharmonic function $v$ which is $\leqslant m$ on $E$ and $\leqslant M$ in $\Omega$ must satisfy $v \leqslant m + (M-m)U^*_{E,\Omega}$ in $\Omega$; the function $U^*_{E,\Omega}$ serves in a version of the Two Constants Theorem for plurisubharmonic functions.

Zakharyuta (1974: §3) used the sublevel sets of the function $U^*_{E,\Omega}$ to define open and compact sets

$$\Omega_\alpha = \{z \in \Omega;\ U^*_{K,\Omega}(z) < \alpha\}, \quad K_\alpha = \{z \in \Omega;\ U^*_{K,\Omega}(z) \leqslant \alpha\}.$$

He proved that they are associated to interpolation of Hilbert spaces. Suppose two Hilbert spaces $H_1$ and $H_0$ are given satisfying

$$\mathscr{O}\big(\,\overline{\Omega}\,\big) \subset H_1 \subset \mathscr{O}(\Omega) \subset \mathscr{O}(K) \subset H_0 \subset AC(K),$$

where $\mathscr{O}(\Omega)$ is the space of holomorphic functions in $\Omega$, $\mathscr{O}(K)$ the inductive limit of $\mathscr{O}(\omega)$ for all open neighborhoods $\omega$ of a compact set $K$, and finally $AC(K)$ is the Banach space obtained by taking the closure of $\mathscr{O}(K)$ in $C(K)$. Then, under certain regularity assumptions,

$$\mathscr{O}(K_\alpha) \subset H^\alpha \subset \mathscr{O}(\Omega_\alpha), \qquad 0 < \alpha < 1, \tag{4.2}$$

where $H^\alpha$ is the interpolation between $H_0$ and $H_1$ defined using a basis $(e_j)_{j\in\mathbb{N}}$ which is common for $H_1$ and the closure of $H_1$ in $H_0$, and determined by the requirement that $\|e_j\|_{H^\alpha} = e^{\alpha a_j}$ if $\|e_j\|_{H_0} = 1$ and $\|e_j\|_{H_1} = e^{a_j}$, $j \in \mathbb{N}$. Thus interpolation in Hilbert spaces approximates very well the interpolation between $K$ and $\Omega$ provided by $U^*_{K,\Omega}$.

Zakharyuta proved (1976/77) the general result on bases common to $\mathscr{O}(\Omega)$ and $\mathscr{O}(K)$ for pluriregular pairs $(K, \Omega)$ using the method of Hilbert scales of (1974); in his paper (1967) this was done for one variable and it was indicated by examples that it would work also for several variables. Nguyen Thanh Van (1972:230) generalized the results of Zakharyuta (1967) for one complex variable.

Vyacheslav Zakharyuta and Nikolai Skiba (1976) used common bases of Hilbert spaces for pairs on open Riemann surfaces of dimension 1 to prove asymptotic formulas for Kolmogorov's width (see Section 6).

A theorem of Poletsky (1991, 1993) and Bu and Schachermayer (1992) states that if $\varphi$ is an upper semi-continuous function on $\Omega$, then

$$\sup\big(u(z);\ u \in PSH(\Omega),\ u \leqslant \varphi\big) = \inf_f \left[\int_{\mathbb{T}} \varphi\circ f\, d\sigma;\ f \in \mathscr{O}(\overline{\mathbb{D}}, \Omega),\ f(0) = z\right]. \tag{4.3}$$

Here $\mathbb{D}$ and $\mathbb{T}$ denote the unit disk and the unit circle in $\mathbb{C}$, $\sigma$ is the arc length measure on $\mathbb{T}$ normalized to 1, and $\mathscr{O}(\overline{\mathbb{D}}, \Omega)$ denotes the set of all analytic disks that extend as holomorphic mappings to some neighborhood of the closed unit disk. Thus the plurisubharmonic envelope of $\varphi$ defined by the left-hand side can be expressed also as an infimum as defined by the right-hand side, an approach from above. The change of viewpoint is similar to that in convexity theory: the convex envelope of a function is defined by taking the supremum of all convex minorants, but can also be expressed as an infimum of linear combinations of function values, thus approximated from above.

If we take $\varphi = 1 - \chi_E = \chi_{\complement E}$, the characteristic function of the complement of an open set $E$, then $\varphi$ is upper semi-continuous, and the left hand side of (4.3) is equal to $U_{E,\Omega}(z)$; we know that $U_{E,\Omega}$ is plursubharmonic in $\Omega$. The integral in the right hand side is equal to $\sigma_f(E) = \sigma\left(f^{-1}(\complement E) \cap \mathbb{T}\right)$. Hence we can say that the function $U_{E,\Omega}$ takes a given value $a \in [0, 1[$ at the point $z \in \Omega$ if and only if for every $\varepsilon > 0$ there exist a closed analytic disk $f$ which maps the origin 0 to $z$ and maps an open subset of the unit cicle $\mathbb{T}$ of arc length at least $2\pi(1 - a - \varepsilon)$ into $E$.

Poletsky (1993: Theorem 7.2) extended the disk formula to pluriregular sets. In the case when $E$ is a pluriregular compact set in a bounded domain $\Omega$, it becomes

$$U_{E,\Omega}(z) = \inf_f \big(\sigma_f(\Omega \smallsetminus E);\ f \in \mathscr{O}(\mathbb{D}, \Omega) \cap C(\overline{\mathbb{D}}, \Omega),\ f(0) = z\big).$$

Observe that in this formula the infimum is taken over all analytic disks that extend continuously to the closed unit disk.

By applying this formula Poletsky (1993: Corollary 7.1), was able to describe polynomial convexity in terms of existence of analytic disks. This is like the result by Lárusson and Sigurdsson (2007) mentioned in Section 2, but the disk functionals used are quite different in the two cases.

The formula of Poletsky and Bu–Schachermayer was generalized to hold for a large class of complex manifolds by Lárusson and Sigurdsson (1998) and to all manifolds by Rosay (2003). The disk formula for (locally) pluriregular sets was generalized to all manifolds by Edigarian and Sigurdsson (2006).

## 5. Separate analyticity

A motivation for Siciak's studies was Hartogs' theorem on separate analyticity (1906:12). Terada (1967) weakened its hypotheses, using Chebyshev polynomials in the proof.

Siciak considered sets in the form of a cross, $X = (\Omega_1 \times K_2) \cup (K_1 \times \Omega_2)$, where $K_j$ is a compact set in a domain of holomorphy (or a Stein manifold) $\Omega_j$, $j = 1, 2$, and he established the existence of holomorphic extensions of separately analytic functions defined on such sets. The conclusion was that every separately analytic function on $X$ can be extended to a holomorphic function in

$$\Omega = \big\{(z, w) \in \Omega_1 \times \Omega_2;\ U^*_{K_1,\Omega_1}(z) + U^*_{K_2,\Omega_2}(w) < 1\big\}.$$

Actually Siciak proved some special cases of that result in his paper (1969), whereas Zakharyuta (1976:64) proved the more general result just quoted, assuming a certain regularity of the $K_j$.

This is just one of several generalizations of Hartogs' theorem on separately analytic functions. Siciak returned to the subject in (1981).

In subsequent studies, the extremal function $V_E$ as well as the relative extremal function have played important roles in the proofs of generalizations of this theorem of Hartogs; see, e.g., Siciak (1969), Nguyen Thanh Van and Zeriahi (1983), Shiffman (1989) and Nguyên (2008). The last-mentioned paper contains new results as well as a careful study of the history of the subject.

Using Siciak's methods, Ozan Öktem (1998) proved a new result to which he was led on the basis of his work on the Radon transformation. In this result, as well as in (1999), he allows singularities in the given function as well as in the extended function. Theorems of this kind have been proved recently by Jarnicki and Pflug; see (2007) and several of their earlier papers.

## 6. Kolmogorov's entropy and width

To single out an element in a finite set $C$, we need $\lceil \log_2 \operatorname{card}(C) \rceil$ bits of information. If $C$ is an infinite subset of a metric space $X$, we specify instead an element within a distance $\varepsilon > 0$: we cover $C$ by finitely many sets $C_j$, each of diameter at most $\varepsilon$ and denote the smallest cardinality of such a covering by $N_\varepsilon(C, X)$. Following Kolmogorov and Tihomirov (1959), we define the *$\varepsilon$-entropy of $C$ in $X$* as

$$H_\varepsilon(C, X) = \log N_\varepsilon(C, X).$$

(We use the natural logarithm rather than the 2-logarithm here.) The question is now how this number depends on $\varepsilon$.

Given a normed space $X$ and a subset $A$, *Kolmogorov's width of $A$ relative to $X$* is the sequence of numbers $(d_s(A,X))_{s\in\mathbb{N}}$ defined by

$$d_s(A,X) = \inf_{L\in\mathscr{L}_s(X)} \sup_{x\in A} \inf_{y\in L} \|y-x\|, \qquad s\in\mathbb{N}.$$

Here $\mathscr{L}_s(X)$ is the family of all vector subspaces of $X$ of dimension $s$.

We shall use this definition writing $H^\infty(D)$ for the space of all bounded holomorphic function in a domain $D$ in $\mathbb{C}^n$ with the supremum norm. We let $K$ be a compact subset of $D$, and $A$ the set of restrictions to $K$ of functions in the unit ball of $H^\infty(D)$, i.e.,

$$A = \mathscr{A}_K^D = \left\{ f\big|_K;\ f\in H^\infty(D),\ \|f\|_\infty \leqslant 1\right\}.$$

Moreover $X$ shall be the Banach space $AC(K)$, the closure of $\mathscr{O}(K)$ in $C(K)$.

The width and the entropy are a kind of inverses to each other: the asymptotic relation

$$-\log d_s\big(\mathscr{A}_K^D, AC(K)\big) = \big(\sigma+o(1)\big)s^{1/n}, \qquad s\to+\infty,$$

is equivalent to

$$H_\varepsilon\big(\mathscr{A}_K^D, AC(K)\big) = \big(\tau+o(1)\big)(-\log\varepsilon)^{n+1}, \qquad \varepsilon\to 0,$$

where $\tau = 2\sigma^{-n}/(n+1)$. Levin and Tihomirov (1968), using results of Mitjagin (1961), proved this fact for the one-dimensional case; Zakharyuta pointed out that their methods can be extended to the case of several variables in his Doctor of Science Thesis presented at Rostov on Don (1984).

## 7. Kolmogorov's question

Andreĭ Nikolaevič Kolmogorov (1903–1987) raised a question in 1955:

*Given an open set $D$ in $\mathbb{C}^n$ and a compact subset $K$ of $D$, does there exist a constant $\tau$ such that*

$$H_\varepsilon\big(\mathscr{A}_K^D, AC(K)\big) = (\tau+o(1))(-\log\varepsilon)^{n+1}, \qquad \varepsilon\to 0$$

*at least if $D$ and $K$ are nice enough? Moreover, for $n=1$, he conjectured that the constant $\tau$ is equal to the Green capacity $C_1(K,D)/(2\pi)$ of the condenser $(K,D)$.*

As we have seen above, the question can equivalently be formulated as follows. *Is it true that*

$$-\log d_s\big(\mathscr{A}_K^D, AC(K)\big) = \big(\sigma+o(1)\big)s^{1/n}, \qquad s\to+\infty \tag{7.1}$$

*for some constant $\sigma$?*

For more than one variable, this question could not yet be formulated in terms of a capacity. Considerably later, a theory of capacities was developed also for several variables, as we shall now try to describe.

## 8. Capacities defined by the relative extremal function

Eric Bedford and Al Taylor defined in their fundamental paper (1982) a capacity

$$C_n(K,\Omega) = \sup_u \left[\int_K (dd^c u)^n;\ u\in PSH(\Omega),\ 0<u<1\right]. \tag{8.1}$$

Here $(dd^c)^n$ is the complex Monge–Ampère operator, which the authors defined for all locally bounded plurisubharmonic functions.

Just as in the case of the global extremal function, the relative extremal function can serve to define a capacity. Bedford (1980a, 1980b) expressed the functional $C_n$ defined in (8.1) in terms of the relative extremal function and proved that actually

$$C_n(K,\Omega) = \int_\Omega (dd^c U^*_{K,\Omega})^n$$

for a compact subset $K$ of $\Omega$, a strongly pseudoconvex domain in a Stein manifold, where $U_{K,\Omega}$ is the relative extremal function defined by (4.1).

Bedford and Taylor proved (1982:32) that the measure $(dd^c U^*_{K,\Omega})^n$ is supported by $K$. Actually $C_n$ plays the role of an inner capacity, so they defined

$$C_n(E,\Omega) = \sup_K \big(C_n(K,\Omega);\ K \text{ is a compact set contained in } E\big) \tag{8.2}$$

for any subset $E$ of $\Omega$, and an outer capacity

$$C^*_n(E,\Omega) = \inf_U \big(C_n(U,\Omega);\ U \text{ is an open set containing } E\big).$$

They proved (1982:23) that $C^*_n$ satisfies Choquet's axioms (3.2); it follows that, for any compact set $K$, $C^*_n(K,\Omega) = C_n(K,\Omega)$ as defined by (8.1). Thus the functional many authors had called a "capacity" was proved to actually be a Choquet capacity. Alexander and Taylor (1984) proved sharp inequalities between the relative capacity $C_n$ of (8.1), (8.2) and the capacity $c$ defined in (3.1). In particular, for a relatively compact subset $E$ of $\Omega$, $C_n(E,\Omega) = 0$ if and only if $c(E) = 0$.

As Zakharyuta's result (4.2) showed, it is natural to think of the sublevel sets $\Omega_\alpha$ and $K_\alpha$ as a kind of interpolation between $K$ and $\Omega$. In particular, if both $\Omega$ and $K$ are convex, one would expect the sublevel sets to be convex, too. This is, however, a highly nontrivial result and was proved by Finnur Lárusson, Patrice Lassère and Ragnar Sigurdsson (1998) using Evgeny Poletsky's theory of holomorphic currents (1993).

These ideas can be developed also on a compact Kähler manifold $M$. One then defines a function to be *quasiplurisubharmonic* with respect to a Kähler form $\omega$ on $M$ if it is upper semicontinuous and $dd^c u + \omega$ is a positive current. This definition depends on the choice of $\omega$; the class will be written $PSH(M,\omega)$.

Many notions from pluripotential theory in strictly pseudoconvex domains prove to be useful on Kähler manifolds. Guedj and Zeriahi (2005) defined the relative extremal function of a Borel subset $E$ of the manifold

$$U_{E,M,\omega}(z) = \sup\,(u(z);\ u \in PSH(M,\omega),\ u \leqslant 0,\ u \leqslant -1 \text{ on } E)$$

and showed that it is related to the Monge–Ampère capacity

$$C_\omega(E,M) = \sup\left[\int_E (\omega + dd^c u)^n;\ u \in PSH(M,\omega),\ 0 \le u \le 1\right],$$

where $n$ is the dimension of $M$, by the formula

$$C^*_\omega(E,M) = \int_M \left(-U^*_{E,M,\omega}\right)\left(\omega + dd^c U^*_{E,M,\omega}\right)^n.$$

This capacity was first introduced by Kołodziej (2003) and corresponds to the relative capacity of Bedford and Taylor (1982). The global extremal function is also defined:

$$V_{E,M,\omega}(z) = \sup\big(u(z);\ u \in PSH(M,\omega),\ u \leqslant 0 \text{ on } E\big).$$

This is an analogue of the Siciak–Zakharyuta extremal function in $\mathbb{C}^n$, and the capacity defined in terms of this function, viz.

$$T_\omega(E) = \exp\bigl(-\sup_M V^*_{E,M,\omega}\bigr)$$

is named after Herbert J. Alexander (1940–1999). Guedj and Zeriahi show that this capacity $T$ can be expressed also with the use of Chebyshev constants related to sections of a positive vector bundle on $M$ with $\omega$ being the curvature form of the given metric. It is proved that the two capacities obey the inequalities of Alexander and Taylor (1984) and this is applied to show that locally pluripolar sets are globally pluripolar on compact Kähler manifolds, thus generalizing Josefson's theorem (1978).

## 9. Zakharyuta's first conjecture

Zakharyuta made Kolmogorov's question more precise by relating it to the notion of capacity also in several variables.

*Given an open set $D$ in $\mathbb{C}^n$ and a compact subset $K$ of $D$, Zakharyuta conjectured that*

$$H_\varepsilon\bigl(\mathscr{A}^D_K, AC(K)\bigr) = (\tau + o(1))(-\log\varepsilon)^{n+1}, \qquad \varepsilon \to 0$$

*for some constant $\tau$. He also conjectured that*

$$\tau = \frac{2C_n(K,D)}{(n+1)!(2\pi)^n},$$

*thus generalizing Kolmogorov's conjecture about the constant from $n = 1$ to arbitrary $n$.*

*Equivalently,*

$$-\log d_s\bigl(\mathscr{A}^D_K, AC(K)\bigr) = \bigl(\sigma + o(1)\bigr)s^{1/n}, \qquad s \to +\infty, \tag{9.1}$$

*for a constant $\sigma$, and*

$$\sigma = \left(\frac{2}{(n+1)\tau}\right)^{1/n} = 2\pi\left(\frac{n!}{C_n(K,D)}\right)^{1/n}.$$

In special cases, the asympotics of $d_s\bigl(\mathscr{A}^D_K, AC(K)\bigr)$ in (9.1) is known, e.g., when the $D$ and $K$ are Reinhardt domains (Aytuna, Rashkovskii and Zakharyuta 2002).

## 10. The pluricomplex Green function with several poles

The classical Green function in a domain in one complex variable is zero on the boundary of the domain and has a logarithmic pole at a given point. Lempert (1981, 1983) introduced an analogous function in a strictly convex domain in several complex variables. It is plurisubharmonic in the domain and has a logarithmic pole at a given point $a \in \Omega$. It solves the homogeneous complex Monge–Ampère equation in $\Omega \smallsetminus \{a\}$ and is therefore a maximal plurisubharmonic function in that open set (Lempert 1981:430).

Zakharyuta in his Doctor of Science Thesis (1984) and independently Klimek (1985) replaced Lempert's construction by a Perron–Bremermann approach: they took the supremum $G_\Omega(z,a)$ of $u(z)$ when $u$ varies in the set of all negative plurisubharmonic functions in $\Omega$ with a logarithmic singularity at a given point $a$, thus with

$u(z) \leqslant \log \|z-a\|$ plus some constant near $a$. This function was defined using analytic disks by Poletsky and Shabat (1986). Demailly (1987) gave several precise results, including the continuity of the function $\exp G_\Omega$ in $\overline{\Omega} \times \Omega$ (1987:534).

More generally, given a plurisubharmonic function $\varphi$ in $\Omega$, Zakharyuta considered in his Doctor of Science Thesis (1984) the supremum $G_{\varphi,\Omega}$ of the family of all functions $u \in PSH(\Omega)$ such that $u \leqslant 0$ in $\Omega$ and such that, near every point $a \in \Omega$ with $\varphi(a) = -\infty$, we have

$$u(z) \leqslant \varphi(z) + \text{some constant.}$$

In this definition, the *polar set of* $\varphi$, i.e., the set

$$P(\varphi) = \{z \in \Omega;\ \varphi(z) = -\infty\},$$

may be large, for example $\varphi$ may have several poles; in particular any finite number of logarithmic poles.

Zakharyuta assumed $\varphi$ to be maximal outside its polar set and the real-valued function $e^\varphi$ to be continuous, and he proved that the Green function $G_{\varphi,\Omega}$ in a hyperconvex Stein manifold $\Omega$ and with any prescribed finite set of singularities is a maximal plurisubharmonic function in $\Omega \smallsetminus P(\varphi)$ and that, near any point $a \in P(\varphi)$, $G_{\varphi,\Omega}(z) \leqslant \varphi(z) +$ some constant. In a paper on relative types of plurisubharmonic functions, Rashkovskii (2006) removed the hypothesis of continuity made by Zakharyuta in (1984).

Also Lelong (1987, 1989) studied these functions—as I believe independently of Zakharyuta.

The pluricomplex Green function has since then been generalized to other situations, e.g., by Lárusson and Sigurdsson (1999) and by Rashkovskii and Sigurdsson (2005a, 2005b) to functions with singularities along a closed analytic subspace.

## 11. Zakharyuta's second conjecture

Zakharyuta reduced his first rather abstract conjecture to a more concrete question concerning the new class of pluricomplex Green functions:

*Given a compact holomorphically convex subset $K$ of a pseudoconvex domain $D$ in $\mathbb{C}^n$, the relative extremal function $U^*_{K,D}$ can be uniformly approximated on any compact subset of $D \smallsetminus K$ by pluricomplex Green functions on $D$ with logarithmic poles contained in $K$.*

Zakharyuta proved that a positive answer to the second conjecture would imply an answer in the affirmative to his first conjecture.

Stéphanie Nivoche (2001, 2004) and Poletsky (2003) proved Zakharyuta's second conjecture. Their proofs were based on ideas that they had developed in cooperation. Thus the first conjecture as well as Kolmogorov's question now have affirmative answers.

In a recent manuscript (2007), Vyacheslav Zakharyuta reviews all these questions and proves new results. He also gives some examples where (9.1) cannot hold with the constant $\sigma$ mentioned, but might be true with some larger constant.

It is with sincere admiration that I conclude this short account of a marvelous research effort.

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