Questions inspired by Mikael Passare's mathematics

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Abstract

Mikael Passare (1959–2011) was a brilliant mathematician. His PhD thesis from 1984 was a breakthrough in the theory of residues in several complex variables. Some time before 1998 he started to work on amoebas and coamoebas. In discussions with him during the last thirty years many questions have emerged—not all of them were resolved at the time of his premature death. The purpose of the paper is to save from oblivion some of the mathematical ideas of Mikael Passare.

In the article some of these unanswered questions are presented, always preceded by a discussion leading up to the question. Some of the questions might present challenges to his nine former PhD students, to his many collaborators around the globe—and to anybody interested.

Is there an associative algebra of residue currents and principal-value distributions? Is there an interesting non-associative algebra of such currents? Meromorphic extension using two parameters often leads to points of indeterminacy—what is the natural choice at such points?

Several questions have bearing on tropical mathematics. Is it possible to build an axiomatic theory for tropical geometry? There are also questions on tropical polynomials as limits of classical polynomials. Can the absolute values of the coefficients of a polynomial be retrieved from its growth function? Some questions are concerned with digital convexity. Finally, there is a question on the constant term in powers of a Laurent polynomial.

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1. Introduction

Mikael Passare (1959–2011) was a brilliant mathematician. He started his studies at Uppsala University in the fall of 1976 while still a high-school student, merely seventeen and a half. He finished high school in June 1978, gave his first seminar talk in November 1978, got his Bachelor Degree in 1979, and presented his PhD thesis at Uppsala University on December 15, 1984. He was appointed a professor at Stockholm University in 1994, on the chair which was created for Sonja Kovalevsky (1850–1891) and held during seven years, 1957–1964, by his mathematical grandfather Lars Hörmander.

Mikael was deeply involved in the development of mathematics in Africa: he was a member of the Board of the International Science Programme (ISP), Uppsala, and a member of the Board of the Pan-African Centre for Mathematics (PACM) in Dar es-Salaam, Tanzania. He was a driving force in the creation of this Pan-African Centre, which is a collaborative project between Stockholm University and the University of Dar es-Salaam, and was actively searching for a director of PACM.

Mikael Passare died from a sudden cardiac arrest in Oman on September 15, 2011. He is buried not far from Sonja's grave.

I was Mikael's advisor when he was a PhD student. During the last thirty years we discussed mathematics. The purpose of the present paper is to present some of the questions I have raised during that period.

In some cases the questions just reflect my ignorance. In other cases they might represent a challenge. I address them now to Mikael's nine PhD students and his collaborators around the globe—and to anybody interested—in the hope of starting a dialogue. I will be grateful to receive any corrections, comments—or answers!

The next two sections are devoted to questions in complex analysis: the non-associativity of multiplication of principal-value distributions and residue currents, followed by a section on constructions using meromorphic extension (questions from the early 1980s and up to 1988). The last six sections are related to Mikael's more recent interest: amoebas and tropical geometry (questions from the period 2003–2010). Tropical geometry has intriguing connections to digital geometry, mathematical morphology, discrete optimization, and crystallography, including the theory of quasicrystals. I believe these connections could be further developed—I hope they will.

Section 8 was written by Timur Sadykov and contains a conjecture formulated by Mikael in December 2010. I am grateful to Timur for permitting me to include this text. Mounir Nisse and Jens Forsgård provided the most recent information on this conjecture.

Section 9 about the constant term in powers of a Laurent polynomial was added by Alain Yger, to whom I am grateful for this contribution.

2. Multiplication of residue currents and principal-value distributions

Let f and g be holomorphic functions of n complex variables. The *principal value* $\mathrm{PV}(f/g)$ of the meromorphic function f/g is a distribution defined by the formula

$$\left\langle \mathrm{PV}\left(\frac{f}{g}\right), \varphi \right\rangle = \lim_{\varepsilon\to 0} \int_{|g|>\varepsilon} \frac{f\varphi}{g} = \lim_{\varepsilon\to 0} \int \frac{\chi f \varphi}{g}, \qquad \varphi \in \mathscr{D}(\mathbf{C}^n),$$

where $\chi = \chi(|g|/\varepsilon)$ and χ is a smooth function on the real axis satisfying $0 \leqslant \chi \leqslant 1$ and $\chi(t) = 0$ for $t \leqslant 1$, $\chi(t) = 1$ for $t \geqslant 2$ (in Passare (1985:727) when $f = 1$ and in (1988:39) in general).

The *residue current* is $\bar\partial\,\mathrm{PV}(f/g)$. Can the products

$$(\mathrm{PV}(f_1/g_1))(\mathrm{PV}(f_2/g_2)), \quad \left(\bar\partial(\mathrm{PV}(f_1/g_1))\right)(\mathrm{PV}(f_2/g_2))$$

and other similar products be defined?

Schwartz proved (1954) that it is in general impossible to multiply two distributions while keeping the associative law. He indicated three distributions $u, v, w \in \mathscr{D}'(\mathbf{R})$ where uv, vw, $(uv)w$ and $u(vw)$ all have a good meaning but where $(uv)w \neq u(vw)$. He took $u = \mathrm{PV}(1/x)$, the principal value of $1/x$; v as the identity, i.e., the smooth function $v(x) = x$, which can be multiplied to any distribution; and $w = \delta$, the Dirac measure placed at the origin. Then we have $uv = 1$, $(uv)w = \delta$, while $vw = 0$, $u(vw) = 0$. Hence there is no associative multiplication in $\mathscr{D}'(\mathbf{R})$. An easy modification proves the same result for $\mathscr{E}'(\mathbf{R})$, the distributions of compact support.

Remark 2.1. There is no need to know distribution theory to encounter a similar non-associativity. Let us consider the convolution product of functions defined on the integers:

$$(f * g)(x) = \sum_{y\in\mathbf{Z}} f(x-y)g(y), \qquad x \in \mathbf{Z},$$

which is well defined if f or g is nonzero only in a finite set.

Take now $f(x) = \mathrm{sgn}(x)$, the sign function, which takes the value -1 for $x \leqslant -1$, 0 for $x = 0$, and 1 for $x \geqslant 1$; $g = \delta_0 - \delta_1$ (a difference operator); and $h(x) = 1$ for all $x \in \mathbf{Z}$. Then $f * g = \delta_0 + \delta_1$, $(f * g) * h = 2$, while $(g * h) = 0$, $f * (g * h) = 0$.

We see that f corresponds to $\mathrm{PV}(1/x)$; g to the identity $\mathbf{R} \ni x \mapsto x \in \mathbf{R}$; and h to the Dirac measure on $\mathbf{R}$. The two examples are actually the same—via the Fourier transformation. □

Mikael's construction of residue currents and principal-value distributions goes as follows. Take $f = (f_1, \ldots, f_{p+q})$, $g = (g_1, \ldots, g_{p+q})$, two $(p+q)$-tuples of holomorphic functions, and consider the limit

$$\lim_{\varepsilon_j \to 0} \frac{f_1}{g_1} \cdots \frac{f_{p+q}}{g_{p+q}} \bar{\partial}\chi_1 \wedge \cdots \wedge \bar{\partial}\chi_p \cdot \chi_{p+1} \cdots \chi_{p+q}, \tag{2.1}$$

where $\chi_j = \chi(|g_j|/\varepsilon_j)$ and the ε_j tend to zero in some way.

Coleff and Herrera (1978:35–36) took $q = 0$ or 1 and assumed that ε_j tends to zero much faster than ε_{j+1}, which in this context means that $\varepsilon_j/\varepsilon_{j+1}^m \to 0$ for all $m \in \mathbf{N}$ and $j = 1, \ldots, p+q-1$; thus it is almost an iterated limit. This gives rise to the strange situation that the limit depends in general on the order of the functions (and is not just an alternating product). However, in the case of complete intersection, the construction is satisfactory, and Mikael's construction then gives the same results as that of Coleff and Herrera, but Mikael's construction (1988) is valid also when we do not have a complete intersection.

Mikael took $\varepsilon_j = \varepsilon^{s_j}$ for fixed $s_1, \ldots, s_{p+q}$. The limit, which will be written as $R^pP^q[f/g](s)$, where we now write $[\ldots]$ for the principal value, does not exist for arbitrary s_j. But he proved (in (1985:728) when the $f_j = 1$ and in (1988:40) in general) that, if we remove finitely many hyperplanes, then $R^pP^q[f/g](s)$ is locally constant in a finite subdivision of the simplex

$$\Sigma = \left\{ s \in \mathbf{R}^{p+q};\ s_j > 0,\ \ \textstyle\sum s_j = 1 \right\},$$

so that the mean value

$$R^pP^q\left[\frac{f}{g}\right] = \fint_{\Sigma} R^pP^q\left[\frac{f}{g}\right](s) = \bar{\partial}\left[\frac{f_1}{g_1}\right] \wedge \cdots \wedge \bar{\partial}\left[\frac{f_p}{g_p}\right] \cdot \left[\frac{f_{p+1}}{g_{p+1}}\right] \cdots \left[\frac{f_{p+q}}{g_{p+q}}\right] \tag{2.2}$$

exists (Definition A in Passare (1987:159)). This defines the product of p residue currents and q principal-value distributions.

In the little paper (1993), based on his talk when accepting the Lilly and Sven Thuréus Prize in 1991, he discusses the possibility of defining the product $\mathrm{PV}(1/x)\delta$ on the real axis, and finds that it should be $-\frac{1}{2}\delta'$, which is the mean value of $-\delta'$ and zero. This is an analogue in real analysis to the mean value over Σ which he considered in the complex case.

Leibniz' rule for the derivative of a product and some other rules of calculus hold; for example we have (1988:43):

$$\left[\frac{1}{z_1}\right]\left[\frac{z_1}{z_2}\right] = \left[\frac{1}{z_2}\right],$$

which yields

$$\left(\bar{\partial}\left[\frac{1}{z_1}\right]\right)\left\{\left[\frac{1}{z_1}\right]\left[\frac{z_1}{z_2}\right]\right\} = \left(\bar{\partial}\left[\frac{1}{z_1}\right]\right)\left[\frac{1}{z_2}\right] \neq 0,$$

while

$$\left\{\left(\bar{\partial}\left[\frac{1}{z_1}\right]\right)\left[\frac{1}{z_1}\right]\right\}\left[\frac{z_1}{z_2}\right] = \frac{1}{2}z_1\left(\bar{\partial}\left[\frac{1}{z_1^2}\right]\right)\left[\frac{1}{z_2}\right] = \frac{1}{2}\left(\bar{\partial}\left[\frac{1}{z_1}\right]\right)\left[\frac{1}{z_2}\right].$$

Thus the associative law does not hold. It is natural to ask whether these currents are just as bad as the general distributions when it comes to multiplication, or whether there is a subclass of them with nicer properties.

Question 2.2. *We saw in Schwartz' example that an associative multiplication is impossible in general; the last example shown here makes us wonder whether it is possible to define an associative multiplication for some residue currents and principal-value distributions. Is there an associative algebra of residue currents and principal-value distributions? Is there an interesting non-associative algebra of such currents?*

3. Constructions using meromorphic extension

3.1. Extending a given meromorphic function

The Riemann ζ-function is a classical example of meromorphic extension: the series

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s},$$

which converges for $s \in \mathbf{C}$, $\operatorname{Re} s > 1$, is extended to a meromorphic function in the whole plane.

The well-known formula

$$\sum_0^\infty \alpha^j = \frac{1}{1-\alpha}, \qquad |\alpha| < 1,$$

can be used to define more or less funny results like

$$\sum_0^\infty (-1)^j = \frac{1}{2}; \quad \sum_0^\infty 2^j = -1; \quad \sum_0^\infty (-2)^j = \frac{1}{3}; \quad \sum_0^\infty 3^j = -\frac{1}{2}; \quad \sum_0^\infty (-3)^j = \frac{1}{4}.$$

This is based on the observation that $1/(1-\alpha)$ is a meromorphic function with a single pole at $\alpha = 1$ but otherwise regular. But why should $\sum \alpha^j$ be meromorphic?

The construction of homogeneous distributions in Hörmander (1990: Section 3.2), in particular of the distributions x_+^a on the real axis, is done by meromorphic extension.

In the three examples mentioned, we have a given meromorphic function in a nonempty open set of the complex plane, and we know that, if it has a meromorphic extension to the whole plane, then the extension is unique. The situation is different when we want to construct an object and have to choose a meromorphic function.

3.2. Finding a meromorphic function

Michael Atiyah proved (1970) that if F is a nonnegative real-analytic function, then $\lambda \mapsto F^\lambda$ can be extended to a meromorphic function in all of $\mathbf{C}$. Bernšteĭn and Gel′fand (1969) proved a similar result for polynomials. Using Atiyah's result, Alain Yger (1987) defined residue currents as meromorphic extensions of $(f\bar{f})^\lambda$ for a holomorphic f, and Mikael compared them (Definition B in (1987:159)) with his own construction (Definition A already mentioned in Section 2, equation (2.2)).

In this case the authors want to obtain $(f\bar{f})^\lambda$ for just one value of λ, viz. $\lambda = -1$, which means that they have to choose a parametrized family; the choice is not obvious.

Another kind of limit of a meromorphic function is

$$\lim_{\varepsilon_j \to 0} \bar{\partial}\frac{\bar{f}_1}{|f_1|^2 + \varepsilon_1} \wedge \cdots \wedge \bar{\partial}\frac{\bar{f}_p}{|f_p|^2 + \varepsilon_p},$$

which is obtained by taking $\chi_j(t) = t/(t+1)$ in (2.1) (the case $q = 0$). It yields the same current as the former construction for complete intersections (Björk & Samuelsson (2010:35); cf. earlier results by Samuelsson (2006: Corollary 26), who may have been inspired to consider averaging from a paper on defining residues of a complete intersection by Passare & Tsikh (1996)).

If we want to evaluate a divergent integral, for instance

$$\int_0^1 x^{-2} dx, \tag{3.1}$$

one method is to embed the integrand into a family of functions depending on a parameter and define the integral as the value of an extension in the parameter space. In the case mentioned, we can define $f(x,\alpha) = x^\alpha$, and since

$$\Phi(\alpha) = \int_0^1 f(x,\alpha)\, dx = \int_0^1 x^\alpha\, dx = \frac{1}{\alpha+1}$$

is well-defined for $\operatorname{Re}\alpha > -1$ and has a meromorphic extension to the whole complex plane, we can define the integral of x^{-2} as $\Phi(-2) = -1$. The question is now: will we get a different answer if we use a different meromorphic function?

It can be remarked that $\Phi(-2)$ is also the finite part of the integral (3.1) in view of the formula

$$\int_\varepsilon^1 x^{-2}\, dx = \frac{1}{\varepsilon} - 1, \quad 0 < \varepsilon < 1,$$

where $1/\varepsilon$ is the infinite part (to be thrown away) and -1 is the finite part (to be kept).

We may conclude that meromorphic extension is an often used method to construct mathematical objects.

3.3. Two-parameter families

While meromorphic functions of one variable can be assigned the value ∞ at a pole, and therefore can be defined as good mappings with values in the Riemann sphere $\mathbf{C} \cup \{\infty\}$, meromorphic functions of two variables may have points of indeterminacy: the function $f(z_1, z_2) = z_2/z_1$ can be assigned the value ∞ at a point $(0, z_2) \neq (0,0)$, but at the origin we cannot do so. This explains the trouble we are in for.

I will consider a divergent integral where we can get different values for different choices of parametrized families.

We denote by $D(c,r)$ the open disk with center at c and with radius r:

$$D(c,r) = \{z \in \mathbf{C};\ |z-c| < r\}.$$

Let us consider the divergent integral

$$\int_{D(1,1)} z^{-2} d\lambda(z). \tag{3.2}$$

A simple idea is to vary the disk. We have

$$\int_{D(c,r)} z^{-2} d\lambda(z) = \pi r^2/c^2 \text{ when } r < |c|,$$

i.e., when the origin is not in the closure of $D(c,r)$. Hence the limit of these values is π as $(c,r) \in \mathbf{C} \times \mathbf{R}$ tends to $(1,1)$ under the restriction $r < |c|$. So this is one possible method.

Another idea is to remove a small disk around the origin, like in the definition of the principal value:

$$\mathrm{PV} \int_{D(1,1)} z^{-2}\, d\lambda(z) = \lim_{\varepsilon \to 0} \int_{z \in D(1,1),\, |z|>\varepsilon} z^{-2}\, d\lambda(z) = 2 \int_0^{\pi/2} \cos 2\theta \log \cos\theta \, d\theta = \tfrac{1}{2}\pi.$$

The last integral is evaluated in Gradšteĭn & Ryžik (1962:598:4.384.7).

Are there other ways to approach the divergent integral? Let us look at a two-parameter family.

Lemma 3.1. *For* $(\alpha, \beta) \in \mathbf{C}^2$ *with* $\mathrm{Re}\,(\alpha+\beta) > -2$ *we have*

$$F(\alpha,\beta) = \int_{D(1,1)} z^{\alpha} \bar{z}^{\beta}\, d\lambda(z) = \frac{\pi \Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+2)}. \tag{3.3}$$

Proof. This follows from Gradšteĭn & Ryžik (1962:490:3.892.2). □

So the extended values of the integral (3.3) define a meromorphic function F in all of $\mathbf{C}^2$, with singularities, e.g., on the hyperplane $\alpha+\beta = -2$. But restrictions of F may be free from singularities: the function $\beta \mapsto F(m,\beta)$ is entire (in fact a polynomial of degreee m) for every $m \in \mathbf{N}$. At the point $(\alpha,\beta) = (-1,-1)$ we can assign the value ∞ to F if we like, but the point $(\alpha,\beta) = (-2,0)$ is a point of indeterminacy.

We take $(\alpha,\beta) = (-2+\varepsilon, \theta-\varepsilon)$ and consider

$$F(\alpha,\beta) = F(-2+\varepsilon, \theta-\varepsilon) = \frac{\pi\varepsilon}{\theta} \frac{\Gamma(1+\theta)}{\Gamma(1+\varepsilon)\Gamma(2+\theta-\varepsilon)} = \frac{\pi\varepsilon}{\theta}(1+o(1))$$

as $(\theta,\varepsilon) \to (0,0)$ under the restriction $\mathrm{Re}\,\theta > 0$. The quantity $\pi\varepsilon/\theta$ has no limit as $(\varepsilon,\theta) \to (0,0)$, but we may introduce a relation between ε and θ to create a one-parameter family of functions which has a limit, e.g., $\varepsilon = \theta$ or $\varepsilon = 0$.

If we take $\varepsilon = \theta$, we obtain

$$G(\theta) = F(-2+\theta, 0) = \int_{D(1,1)} z^{-2+\theta}\, d\lambda = \pi, \quad \operatorname{Re}\theta > 0.$$

If we take $\varepsilon = 0$, we get

$$H(\theta) = F(-2, \theta) = \int_{D(1,1)} z^{-2}|z|^{\theta}\, d\lambda = 0, \quad \operatorname{Re}\theta > 0.$$

More generally, we may take $\varepsilon = c\theta$ and get the limit $c\pi$ for certain values of c, or $\theta = \varepsilon^2$ and get infinity.

Inspired by the work of Mikael and others I asked a question:

Question 3.2. *Meromorphic extension using two parameters easily leads to points of indeterminacy, and so gives rise to infinitely many one-parameter families. Sometimes we will have to accept all limits that can appear as solutions to a certain problem; sometimes, as we have seen, mathematicians do make a choice. And which are then the criteria for a choice?*

The text in this section was essentially written on 1988-10-17 and sent out to some people, among them Mikael Passare and Bo Berndtsson. Bo expressed surprise.

4. The axioms of tropical geometry

4.1. Tropicalization

Roughly speaking, *tropical mathematics* is the mathematics of a structure with addition and maximum as binary operations. The simplest example is the semiring of real numbers $(\mathbf{R}, +, \vee)$, where $+$ denotes usual addition and $\vee$ is the maximum operation, $x \vee y = \max(x, y)$. Note the distributive law $a + (b \vee c) = (a+b) \vee (a+c)$: addition is distributive over maximum. Sometimes $\mathbf{R}$ is augmented by adding an element $-\infty$, the neutral element for the maximum operation: $x \vee (-\infty) = x$. Another name is *idempotent mathematics*, used because of the idempotency of the maximum operation: $x \vee x = x$.

Tropicalization means that in a semiring with multiplication and addition we replace multiplication by addition, and addition by the maximum operation. This is somewhat reminiscent of taking the logarithm. Start with the semiring $(\mathbf{R}_+, \times, +)$, where $\mathbf{R}_+ = \{x \in \mathbf{R}; x > 0\}$ is the set of positive real numbers, and take the logarithm. Then

$$\log(x \times y) = \log x + \log y, \text{ and}$$

$$(\log x) \vee (\log y) \leqslant \log(x+y) \leqslant \log 2 + ((\log x) \vee (\log y)),$$

so that multiplication gives addition of the logarithms, and addition comes close to the maximum of the logarithms—a good approximation if $x \gg 1$ or $y \gg 1$.

If we introduce $s = \log x$ and $t = \log y$, and make a change of scale, we see that

$$\log(e^s \times e^t) = s + t.$$

In the limit,

$$h \log(e^{s/h} \times e^{t/h}) = s + t \to s + t \quad \text{and} \quad h \log(e^{s/h} + e^{t/h}) \to s \vee t \text{ as } h \to 0+.$$

Thus we may say that tropicalization is a limiting case of logarithmization. Here $h > 0$ is an analogue of Planck's constant—hence the name *dequantization* (Litvinov (2005, 2007); Viro (2001: Section 2.1)).

For basic concepts of tropical geometry, see also Viro (2010, 2011). The book by Itenberg et al. (2007) presents fundamental ideas and key results in tropical algebraic geometry.

4.2. Tropical straight lines

A polynomial function of degree one has the form

$$(x, y) \mapsto f(x, y) = ax + by + c.$$

If we tropicalize it, we get

$$f_{\text{trop}}(x, y) = (a + x) \vee (b + y) \vee c.$$

Thus f_{trop} is a convex function with a simple structure. It is piecewise affine outside the three lines $x + a = c$, $y + b = c$, and $x + a = y + b$, which meet at the point $p = (p_1, p_2) = (c - a, c - b)$. More precisely, it is affine outside the three rays emanating from that point in the directions $(-1, 0)$, $(0, -1)$ and $(1, 1)$. We shall call this the *tropical straight line* $\mathrm{TSL}(p)$ with *vertex* p.

A real line is the set of zeros of such a function f, but if we replace the coefficients (a, b, c) by $(\lambda a, \lambda b, \lambda c)$ we get the same line for all real $\lambda \neq 0$. This implies by analogy that if we replace the coefficients (a, b, c) in f_{trop} by $(a+\lambda, b+\lambda, c+\lambda)$, we should also get the same line, in other words, the functions $(x, y) \mapsto (a+\lambda+x) \vee (b+\lambda+y) \vee (c+\lambda)$ should define the same line as f_{trop} for all real λ.

So what is independent of λ? It is the three rays going out from the point $(c - a, c - b)$, which is also the support of the distribution Δf_{trop}, the Laplacian of f_{trop}. It is often called the *corner locus*, e.g., by Izhakian (2009:1447), or *tropical zero locus*, e.g., by Grigg & Manwaring (2007:7). So it is justified to say that a tropical straight line *is* these three rays, which are parametrized by the point $p = (c-a, c-b)$. Given any point $p = (p_1, p_2)$, we define $f_{\text{trop}}(x, y) = (x - p_1 + \lambda) \vee (y - p_2 + \lambda) \vee \lambda$ for any constant λ, which is affine outside the union of the three rays constituting $\mathrm{TSL}(p)$.

We see that in general two different tropical lines intersect in a single point. An exception occurs when the lines have their vertices at points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ with $p_1 = q_1$ or $p_2 = q_2$ or $p_2 - p_1 = q_2 - q_1$, i.e., when $q \in \mathrm{TSL}(p)$ or $p \in \mathrm{TSL}(q)$. Then there are infinitely many points in the intersection, but Mikael explained that we should require stability under small perturbations, which means that we should define the intersection as the limit of the unique intersection of the lines $\mathrm{TSL}(p)$ and $\mathrm{TSL}(q + (\delta, \varepsilon))$ as $(\delta, \varepsilon) \to (0, 0)$ while avoiding the exceptional values (cf. Richter-Gebert et al. (2005: Theorem 4.3) and Tabera (2008: Definition 4)).

In this way two distinct tropical straight lines always have a unique intersection, just like in spherical geometry. And, similarly, two distinct points always define a tropical straight line, except in certain cases, where again we use stability to impose uniqueness.

Question 4.1. *So we may ask about all of Euclid's axioms! Is it possible to build up an axiomatic theory for tropical geometry? What are the similarities with spherical geometry?*

I asked Mikael these questions a few years ago.

5. Tropical functions

5.1. Largest tropical minorants

A tropical power series function of one real variable is of the form

$$f(x) = \sup_{j\in\mathbf{Z}}(a_j + jx), \qquad x \in \mathbf{R}.$$

It is typically a piecewise affine function, where each piece has integer slope.

To any function $G\colon \mathbf{R} \to \mathbf{R}$ we associate its largest tropical minorant f. The example

$$G(x) = 0 \vee (a + \tfrac{1}{2}x) \vee x, \qquad x \in \mathbf{R},$$

shows that the difference $G - f$ can be arbitrarily large even if G is convex: in this case we have $f(x) = 0 \vee x$, so that $G(0) - f(0) = 0 \vee a$.

On the other hand, if

$$G(x) = \sup_{|z|=e^x} \log|h(z)|, \qquad x \in \mathbf{R}, \tag{5.1}$$

for some holomorphic function h, I proved (1984:168) that

$$f \leqslant G \leqslant f + \log 3.$$

Question 5.1. *Which is the smallest constant c such that $f \leqslant G \leqslant f + c$ if G is of the form* (5.1) *for some polynomial h?*

We know that $c \leqslant \log 3 \approx 1.09861$. The example $h(z) = 1 + z$ shows that $c \geqslant \log 2 \approx 0.69315$: in this case $G(0) = \log 2$ and $f(0) = 0$.

I sent this question to Mikael Passare, Jan Boman, David Jacquet, Hans Rullgård, Erik Melin, and Markku Ekonen in a letter of 2003-10-21.

5.2. Approximation of the exponential function

The Fenchel transformation is a tropical analogue of the Fourier transformation or the Laplace transformation.

The *Fenchel transform* $\tilde{f}$ of a function $f\colon \mathbf{R} \to [-\infty, +\infty]$ is defined by

$$\tilde{f}(\xi) = \sup_{x\in\mathbf{R}} \Big(\xi x - f(x)\Big), \qquad \xi \in \mathbf{R}.$$

We have $\tilde{\tilde{f}} \leqslant f$ with equality if and only if f is convex, lower semicontinuous, and takes the value $-\infty$ only if it is identically $-\infty$. The equality $f = \tilde{\tilde{f}}$ means that f is represented as a supremum of affine functions—a tropical integral of the simplest convex functions—just as the Fourier inversion formula $f = \mathscr{F}^{-1}(\mathscr{F}(f))$ represents f as an integral of the simplest oscillations.

Now take $f(x) = e^x$, $x \in \mathbf{R}$. The Fenchel transform of this function is

$$\tilde{f}(\xi) = \begin{cases} +\infty, & \xi < 0, \\ 0, & \xi = 0, \\ \xi \log \xi - \xi, & \xi > 0. \end{cases}$$

Then, for $\xi > 0$,

$$e^{\tilde{f}(\xi)} = \xi^\xi e^{-\xi} = \sup_{x\in\mathbf{R}} e^{\xi x - e^x} = \sup_{y>0} y^\xi e^{-y} \approx \int_0^\infty y^\xi e^{-y} dy = \Gamma(\xi + 1),$$

for the integral defining the Gamma function is approximately equal to a tropical integral, i.e., to a supremum. This is a crude form of Stirling's formula.

Furthermore, since $\tilde{\tilde{f}} = f$,

$$e^{e^x} = e^{\tilde{\tilde{f}}(x)} = \sup_{\xi>0} e^{\xi x - \tilde{f}(\xi)} = \sup_{\xi>0} \frac{e^{\xi x}}{e^{\tilde{f}(\xi)}} \approx \sup_{\xi>0} \frac{e^{\xi x}}{\Gamma(\xi+1)} \approx \sum_{k\in\mathbf{N}} \frac{e^{kx}}{k!} = e^{e^x},$$

where we have used the former approximation that $e^{\tilde{f}(\xi)} \approx \Gamma(\xi + 1)$ and a new tropical approximation: the sum defining the exponential function is approximately equal to a tropical sum, i.e., to a supremum.

Question 5.2. *We have thus showed that e^{e^x} is approximatively equal to e^{e^x}. This is not so remarkable. But the surprising fact is that it is an exact equality. After two approximations we return to the exact value. Is there an explanation? To be precise: is there a more direct explanation why*

$$\sup_{\xi>0} \frac{e^{\xi x}}{e^{\tilde{f}(\xi)}} = \sum_{k=0}^{\infty} \frac{e^{kx}}{k!}, \qquad x \in \mathbf{R}?$$

Are there other, similar examples?

I sent this question to Mikael Passare, Jan Boman, David Jacquet, Hans Rullgård, Erik Melin, and Markku Ekonen on 2003-10-21.

5.3. Representation of tropical functions by tropical power series

Lemma 5.3. *A tropical function of the form*

$$f(x) = \sup_{\alpha \in \mathbf{Z}^n} (x \cdot \alpha + a_\alpha), \qquad x \in \mathbf{R}^n, \tag{5.2}$$

with coefficients $a_\alpha \in [-\infty, +\infty]$ can sometimes be represented in more than one way by tropical power series, i.e., by using different choices for the coefficients a_α (see Section 6). The representation

$$f(x) = \sup_{\alpha \in \mathbf{Z}^n} (x \cdot \alpha + b_\alpha), \qquad x \in \mathbf{R}^n,$$

with $b_\alpha = -\tilde{f}(\alpha)$, $\alpha \in \mathbf{Z}^n$, is the one with the largest coefficients.

Proof. Let (5.2) be any representation of f and define $\varphi(\alpha) = -a_\alpha$ for $\alpha \in \mathbf{Z}^n$; $\varphi(\xi) = +\infty$ for $\xi \in \mathbf{R}^n \smallsetminus \mathbf{Z}^n$. Then $f = \tilde{\varphi}$. This implies

$$\tilde{f} = \tilde{\tilde{\varphi}} \text{ and } \tilde{\tilde{f}} = \tilde{\tilde{\tilde{\varphi}}} = \tilde{\varphi} = f.$$

(The third Fenchel transform is always equal to the first.) Moreover, it is clear that, in the definition of the second Fenchel transform of f, it is enough to take $\xi \in \mathbf{Z}^n$:

$$f(x) = \tilde{\tilde{f}}(x) = \sup_{\xi \in \mathbf{R}^n} \Big(x \cdot \xi - \tilde{f}(\xi) \Big) = \sup_{\alpha \in \mathbf{Z}^n} \Big(x \cdot \alpha - \tilde{f}(\alpha) \Big), \qquad x \in \mathbf{R}^n.$$

Now define $b_\alpha = -\tilde{f}(\alpha)$, $\alpha \in \mathbf{Z}^n$. Then $b_\alpha = -\tilde{f}(\alpha) = -\tilde{\tilde{\varphi}}(\alpha) \geqslant -\varphi(\alpha) = a_\alpha$. (Cf. Grigg & Manwaring (2007: Lemma 3.3).) ☐

5.4. The exponential of a tropical polynomial function

Let $\varphi\colon \mathbf{Z}^n \to [-\infty, +\infty]$ be a function on the integer points which is $< +\infty$ only at finitely many of them. (This can be expressed by saying that $\exp(-\varphi)$ has finite support.) We may also take φ defined on $\mathbf{R}^n$ and with the value $+\infty$ at all points in $\mathbf{R}^n \smallsetminus \mathbf{Z}^n$.

We define $f = \tilde{\varphi}$, the Fenchel transform of φ,

$$f(x) = \tilde{\varphi}(x) = \sup_{\alpha \in \mathbf{Z}^n} \Big(x \cdot \alpha - \varphi(\alpha) \Big), \qquad x \in \mathbf{R}^n.$$

It is a tropical polynomial function.

Passing to the exponential, we see that

$$e^{f(x)} = e^{\tilde{\varphi}(x)} = \sup_{\alpha \in \mathbf{Z}^n} e^{x \cdot \alpha} e^{-\varphi(\alpha)} \leqslant \sum_{\alpha \in \mathbf{Z}^n} e^{x \cdot \alpha} e^{-\varphi(\alpha)} = g(y),$$

where $y_j = e^{x_j}$ and

$$g(y) = \sum_{\alpha \in \mathbf{Z}^n} e^{-\varphi(\alpha)} y^\alpha$$

is a classical Laurent polynomial majorizing $e^{f(x)} = e^{\tilde\varphi(x)}$, but actually often rather close to it.

Summing up, we see that φ contains all information, from which $f = \tilde\varphi$ and g can be constructed. Also g determines its coefficients $\exp(-\varphi(\alpha))$, thus also φ and f. On the other hand, $f = \tilde\varphi$ contains less information, from which φ cannot in general be retrieved.

Question 5.4. *Is it possible to pass from $e^f = e^{\tilde\varphi}$ to g using some other structure? (Cf. Section 6.)*

Question 5.5. *Is it possible to pass to the limit in some way so that the classical polynomials tend to the tropical one?*

6. Ghosts in tropical mathematics

In a polynomial function $f(z) = \sum a_j z^j$ all coefficients can be retrieved from the values of f, both in the real case and in the complex case: if $\sum a_j z^j = \sum b_j z^j$ for sufficiently many z, then $a_j = b_j$ for all j.

But in a tropical Laurent polynomial function $f(x) = \sup_{j\in\mathbf{Z}}(a_j + jx)$, a coefficient a_k cannot be retrieved from the values of f if $a_k \leqslant \frac{1}{2}(a_{k-1} + a_{k+1})$: if this is so, we can replace a_k by a smaller value without changing the values of f at any point. In fact, under this hypothesis,

$$a_k + kx \leqslant \big(a_{k-1} + (k-1)x\big) \vee \big(a_{k+1} + (k+1)x\big) \text{ for all } x \in \mathbf{R}. \tag{6.1}$$

Thus such a coefficient is invisible. We see that many tropical Laurent polynomials $P(X) = \bigvee_{j\in\mathbf{Z}}(a_j + jX)$ have the same evaluations on the real axis.

On the other hand, if the function $\mathbf{Z} \ni j \mapsto a_j$ is strictly concave in the sense that $a_j > \frac{1}{2}a_{j-1} + \frac{1}{2}a_{j+1}$ for all $j \in \mathbf{Z}$, then the coefficients can be retrieved from the values of the function; in fact, under this hypothesis, $a_j = -\tilde f(j)$, $j \in \mathbf{Z}$.

I found this slightly disturbing, and asked Mikael the following question in a letter of 2010-03-26.

Question 6.1. *Is there some structure which will allow us to retrieve all coefficients of a tropical polynomial from its point evaluations?*

It is clear that we need more information than just the values on the real axis.

In his answer of 2010-03-29, Mikael directed me to the preprint by Izhakian & Rowen (2009), published as (2010). (Perhaps the paper by Izhakian (2009) is easier to start with.) It seemed to me that Mikael hinted at a solution in that the ghost elements could be used to retrieve the coefficients.

A first lesson is that, just as for classical polynomials, we must distinguish between a polynomial and the function given by a polynomial. A polynomial $P(X) = \sum a_j X^j$ is a formal expression containing an indeterminate X. If we give X a value as a variable in some ring, we get a polynomial function. For instance, if $P(X) = X^3 - 3X^2 + 2X$, and we replace X by a variable in the finite field

$\mathbf{Z}_3 = \mathbf{Z}/3\mathbf{Z}$, then the value is everywhere zero. The situation is similar for tropical polynomials: a tropical polynomial is a formal expression $P(X) = \bigvee_{j\in\mathbf{N}}(a_j + jX)$, where information about all the coefficients a_j is preserved. That the coefficients are not retrievable from an evaluation in $\mathbf{R}$ is no more upsetting than the example with evaluation in $\mathbf{Z}_3$.

In the quoted papers by Izhakian and Rowen, the authors construct a space $\mathbf{T} = (\mathbf{R} \times \{0, 1\}) \cup \{-\infty\}$, where we have two copies of $\mathbf{R}$; the first copy consists of the usual real numbers, represented as $(x, 0)$, the second of the ghost elements, represented as $(x, 1)$. We can evaluate a tropical polynomial at the points of $\mathbf{T}$.

However, Zur Izhakian explains:

> In the tropical framework there is no injection of the polynomial semiring into the function semiring. Namely, a function could have several polynomial descriptions. In particular there are monomials (called *inessential*) which can be omitted without changing the function determined by the polynomial.
>
> This phenomena is obtained due to convexity considerations involved in this setting, which cause a loss of information. Accordingly, a full recovery of the exact coefficients of a polynomial from the corresponding function is not always possible. (Zur Izhakian, personal communication 2011-10-26)

So this is just the observation I made concerning evaluation at real numbers in the beginning of this section (see formula (6.1)), but now extended to the larger space $\mathbf{T}$.

This means that my original Question 6.1 remains unanswered.

Question 6.2. *It is well known that the coefficients of a holomorphic function can be retrieved from its values:*

$$\text{if } h(z) = \sum_{j\in\mathbf{N}} a_j z^j, \ z \in \mathbf{C}, \quad \text{then} \quad a_k = \frac{1}{2\pi i}\int_{|z|=r} \frac{h(z)}{z^{k+1}}dz, \quad k \in \mathbf{N}.$$

What can be said if we know only the values of the growth function

$$g(r) = \sup_{|z|=r} |h(z)|, \qquad r \geqslant 0? \tag{6.2}$$

If we have two entire functions h_1 and h_2 with growth functions g_1 and g_2, does it follow that $g_1(r) = g_2(r)$ for all r only if the coefficients of h_1 and h_2 have the same absolute values? Is there even a formula that yields the $|a_j|$ from g?

We can at least retrieve the absolute values of the first and second nonzero coefficients from the growth function:

Proposition 6.3. *Let h be an entire function with Taylor series*

$$h(z) = \sum_{q}^{\infty} a_j z^j, \qquad z \in \mathbf{C},$$

and let g be its growth function defined in (6.2)*. Then*

$$g(r)r^{-q} \to |a_q| \quad \text{and} \quad \frac{g(r)r^{-q} - |a_q|}{r} \to |a_{q+1}| \ \text{as } r \to 0+.$$

Corollary 6.4. *If h is a polynomial of the form*

$$h(z) = \sum_q^{q+3} a_j z^j,$$

then the absolute values of all four coefficients $a_q, a_{q+1}, a_{q+2}, a_{q+3}$ can be determined from the growth function.

Proof. To determine $|a_{q+2}|$ and $|a_{q+3}|$ we look at $z^{q+3}h(1/z)$. ☐

When the coefficients are real, Jean-Pierre Kahane could give an affirmative answer:

Proposition 6.5. (Jean-Pierre Kahane, 2011-11-11.) *If $h(z) = \sum a_j z^j$ is an entire function with real coefficients a_j and if a_0 and a_1 are positive, then all coefficients can be determined from the Taylor expansion at the origin of the square of the growth function.*

Proof. From the following lemma we have that $h(r)\overline{h(r)} = g(r)^2$ for small r. It is easy to see that the coefficients a_j of h can be read off from the Taylor expansion of $h\bar{h}$ at the origin. ☐

Lemma 6.6. (Jean-Pierre Kahane, 2011-11-11.) *If $h(z) = \sum a_j z^j$ is an entire function with real coefficients a_j and if a_0 and a_1 are positive, then $g(r) = |h(r)|$ for sufficiently small $r \geqslant 0$.*

Proof. We have, writing $z = re^{it}$,

$$|h(z)|^2 = \sum_{j,k\in\mathbf{N}} a_j a_k z^j \bar{z}^k = \sum_{m=0}^{\infty} r^m \sum_{j=0}^{m} a_j a_{m-j} e^{i(2j-m)t}.$$

Taking the real part, we get

$$|h(z)|^2 = \operatorname{Re}|h(z)|^2 = \sum_{m=0}^{\infty} r^m \sum_{j=0}^{m} a_j a_{m-j} \cos(2j-m)t.$$

The partial derivative with respect to t is

$$\frac{\partial}{\partial t}|h(re^{it})|^2 = -2ra_0a_1 \sin t + \sum_{m=2}^{\infty} r^m \sum_{j=0}^{m} a_j a_{m-j}(m-2j)\sin(2j-m)t.$$

Here the first term $-2ra_0a_1\sin t$, where $a_0a_1 > 0$, dominates all the others when r is small. This follows from the inequality $|\sin(m-2j)t| \leqslant |(m-2j)\sin t|$, $-\pi \leqslant t \leqslant \pi$. Since the coefficients are bounded, the sum can be estimated by a constant times

$$\sum_{m=2}^{\infty} m^2 r^m |\sin t| \leqslant Cr^2|\sin t|, \qquad 0 \leqslant r \leqslant \tfrac{1}{2},$$

which is strictly less than $2ra_0a_1|\sin t|$ when $r < 2a_0a_1/C$. Hence the partial derivative of $|h(re^{it})|^2$ with respect to t has the same sign as $-\sin t$, which shows that $[-\pi,\pi] \ni t \mapsto |h(re^{it})|^2$ attains its maximum for $t = 0$, i.e., for real $z = re^{it}$. ☐

7. The discrete Prékopa problem

If $F\colon \mathbf{R}^2 \to [-\infty, +\infty]$ is a function of two real variables, its *marginal function* $H\colon \mathbf{R} \to [-\infty, +\infty]$ is defined by

$$H(x) = \inf_{y\in\mathbf{R}} F(x,y), \qquad x \in \mathbf{R}.$$

It is well known and easy to prove that H is convex if F is convex.

There is a more general marginal function H_p, called the *p-marginal function*, of $F\colon \mathbf{R}^2 \to [-\infty, +\infty]$. It is defined by

$$e^{-pH_p(x)} = \int_{\mathbf{R}} e^{-pF(x,y)}dy, \qquad x \in \mathbf{R},$$

for any positive real number p. Prékopa's theorem, first presented by András Prékopa in Budapest in 1972 (Christer Borell, personal communication 2011-10-24) and published by Prékopa in (1973: Theorem 6), says that H_p is convex if F is convex. For an elegant proof of a more general theorem, see Ledoux (2001: Theorem 2.13).

In the discrete case, with $f\colon \mathbf{Z}^2 \to [-\infty, +\infty]$, we define the marginal function h by

$$h(x) = \inf_{y\in\mathbf{Z}} f(x,y), \qquad x \in \mathbf{Z},$$

and the p-marginal function h_p by

$$e^{-ph_p(x)} = \sum_{y\in\mathbf{Z}} e^{-pf(x,y)}, \qquad x \in \mathbf{Z}. \tag{7.1}$$

The classical marginal function $h = h_\infty$ is a limiting case when $p \to +\infty$ and may be defined by

$$e^{-h_\infty(x)} = \sup_{y\in\mathbf{Z}} e^{-f(x,y)}, \qquad x \in \mathbf{Z}. \tag{7.2}$$

So we may say that (7.2) is a dequantization of the sums of the exponential functions in (7.1): replacing the sum by the sup. We have $h_p \leqslant h_\infty$.

Question 7.1. *Is it possible to go from $\sum_{y\in\mathbf{Z}} e^{-f(x,y)}$ to $\sup_{y\in\mathbf{Z}} e^{-f(x,y)}$ under some reasonable hypotheses on f? (Cf. Question 5.5.)*

In the digital case, it is not enough to assume that f has a convex extension to all of $\mathbf{R}^2$ to conclude that h has a convex extension to $\mathbf{R}$. But a stronger convexity property, now called rhomboidal convexity, implies that h has a convex extension to $\mathbf{R}$ (Kiselman 2008). I call $f\colon \mathbf{Z}^2 \to \mathbf{R}$ *rhomboidally convex* if its second differences satisfy six conditions:

$$D_bD_af \geqslant 0 \text{ for all } (a,b) \in \{((1,0),(1,b_2));\ b_2 = -1,0,1\},$$

as well as

$$D_bD_af \geqslant 0 \text{ for all } (a,b) \in \{((0,1),(b_1,1));\ b_1 = -1,0,1\}.$$

Here D_a is the difference operator $(D_a f)(x) = f(x+a) - f(x)$. It is not known whether the result holds for the p-marginal function. This is the *discrete Prékopa problem:*

Question 7.2. *Is it true that the p-marginal function h_p, defined by* (7.1)*, has a convex extension to* $\mathbf{R}$ *if f is rhomboidally convex?*

I asked Mikael Passare and Bo Berndtsson this question in the spring of 2008.

It is enough here to take $p = 1$. Examples show that it is not enough that f admits a convex extension to all of $\mathbf{R}^2$, and that rhomboidal convexity is sufficient in some special classes.

8. A conjecture on coamoebas and Newton polytopes

This section was written by Timur Sadykov and is included here with his permission. It also contains recent information from Mounir Nisse and Jens Forsgård.

Definition 8.1. A *Laurent polynomial* is a polynomial in z_j and z_j^{-1}, $j = 1, \dots, n$. It thus has the form

$$f(z) = \sum_{\alpha \in A} a_\alpha z^\alpha$$

for some finite subset A of $\mathbf{Z}^n$. The *Newton polytope* of a Laurent polynomial is defined to be the convex hull in $\mathbf{R}^n$ of the set $\{\alpha \in A;\ a_\alpha \neq 0\}$. We will denote this polytope by Δ_f. A Laurent polynomial is said to be *maximally sparse* if the number of its nonzero terms is equal to the number of vertices of its Newton polytope. □

Definition 8.2. The *amoeba* of a function f defined in $(\mathbf{C} \smallsetminus \{0\})^n$ is a set in $\mathbf{R}^n$ defined as follows. We define a mapping

$$\operatorname{Log}\colon (\mathbf{C} \smallsetminus \{0\})^n \to \mathbf{R}^n \quad \text{by} \quad \operatorname{Log}(z) = (\log|z_1|, \log|z_2|, \dots, \log|z_n|).$$

Then the *amoeba* $\mathscr{A}_f$ of f is the image under Log of its set of zeros. The *coamoeba* $\mathscr{A}'_f$ is defined analogously but with the mapping Log replaced by the mapping

$$\operatorname{Arg}(z) = (\arg z_1, \arg z_2, \dots, \arg z_n).$$

The amoeba of a Laurent polynomial f is said to be *solid* if the number of components of its complement is as small as it can possibly be, that is, if it equals the number of vertices of the Newton polytope Δ_f. □

Mikael wanted to establish formally the duality between amoebas and coamoebas, and he started to write a paper with Mounir Nisse (2012), which Mounir has now finished (Mounir Nisse, personal communication 2011-11-13, 2012-06-24).

Maximally sparse polynomials enjoy certain minimality properties. For instance, it has been proved by Mounir Nisse that the amoeba of a maximally sparse polynomial is necessarily solid; see Nisse (2008, 2009:33). This was earlier conjectured by Mikael and others.

The solidness of the amoeba is also one of the characteristic properties of discriminants according to Passare et al. (2005).

Conjecture 8.3. (Mikael Passare, 2010-12.) *Let f be a maximally sparse Laurent polynomial in n variables. Then the number of components in the complement of the closed coamoeba $\overline{\mathscr{A}'_f}$ is equal to $n!\,\mathrm{Vol}(\Delta_f)$.*

The conjecture was formulated by Mikael in Stockholm in December 2010 and written down on a napkin. Timur kept this napkin and reconstructed the conversation.

However, the conjecture is false: Jens Forsgård and Petter Johansson found counterexamples in dimensions 2 and 3. In two dimensions their polynomial is of the form

$$f(z, w) = 1 + z^2 + w^3 + azw^3 + bz^2w^2,$$

where a and b are constants. The normalized area of the Newton polytope is 11, while the maximal number of components in the complement of the closed coamoeba is 10. In three dimensions, the Newton polytope is the cube with side length 1. The normalized volume is then 6, while the maximal number of components in the complement is 4. Part of the results is described in Broms (2012). (Jens Forsgård, personal communication 2012-06-26.)

Also Mounir Nisse and Frank Sottile found a counterexample in dimension two. More precisely, they proved that there exists a 2-dimensional polygon Δ such that, for any complex plane curve with Δ as Newton polygon, the number of components in the complement of its coamoeba is strictly less than $2\mathrm{Area}(\Delta)$ (in particular when it is defined by a maximally sparse polynomial). (Mounir Nisse, personal communication 2012-06-24.)

Recently however, Jens Forsgård and Petter Johansson (2012) could prove that the conjcture is true if the Newton polytope has $n + 2$ vertices.

9. The constant term in powers of a Laurent polynomial

Let $P(X) = \sum_{\alpha \in A} a_\alpha X^\alpha$, A a finite subset of $\mathbf{Z}^n$, be a Laurent polynomial and assume that its Newton polytope, defined in Definition 8.1, contains the origin in its interior. We consider powers $P(X)^k$, $k \in \mathbf{N}$, of $P(X)$ and denote by $c_k(P)$ the constant term in $P(X)^k$. The question is whether there are infinitely many k such that $c_k(P)$ is nonzero. This seems plausible, and for $n = 1$ it has an elementary proof. Mikael lectured on this problem in the Pluricomplex Seminar on 2000-03-14.

Alain Yger points out that Hans Duistermaat and Wilberd van der Kallen (1998) proved that the answer is in the affirmative as well as a more precise result: the radius of convergence of the formal power series $\sum_{k=1}^{\infty} c_k(P)t^k$ is finite (Alain Yger, personal communication 2011-12-01). However, the proof relies on a very heavy machinery when $n \geqslant 2$.

Alain writes that Mikael was "deeply concerned" about finding a simpler proof of this result. So we may list a new question:

Question 9.1. *Is there a more elementary proof of the result of Duistermaat and van der Kallen (1998) that the constant term $c_k(P)$ in the k-th power of a Laurent polynomial is different from zero for infinitely many values of k?*

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