Convexity of marginal functions in the discrete case

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Abstract

We define, using difference operators, a class of functions on the set of points with integer coordinates which is preserved under the formation of marginal functions.

1. Introduction

A simple everyday observation is that the shadow of a convex body is convex. Mathematically this means that the image under an affine mapping of a convex subset of a vector space is convex. It is often convenient to express this in terms of marginal functions: if F is a convex function defined on $\mathbb{R}^n \times \mathbb{R}^m$ and with values in

$$\mathbb{R}_! = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\},$$

then its *marginal function* $H\colon \mathbb{R}^n \to \mathbb{R}_!$, defined by

$$H(x) = \inf_{y \in \mathbb{R}^m} F(x, y), \qquad x \in \mathbb{R}^n,$$

is convex in $\mathbb{R}^n$. This result has manifold uses in the theory of convexity of real variables, and it would be of interest to establish a similar result for functions $f\colon \mathbb{Z}^n \times \mathbb{Z}^m \to \mathbb{R}_!$, i.e., functions defined at the points in $\mathbb{R}^n \times \mathbb{R}^m$ with integer coordinates. However, for the most obvious attempt at defining a convex function of integer variables, such a result fails in a very conspicuous way, even in low dimensions, $n = m = 1$:

Example 1.1. Define $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) = |x_1 - 2mx_2|, \qquad x = (x_1, x_2) \in \mathbb{Z} \times \mathbb{Z},$$

where m is a positive integer. Then its marginal function

$$h(x_1) = \inf_{x_2 \in \mathbb{Z}} f(x_1, x_2), \qquad x_1 \in \mathbb{Z},$$

is a periodic function of period $2m$ which is equal to $|x_1|$ for $-m \leqslant x_1 \leqslant m$. This means that it is a saw-tooth function with teeth as large as we like. We remark also that if we define f in $\mathbb{R} \times \mathbb{Z}$ by the same expression, then the same phenomenon appears. $\square$

We shall say that a function $f\colon \mathbb{Z}^n \to \mathbb{R}_!$ is *convex extensible* if it is the restriction of a convex function $F\colon \mathbb{R}^n \to \mathbb{R}_!$. (This term has been used in another, narrower sense by Murota [10], page 93.) With this terminology we see that the function f in Example 1.1 is actually convex extensible; indeed, an extension is given by the same expression, while h is not convex extensible (or convex in any reasonable sense). Our conclusion is that the property of being convex extensible is too weak to be of use in this context. In view of this observation, one of us has studied a class of functions defined on $\mathbb{Z} \times \mathbb{Z}$ which is suitable for this and other important properties in convexity theory [3, 4, 6].

In the case of two integer variables, there are several equivalent ways to define integral convexity; see [3, 6]. From a characterization using difference operators it is obvious that the class is closed under addition. It was also proved that in a certain sense integral convexity is necessary for marginal functions to be convex extensible.

The purpose of the present paper is to extend this study to higher dimensions, i.e., functions on $\mathbb{Z}^n \times \mathbb{Z}^m$.

In addition to the result on marginal functions, there are two other results in convexity theory of real variables which would be of interest to have also in the discrete case, viz. that a local minimum is global, and that two disjoint convex sets can be separated by a hyperplane. Also in the other two cases it can be proved by simple examples that convex extensibility of the functions is not enough; see [6] and Murota [10], page 15.

Several kinds of discrete convexity have been studied. Miller [7], page 168, introduced discretely convex functions for which local minima are global. These functions are neither convex extensible nor is the class closed under addition.

Another kind of convexity was introduced by Favati and Tardella [1] using locally convex functions. These functions were called integrally convex and for them local minima are global. They are all convex extensible. However, there are some examples which show that the class of integrally convex function is not closed under addition (see Murota and Shioura [12]).

Two other concepts of convexity were presented by Murota in [8] and [9]. They are called M- and L-convexity, respectively. For these two classes of functions, local minima are global. Two other classes of functions are obtained by a special restriction of M- and L-convex functions to a space of one dimension less. These functions are called M$^\natural$ and L$^\natural$-convex and were introduced by Murota and Shioura [11] and Fujishige and Murota [2], respectively. The class of M$^\natural$-convex (L$^\natural$-convex) functions properly contains M-convex (L-convex) functions. These classes of functions have been studied with respect to some operations such as infimal convolution, addition, and addition by an affine function (Murota and Shioura [12]).

The class of M- and L-convex functions form two distinct classes of discrete convexity which are conjugate to each other under the Fenchel transformation. Similarly we have a conjugacy relation between the classes of $\mathrm{M}^\natural$- and $\mathrm{L}^\natural$-convex functions. An $\mathrm{M}^\natural$-convex function is supermodular, whereas an $\mathrm{L}^\natural$-convex function is submodular (Murota [10]:145, 189). On the other hand, in dimensions $n \geqslant 3$, it is not in general true that the Fenchel transform of a supermodular function is submodular; however, the converse is always true. Thus, these two concepts, submodularity and supermodularity, are not symmetric under this transformation. On the other hand, adding an exchange axiom to supermodularity and a linearity to submodularity makes the behavior under the Fenchel transformation different.

Moreover, an integrally convex function is $\mathrm{L}^\natural$-convex if and only if it is submodular (Fujishige and Murota [2]; Murota and Shioura [12]:172); also an $\mathrm{M}^\natural$-convex function is integrally convex (Murota and Shioura [12]:170). Various classes of discrete convexity were compared in Murota and Shioura [12].

2. The convex envelope and the canonical extension

To each function $f\colon \mathbb{Z}^n \to \mathbb{R}_!$ we shall associate two functions defined on $\mathbb{R}^n$. The first is the convex envelope of f, the second the canonical extension of f.

Definition 2.1. The *convex envelope* of a function $f\colon \mathbb{Z}^n \to \mathbb{R}_!$ is the largest convex function $G\colon \mathbb{R}^n \to \mathbb{R}_!$ such that $G|_{\mathbb{Z}^n} \leqslant f$. We shall denote it by $\mathbf{cvxe}(f)$. □

The convex envelope is well defined, because the supremum of all functions G which are convex and satisfy $G|_{\mathbb{Z}^n} \leqslant f$ has the same properties.

As already mentioned, we shall say that $f\colon \mathbb{Z}^n \to \mathbb{R}_!$ is *convex extensible* if f is the restriction of a convex function $F\colon \mathbb{R}^n \to \mathbb{R}_!$. This happens if and only if $\mathbf{cvxe}(f)$ is an extension of f. Indeed, if f admits a convex extension, then also $\mathbf{cvxe}(f)$ is a convex extension.

Definition 2.2. The *canonical extension* of a function $f\colon \mathbb{Z}^n \to \mathbb{R}_!$ is defined, in each cube $p + [0,1]^n$, $p \in \mathbb{Z}^n$, as the convex envelope of $f|_{p+\{0,1\}^n}$, the restriction of f to the 2^n vertices of the cube $p + [0,1]^n$. We shall denote it by $\mathbf{can}(f)$. □

It may of course happen that a point x belongs to two cubes $p+[0,1]^n$ and $q+[0,1]^n$ for some $p,q\in\mathbb{Z}^n$, $p\neq q$. But then the definition yields the same result in both cases. To see this, we define the *integer neighborhood* of a real number a, denoted by $N(a)$, as the set $\{\lfloor a\rfloor, \lceil a\rceil\}$. We define the *integer neighborhood* of a point $a=(a_1,\dots,a_n)\in\mathbb{R}^n$ as the set

$$N(a)=N(a_1)\times\cdots\times N(a_n).$$

It has 2^k elements, where k is the number of indices j such that $a_j\in\mathbb{R}\smallsetminus\mathbb{Z}$. Equivalently,

$$N(a)=\big(a+B^\infty_<(0,1)\big)\cap\mathbb{Z}^n,\qquad a\in\mathbb{R}^n,$$

where $B^\infty_<(c,r)$ denotes the strict ball for the l^∞ norm with center at c and of radius r. The mapping

$$\mathbb{R}^n\supset A\mapsto\bigcup_{a\in A}N(a)\subset\mathbb{Z}^n$$

is a dilation and one of many digitizations of $\mathbb{R}^n$.

Proposition 2.3. *For any function $f\colon\mathbb{Z}^n\to\mathbb{R}_!$ and any point $a\in\mathbb{R}^n$, $\mathbf{can}(f)(a)$ is equal to the convex envelope of $f|_{N(a)}$. In particular, $\mathbf{can}(f)$ is an extension of f: when $a\in\mathbb{Z}^n$, $N(a)=\{a\}$ and $(\mathbf{can}(f))(a)=f(a)$.*

Proof. We shall denote by $\mathbf{cvxh}(A)$ the convex hull of a set $A\subset\mathbb{R}^n$. Denote by $K\colon\mathbf{cvxh}(N(a))\to\mathbb{R}_!$ the function which is equal to $\mathbf{can}(f)$ in $\mathbf{cvxh}(N(a))$, and by $L\colon\mathbf{cvxh}(N(a))\to\mathbb{R}_!$ the function which is equal to the convex envelope of $f|_{N(a)}$. Clearly $K\leqslant L$.

To prove the opposite inequality $L\leqslant K$ we shall prove that L can be extended to a convex minorant of f restricted $p+\{0,1\}^n$, where $p+[0,1]^n$ is any cube containing $N(a)$.

Let J be the set of indices j such that $a_j\in\mathbb{Z}$. By a change of coordinates we may assume that $a_j=0$ for $j\in J$ and that $0<a_j<1$ for $j\notin J$. This change is actually just translation by the vector $(\lfloor a_1\rfloor,\dots,\lfloor a_n\rfloor)$ (we must note here that all definitions commute with the translation by a vector with integer components). We define a convex function

$$G(x)=\begin{cases} +\infty & \text{when } \sum_{j\in J}x_j<0;\\ (\mathbf{cvxe}(L))(x) & \text{when } \sum_{j\in J}x_j=0;\\ -\infty & \text{when } \sum_{j\in J}x_j>0.\end{cases}$$

We note that the restriction to $\mathbf{cvxh}(N(a))$ of $\mathbf{cvxe}(L)$ is equal to L. We see that G is a convex minorant of $f|_{p+\{0,1\}^n}$, hence of $\mathbf{can}(f)$ restricted to $p+[0,1]^n$, so $L \leqslant K$. $\square$

We shall say with Favati and Tardella [1], page 9, that f is *integrally convex* if $\mathbf{can}(f)$ is convex.

We always have $\mathbf{cvxe}(f) \leqslant \mathbf{can}(f)$ with equality if and only if f is integrally convex.

3. Lateral convexity

The following definition generalizes that for two variables in [3], Definition 2.1; cf. Theorem 2.4 there. See also [5, 6].

Definition 3.1. Given $a \in \mathbb{R}^n$ we define a difference operator

$$D_a \colon \mathbb{R}^{\mathbb{R}^n} \to \mathbb{R}^{\mathbb{R}^n}$$

by

$$(D_aF)(x) = F(x+a) - F(x), \qquad x \in \mathbb{R}^n, \quad F \in \mathbb{R}^{\mathbb{R}^n}. \tag{3.1}$$

$\square$

If $a \in \mathbb{Z}^n$, it operates from $\mathbb{R}^{\mathbb{Z}^n}$ to $\mathbb{R}^{\mathbb{Z}^n}$ and from $\mathbb{Z}^{\mathbb{Z}^n}$ to $\mathbb{Z}^{\mathbb{Z}^n}$. In particular, $D_{e(j)}$ is the difference operator in the j^{th} coordinate.

Definition 3.2. Given a set $A \subset \mathbb{Z}^n \times \mathbb{Z}^n \cong \mathbb{Z}^{2n}$ we shall say that a function $f \colon \mathbb{Z}^n \to \mathbb{R}$ is *A-laterally convex* if

$$(D_bD_af)(x) \geqslant 0, \qquad x \in \mathbb{Z}^n, \quad (a,b) \in A. \tag{3.2}$$

$\square$

From the definition it is obvious that this class is closed under addition and multiplication by a nonnegative scalar. From the formulas

$$(D_{-a}f)(x) = -(D_af)(x-a), \qquad (D_{-b}D_{-a}f)(x) = (D_bD_af)(x-a-b)$$

it follows that

$$-A = \{(-a,-b); (a,b) \in A\}$$

defines the same class as A. The same is true of

$$A^{\smile} = \{(b,a); (a,b) \in A\}.$$

We define

$$A^{\mathrm{sym}} = A \cup (-A) \cup A^{\smile} \cup (-A)^{\smile},$$

which may have up to four times as many elements as A but still defines the same class.

The formula

$$D_b D_{-a} f(x) = -D_b D_a f(x-a)$$

shows that f is $\{(-a,b)\}$-laterally convex if and only if $-f$ is $\{(a,b)\}$-laterally convex. So the concepts introduced will enable us to study also A-laterally concave functions and A-laterally affine functions.

The formula

$$(D_b f)(x) + (D_c f)(x+b) = (D_{b+c} f)(x)$$

applied to $D_a f$ yields

$$(D_b D_a f)(x) + (D_c D_a f)(x+b) = (D_{b+c} D_a f)(x), \tag{3.3}$$

which implies that if $D_b D_a f \geqslant 0$ and $D_c D_a f \geqslant 0$, then we also have $D_{b+c} D_a f \geqslant 0$. This means that the set of pairs $\{(a,b) \in \mathbb{Z}^n \times \mathbb{Z}^n\}$ such that the inequality holds is closed under partial addition:

$$(a,b) +_2 (a,c) = (a, b+c),$$

i.e., if the first elements agree, we may add the second elements. For sets we define

$$B +_2 C = \{(a, b+c); (a,b) \in B, (a,c) \in C\}.$$

Similarly we can define of course $(a,b) +_1 (c,b) = (a+c, b)$ and

$$B +_1 C = \{(a+c, b); (a,b) \in B, (c,b) \in C\}$$

when the two second elements are the same.

By repeated use of these formulas we see that A-lateral convexity is equivalent to B-lateral convexity, where B is any class such that $A \subset B \subset \tilde{A}$, denoting by $\tilde{A}$ the smallest set which contains

$$(\mathbb{Z}^n \times \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \times \mathbb{Z}^n) \cup A^{\mathrm{sym}}$$

and is closed under both partial additions

$$(B, C) \mapsto B +_1 C, \quad (B, C) \mapsto B +_2 C.$$

Thus $\tilde{A}$ contains the sets $A^{\mathrm{sym}} +_1 A^{\mathrm{sym}}$, $A^{\mathrm{sym}} +_2 (A^{\mathrm{sym}} +_1 A^{\mathrm{sym}})$ and so on.

We sum up the discussion on $\tilde{A}$ in the following lemma.

Lemma 3.3. *Let A be any subset of $\mathbb{Z}^n \times \mathbb{Z}^n$.*

1. *For any $a \in \mathbb{Z}^n$, $(a, \mathbf{0})$ and $(\mathbf{0}, a)$ belong to $\tilde{A}$.*
2. *If $(a, b) \in \tilde{A}$, then $(b, a), (-a, -b), (-b, -a)$ all belong to $\tilde{A}$.*
3. *If $(a, b), (c, b) \in \tilde{A}$, then $(a, b) +_1 (c, b) = (a + c, b)$ belongs to $\tilde{A}$.*
4. *If $(a, b), (a, c) \in \tilde{A}$, then $(a, b) +_2 (a, c) = (a, b + c)$ belongs to $\tilde{A}$.*
5. *For a given function f, let us denote by C_f the set of all pairs $(a, b) \in \mathbb{Z}^n \times \mathbb{Z}^n$ such that $D_b D_a f \geqslant 0$. Then if C_f contains A, it also contains $\tilde{A}$.*

When $n = 1$ and $A = \{(1, 1)\}$, f is A-laterally convex if and only if it is convex extensible. This is the only reasonable definition of convexity in one integer variable. We note that it is equivalent to B-lateral convexity for any B such that

$$(1, 1) \in B \subset \tilde{A} \text{ or } (-1, -1) \in B \subset \tilde{A}.$$

In this case, $\tilde{A}$ is easy to determine: it is equal to $\{(s, t) \in \mathbb{Z}\times\mathbb{Z}; st \geqslant 0\}$.

More generally, for any n and any $j \in [1, n]_{\mathbb{Z}}$, if $A = \{(e^{(j)}, e^{(j)})\}$, then a function is A-laterally convex if and only if it is convex extensible in the variable x_j when the others are kept fixed. Since this is a convenient property, we shall normally require that

$$(e^{(j)}, e^{(j)}) \in A, \qquad j = 1, \dots, n. \tag{3.4}$$

If so, all A-laterally convex functions are separately $\{(1, 1)\}$-laterally convex.

Example 3.4. If f is a polynomial of degree at most two,

$$f(x) = \alpha + \sum_{j=1}^{n} \beta_j x_j + \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{jk} x_j x_k, \qquad x \in \mathbb{R}^n,$$

with $\gamma_{jk} = \gamma_{kj}$, we see that

$$(D_b D_a f)(x) = 2 \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{jk} a_j b_k,$$

so that f is A-laterally convex if and only if the last expression is nonnegative for all $(a, b) \in A$.

In particular, all affine functions are A-laterally convex.

We also see that the special polynomial $f(x) = x_j^2$ is A-laterally convex if and only if $a_j b_j \geqslant 0$ for all $(a, b) \in A$. Conversely, if $a_j b_j \geqslant 0$ and g is any convex extensible function of one variable, then the function $x \mapsto g(x_j)$ is $\{(a, b)\}$-laterally convex. $\square$

In view of this example we shall normally require that

$$(a,b) \in A \text{ implies } a_j b_j \geqslant 0, \qquad j = 1, \dots, n. \tag{3.5}$$

4. Two variables

Let us see what the definitions mean for functions of two variables.

When $n = 2$ and $A = \{(e^{(1)}, e^{(2)})\}$, a function is A-laterally convex if and only if it is submodular. (Cf. Murota [10]:26, 206–207.)

When $n = 2$ and

$$A = \{((1,0),(1,t)); t \in [-1,1]_{\mathbb{Z}}\} \cup \{((0,1),(s,1)); s \in [-1,1]_{\mathbb{Z}}\},$$

then a function is A-laterally convex if and only if it is integrally convex; see [3, 6].

Given a function f, we consider the set C_f of all pairs $(a,b) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ such that $D_b D_a f \geqslant 0$. Then we have to take into account several conditions, e.g., the two *one-variable conditions*

$$(e^{(1)}, e^{(1)}),\ (e^{(2)}, e^{(2)}) \in C_f \tag{4.1}$$

(which we always require in order to avoid uninteresting cases—see (3.4)), the two *diagonal conditions*

$$((-1,1),(-1,1)),\ ((1,1),(1,1)) \in C_f, \tag{4.2}$$

the *left and right horizontal lozenge conditions*[1]

$$((-1,0),(-1,1)),\ ((1,0),(1,1)) \in C_f, \tag{4.3}$$

and finally the *left and right vertical lozenge conditions*,

$$((0,1),(-1,1)),\ ((0,1),(1,1)) \in C_f. \tag{4.4}$$

We note that, by partial addition, $((1,0),(1,1)) +_1 ((0,1),(1,1)) = ((1,1),(1,1))$: the right horizontal lozenge condition and the right vertical lozenge condition yield the diagonal condition for $((1,1),(1,1))$. So this means that we often do not need to consider the diagonal conditions.

It follows from simple examples in [6] that all four lozenge conditions are necessary if we want to obtain a reasonable result on the convex extensibility of marginal functions.

[1] We are aware that *lozenge* and *rhombus* are considered to be synonyms, but we are brave enough to call a set like $\mathbf{cvxh}\{(0,0),(1,0),(1,1),(2,1)\}$ a lozenge, although its sides have Euclidean lengths 1 and $\sqrt{2}$. But their l^∞ lengths are all equal, so it is actually a rhombus as well as a lozenge for the l^∞ metric. A slightly generalized lozenge ...

5. The set where the infimum is attained

We shall first study the relation between A-lateral convexity and the interval (possibly empty) where the infimum defining the marginal function is attained.

Theorem 5.1. *Let us define, for any function $f\colon \mathbb{Z}^n \to \mathbb{R}$,*

$$M_f(x_1, \dots, x_{n-1}) = \{b \in \mathbb{Z}; f(x_1, \dots, x_{n-1}, b) = \inf_{t\in\mathbb{Z}} f(x_1, \dots, x_{n-1}, t)\},$$

where $(x_1, \dots, x_{n-1}) \in \mathbb{Z}^{n-1}$. We also define

$$f_\beta(x) = f(x) - \beta x_n, \qquad x = (x_1, \dots, x_n) \in \mathbb{Z}^n, \quad \beta \in \mathbb{R}.$$

Now let $a = (a', a_n) \in \mathbb{Z}^n$, where $a' = (a_1, \dots, a_{n-1})$ and $a_n \geqslant 0$, and define

$$A = \{(e^{(n)}, e^{(n)}), ((a', a_n), e^{(n)}), ((-a', a_n), e^{(n)})\}.$$

Then f is A-laterally convex if and only if $t \mapsto f(x,t)$ is convex extensible for every x and

$$M_{f_\beta}(x + a') \subset M_{f_\beta}(x) + [-a_n, a_n]_{\mathbb{Z}}, \qquad x \in \mathbb{Z}^{n-1}, \quad \beta \in \mathbb{R}.$$

Proof. Assume first that f is A-laterally convex. Since A contains $(e^{(n)}, e^{(n)})$, $\mathbb{Z} \ni t \mapsto f(x,t)$ is convex extensible for every x.

We note that for a function which is convex extensible in the last variable,

$$b \in M_f(x) \text{ if and only if } D_{e^{(n)}} f(x, b-1) \leqslant 0 \leqslant D_{e^{(n)}} f(x,b). \tag{5.1}$$

Moreover

$$b, b+1 \in M_f(x) \text{ if and only if } D_{e^{(n)}} f(x,b) = 0. \tag{5.2}$$

Let now f satisfy $D_a D_{e^{(n)}} f \geqslant 0$ and consider two points x and $x + a'$ in $\mathbb{Z}^{n-1}$. Then for any $b \in M_f(x)$ we have, since also $(e^{(n)}, (-a', a_n))$ is in A,

$$\begin{aligned} D_{e^{(n)}} f(x + a', b - a_n - 1) \leqslant D_{e^{(n)}} f(x, b-1) \leqslant 0 \\ \leqslant D_{e^{(n)}} f(x,b) \leqslant D_{e^{(n)}} f(x + a', b + a_n), \end{aligned} \tag{5.3}$$

which implies that there is a point $c \in [b - a_n, b + a_n]_{\mathbb{Z}}$ with

$$D_{e^{(n)}} f(x + a', c - 1) \leqslant 0 \leqslant D_{e^{(n)}} f(x + a', c).$$

In view of (5.1) this means that $c \in M_f(x+a')$. We have proved that $b \in c+[-a_n, a_n]_{\mathbb{Z}} \subset M_f(x+a')+[-a_n, a_n]_{\mathbb{Z}}$, and, since b was any point in $M_f(x)$, that $M_f(x) \subset M_f(x+a')+[-a_n, a_n]_{\mathbb{Z}}$. We are done, since the whole argument holds also for f_β.

Conversely, suppose that the function f satisfies $D_{e^{(n)}} D_{e^{(n)}} f \geqslant 0$ but is not A-laterally convex. Then it either does not satisfy $D_a D_{e^{(n)}} f \geqslant 0$ or $D_{-a} D_{e^{(n)}} f \geqslant 0$. It suffices to consider one of these cases. We thus assume that there exist $(x, b) \in \mathbb{Z}^{n-1} \times \mathbb{Z}$ such that

$$D_{e^{(n)}} f(x+a', b+a_n) < D_{e^{(n)}} f(x, b).$$

We shall reach a contradiction to the Lipschitz property.

We take a real number β such that

$$D_{e^{(n)}} f(x+a', b+a_n) < \beta < D_{e^{(n)}} f(x, b).$$

If we rewrite this for the function f_β, for which $D_{e^{(n)}} f_\beta = D_{e^{(n)}} f - \beta$, we obtain

$$D_{e^{(n)}} f_\beta(x+a', b+a_n) < 0 < D_{e^{(n)}} f_\beta(x, b), \tag{5.4}$$

which implies that $M_{f_\beta}(x+a') \subset [b+a_n+1, +\infty[_{\mathbb{Z}}$ and that $M_{f_\beta}(x) \subset]-\infty, b]_{\mathbb{Z}}$. Hence

$$M_{f_\beta}(x+a') + [-a_n, a_n]_{\mathbb{Z}} \subset [b+1, +\infty[_{\mathbb{Z}}$$

and $M_{f_\beta}(x) + [-a_n, a_n]_{\mathbb{Z}} \subset]-\infty, b+a_n]_{\mathbb{Z}}$. Thus $M_{f_\beta}(x+a')$ is not contained in $M_{f_\beta}(x) + [-a_n, a_n]_{\mathbb{Z}}$ unless it is empty, and $M_{f_\beta}(x)$ is not contained in $M_{f_\beta}(x+a') + [-a_n, a_n]_{\mathbb{Z}}$ unless it is empty. As soon as one of them is nonempty, we get a contradiction to the Lipschitz property.

So the case when both sets are empty remains to be considered—so far, there is no contradiction in this case. Since $D_{e^{(n)}} f_\beta(x+a', b+a_n) < 0$, we must then have, in view of (5.1),

$$D_{e^{(n)}} f_\beta(x+a', t) < 0 \text{ for all } t \in \mathbb{Z}.$$

Now define $\gamma = D_{e^{(n)}} f(x, b) > \beta$. Then, by (5.2), $M_{f_\gamma}(x)$ is certainly nonempty; it contains b and $b+1$. And since $\gamma > \beta$ we have

$$D_{e^{(n)}} f_\gamma(x+a', t) = D_{e^{(n)}} f_\beta(x+a', t) + \beta - \gamma < D_{e^{(n)}} f_\beta(x+a', t) < 0$$

for all $t \in \mathbb{Z}$, so that (5.1) shows that $M_{f_\gamma}(x+a')$ is empty. This contradicts the inclusion $M_{f_\gamma}(x) \subset M_{f_\gamma}(x+a') + [-a_n, a_n]_{\mathbb{Z}}$. We are done. $\square$

By permuting the variables we easily obtain the following corollary.

Corollary 5.2. *Given a function* $f\colon \mathbb{Z}^n \to \mathbb{R}$, *we define, for* $1 \leqslant j \leqslant n$ *and* $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \mathbb{Z}^{n-1}$,

$$M_{j,f}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}) \\ = \big\{b \in \mathbb{Z}; f(x_1, \dots, x_{j-1}, b, x_{j+1}, \dots, x_{n-1}, x_n) = \inf_{x_j \in \mathbb{Z}} f(x)\big\}.$$

We also define

$$f_{j,\beta}(x) = f(x) - \beta x_j, \quad x = (x_1, \dots, x_n) \in \mathbb{Z}^n, \quad j = 1, \dots, n, \quad \beta \in \mathbb{R}.$$

Fix a set A *which contains* $(a, e^{(j)})$ *and* $(\bar{a}, e^{(j)})$, *where*

$$\bar{a} = 2a_j e^{(j)} - a = (-a_1, \dots, a_j, \dots, -a_n),$$

and satisfies (3.4) *and* (3.5). *If* f *is* A*-laterally convex, then* f *is convex extensible in each variable separately and we have*

$$M_{j,f_{j,\beta}}(x + a') \subset M_{j,f_{j,\beta}}(x) + [-a_j, a_j]_{\mathbb{Z}}, \qquad x \in \mathbb{Z}^{n-1},$$

where now $a' = (a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n)$.

6. Lateral convexity of marginal functions

In [3], Theorem 3.1, and [6] it was shown that for integrally convex functions of two integer variables, the marginal function is convex extensible. We shall now study the marginal function of A-laterally convex functions.

Theorem 6.1. *Let* $A \subset \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$ *and* $B \subset \mathbb{Z}^n \times \mathbb{Z}^n$ *be given. We assume that* (3.4) *and* (3.5) *hold both for* A *and* B. *Assume that*

(6.1) *If* $(a, b) \in A$ *and* $s \in [-1, 1]_{\mathbb{Z}}$, *then* $((a, s), (b, 0))$ *belongs to* $\tilde{B}$;

that

(6.2)
If there exists $c \in \mathbb{Z}^{n-1}$ *such that* $(a, c) \in A$, *then* $((a, 1), e^{(n)}) \in \tilde{B}$;

and finally that

(6.3) *If* $((a, 1), e^{(n)}) \in B$, *then* $((-a, 1), e^{(n)}) \in \tilde{B}$.

If $f\colon \mathbb{Z}^n \to \mathbb{R}$ *is* B*-laterally convex, then its marginal function*

$$h(x) = \inf_{t \in \mathbb{Z}} f(x, t), \qquad x \in \mathbb{Z}^{n-1},$$

is A*-laterally convex, provided that it does not take the value* $-\infty$.

Lemma 6.2. *Let A and B satisfy the hypotheses in Theorem 6.1. Then*

$$\text{If } (a,b) \in A, \text{ then } ((a,-1),(b,-1)),((a,1),(b,1)) \in \tilde{B}. \tag{6.4}$$

Proof. From the conditions (6.1) and (6.2) we know that $((a,1),(b,0))$ and $((a,1),e^{(n)})$ belong to $\tilde{B}$. By partial addition $+_2$ we conclude that so does $((a,1),(b,1))$.

From the condition (6.2) we know that $((a,1),e^{(n)})$ and, consequently, in view of (6.3), also $((-a,1),e^{(n)})$ belongs to $\tilde{B}$. So does the opposite pair $-((-a,1),e^{(n)}) = ((a,-1),-e^{(n)})$. By condition (6.1) we find that $((a,-1),(b,0))$ is in $\tilde{B}$, and we now only have to form the partial sum

$$((a,-1),-e^{(n)}) +_2 ((a,-1),(b,0)) = ((a,-1),(b,-1))$$

to conclude. □

By this lemma we know that if A and B satisfy the hypotheses of the theorem and if $(a,b) \in A$, then there are pairs of the form $((a,s),(b,t))$ in $\tilde{B}$ with $-1 \leqslant s,t \leqslant 1$ and the sum $s+t$ taking any of the five values $-2,-1,0,1,2$.

Proof of Theorem 6.1. It is enough to prove the theorem for functions such that the infimum defining h is attained at a unique point. Indeed, if $t \mapsto f(x,t)$ is convex extensible, then for any positive $\varepsilon > 0$, the infimum defining the marginal function h_ε of $f_\varepsilon(x,t) = f(x,t) + \varepsilon t^2$ is attained at a unique integer $t = \varphi_\varepsilon(x)$, and h_ε tends to h as $\varepsilon \to 0$, preserving the A-lateral convexity of h_ε. We observe that f_ε is B-laterally convex with f provided that $(e^{(n)},e^{(n)}) \in B$, which we assume.

We may therefore suppose that $h(x) = f(x,\varphi(x))$ for some function $\varphi\colon \mathbb{Z}^{n-1} \to \mathbb{Z}$. Moreover, we know that φ is Lipschitz in the sense that

$$|\varphi(x+a) - \varphi(x)| \leqslant 1, \qquad x \in \mathbb{Z}^{n-1}, \tag{6.5}$$

for certain values of $a \in \mathbb{Z}^{n-1}$, viz. when $((a,1),e^{(n)})$ and $((-a,1),e^{(n)})$ both belong to $\tilde{B}$. For this to happen, it is enough that there exists a c such that $(a,c) \in A$.

Similarly, we know that

$$|\varphi(x+b) - \varphi(x)| \leqslant 1 \qquad x \in \mathbb{Z}^{n-1}, \tag{6.6}$$

for certain values of $b \in \mathbb{Z}^{n-1}$, viz. when $((b,1),e^{(n)})$ and $((-b,1),e^{(n)})$ both belong to $\tilde{B}$. For this it is enough that there exists a d such that $(b,d) \in A$.

In particular, if (a, b) is in A, we can take $c = b$ and $d = a$ above to conclude that the two Lipschitz conditions (6.5) and (6.6) hold.

We have

$$\begin{aligned} &D_b D_a h(x) \\ &\quad = f(x+a+b, \varphi(x+a+b)) - f(x+a, \varphi(x+a)) \\ &\qquad - f(x+b, \varphi(x+b)) + f(x, \varphi(x)). \end{aligned} \tag{6.7}$$

The formula holds of course for all $x, a, b \in \mathbb{Z}^{n-1}$, but we shall need it only when $(a, b) \in A$. We shall compare (6.7) with

$$\begin{aligned} &D_{(b,t)} D_{(a,s)} f(x, \varphi(x)) \\ &\quad = f(x+a+b, \varphi(x)+s+t) - f(x+a, \varphi(x)+s) \\ &\qquad - f(x+b, \varphi(x)+t) + f(x, \varphi(x)) \end{aligned} \tag{6.8}$$

for suitable s and t. This expression is nonnegative if $((a, s), (b, t)) \in \tilde{B}$.

By the definition of φ we have $-f(x+a, \varphi(x+a)) \geqslant -f(x+a, s)$ and $-f(x+b, \varphi(x+b)) \geqslant -f(x+b, t)$ for any s and t, so we get

$$D_b D_a h(x) \geqslant D_{(b,t)} D_{(a,s)} f(x, \varphi(x))$$

as soon as $s + t = \varphi(x+a+b) - \varphi(x)$.

In view of (6.5) and (6.6), which, as we have remarked, are applicable,

$$|\varphi(x+a+b) - \varphi(x)| \leqslant |\varphi(x+a+b) - \varphi(x+a)| + |\varphi(x+a) - \varphi(x)| \leqslant 2,$$

and we know from Lemma 6.2 that there are numbers s, t such that $s + t = \varphi(x+a+b) - \varphi(x)$ and $((a, s), (b, t)) \in \tilde{B}$. We are done. □

By iteration we easily obtain the following result.

Corollary 6.3. *Let us define $B^{(0)} = \{(0, 0)\}$, $B^{(1)} = \{(1, 1)\}$, and generally $B^{(n)} \subset \mathbb{Z}^n \times \mathbb{Z}^n$ such that $B^{(n-1)}$ and $B^{(n)}$ satisfy the conditions for A and B in Theorem 6.1 for $n \geqslant 2$. If $f : \mathbb{Z}^n \to \mathbb{R}$ is a given $B^{(n)}$-laterally convex function, then the successive marginal functions $h_n = f$,*

$$h_m(x) = \inf_{t \in \mathbb{Z}} h_{m+1}(x, t), \qquad x = (x_1, \dots, x_m) \in \mathbb{Z}^m, \quad m = n-1, \dots, 0,$$

are $B^{(m)}$-laterally convex, provided that the constant h_0 is not $-\infty$. In particular, the marginal function h_1 of one variable is $\{(1, 1)\}$-laterally convex, equivalently convex extensible.

In condition (6.1) it is often preferable to replace the pair

$$((a,-1),(b,0)) \text{ by its opposite } ((-a,1),(-b,0)),$$

which determines the same condition. This is to be able to continue as in Corollary 6.3, where the last component should be nonnegative—this is needed in Theorem 5.1. We denote the set B so constructed by $\Phi^n(A)$. We can now define $B^{(n)} = \Phi^n(B^{(n-1)})$ and get Corollary 6.3 to work.

When $n = 2$, the corollary is about three functions: $h_2 = f$ defined on $\mathbb{Z}^2$, $h_1(x_1) = \inf_{x_2\in\mathbb{Z}} f(x_1,x_2)$ defined on $\mathbb{Z}^1$, and the constant $h_0 = \inf_{x\in\mathbb{Z}^2} f(x)$ defined as a function on $\mathbb{Z}^0 = \{0\}$. But here we do not say anything about the marginal function $k_1(x_2) = \inf_{x_1\in\mathbb{Z}} f(x_1,x_2)$. To do so, we should permute the variables. However, it turns out, perhaps surprisingly, that this is not necessary, for the conditions are symmetric in the two variables.

If we start with $A = \{(1,1)\} \subset \mathbb{Z}^1 \times \mathbb{Z}^1$ in one variable, the construction in Theorem 6.1 yields, in order,

$((e^{(2)},e^{(2)}))$ (applying (3.4));

$((1,-1),(1,0))$, $((1,0),(1,0))$, $((1,1),(1,0))$ (applying (6.1));

$((1,1),e^{(2)})$ (applying (6.2)); and

$((-1,1),e^{(2)})$ (applying (6.3)).

However, as already mentioned, we should replace $((1,-1),(1,0))$ by $((-1,1),(-1,0)$. We thus have

$$\begin{aligned} B = \{((e^{(1)},e^{(1)}),\ (e^{(2)},e^{(2)}),\ ((1,1),(1,0)),\}, \\ \cup \{((-1,1),(-1,0)),\ ((-1,1),(0,1)),\ ((1,1),(0,1))\}, \end{aligned}$$

This means that the two one-dimensional conditions and the four lozenge conditions are all satisfied, while the two diagonal conditions need not be listed since they follow from the others. We see that if we permute the variables, the conditions remain the same.

We see that $B = \Phi^2(A)$ consists of 6 pairs, and that $\Phi^3(B)$ consists of $6^2 = 36$ pairs.

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