Dedicated to Professor Qikeng Lu on
the occasion of his eightieth birthday

Functions on discrete sets holomorphic in the sense of Ferrand, or monodiffric functions of the second kind

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Contents:

Abstract. We study the class of functions called monodiffric of the second kind by Rufus Philip Isaacs. They are discrete analogues of holomorphic functions of one or two complex variables. Discrete analogues of the Cauchy–Riemann operator, of domains of holomorphy in one discrete variable, and of the Hartogs phenomenon in two discrete variables are investigated. Two fundamental solutions to the discrete Cauchy–Riemann equation are studied: one with support in a quadrant, the other with decay at infinity. The first is easy to construct by induction; the second is accessed via its Fourier transform.

Keywords: Monodiffric function, holomorphic function on a discrete set, difference operator, Cauchy–Riemann operator, domain of holomorphy, the Hartogs phenomenon.

MSC (2000): 39A12, 47B39, 32A99.

1. Introduction

The pioneer in the study of holomorphic functions on discrete sets is Rufus Philip Isaacs, who introduced two difference equations which are discrete counterparts of the Cauchy–Riemann equation in one complex variable. He thus defined two classes of holomorphic functions on the Gaussian integers $\mathbf{Z}[i] = \mathbf{Z} + i\mathbf{Z}$, called monodiffric functions of the first and second kind, respectively (1941:179). In a later paper (1952) he pursued the study of the monodiffric functions of the first kind.

Jacqueline Ferrand (1944) investigated the monodiffric functions of the second kind, which she called *préholomorphes* 'preholomorphic'. See also her book Lelong-Ferrand (1955).

The term *monodiffric function* shall be understood as an analogue of *monogenic function*, used to designate a function the derivatives of which do not depend on the direction. The latter was used already by Cauchy (*fonction monogène*) and was later replaced by *fonction analytique*. For a monodiffric function it is the difference quotient that is independent of the direction.

For a recent survey of the theory of discrete holomorphic functions of one variable, see Jeong (2007).

In an earlier paper, Kiselman (2005b), I studied monodiffric functions of the first kind. See also the references mentioned there. The purpose of the present paper is to prove similiar results for monodiffric functions of the second kind. In particular we shall study the Cauchy–Riemann equation in one variable and the overdetermined system of Cauchy–Riemann equations in two variables (Section 3).

There exists a fundamental solution to the Cauchy–Riemann equation with support in a quadrant (Section 4). It can be easily defined explicitly by induction, and is closely related to the Delannoy numbers known in combinatorics since 1895. It grows very fast and leads therefore to bad estimates of the growth of solutions. There is another fundamental solution which tends to zero at infinity (Section 5). We shall prove its existence using Fourier analysis. Its explicit values are elusive.

Only very special domains are domains of holomorphy in one discrete variable (Section 6). The Hartogs phenomenon in two complex variables has a counterpart in the discrete setting (Section 7).

2. Definitions

If a function is defined on four distinct points a, b, c, d in the complex plane, we impose the condition

$$f(a)(b-d) + f(b)(c-a) + f(c)(d-b) + f(d)(a-c) = 0,$$

which may be written

$$\frac{f(c)-f(a)}{c-a} = \frac{f(d)-f(b)}{d-b}.$$

This is the definition studied by Ferrand (1944). It expresses the fact that the difference quotient when going from a to c is the same as the difference quotient taken from b to d. For a discussion concerning this and more general conditions, see Ferrand (1944) and Kiselman (2005b).

In particular, if $b = a + 1$, $c = a + 1 + i$ and $d = a + i$, we get

$$\frac{f(a+1+i)-f(a)}{1+i} = \frac{f(a+i)-f(a+1)}{i-1}. \tag{2.1}$$

Definition 2.1. A function f defined on a subset A of $\mathbf{Z}[i]$ shall be said to be *monodiffric of the second kind* or *holomorphic in the sense of Ferrand* if (2.1) holds for all $a \in A$ such that also $a+1$, $a+i$, and $a+1+i$ all belong to A. We shall write $\mathscr{O}_{\mathrm{F}}(A)$ for this class.

In many respects the class just defined has properties similar to those of the class of monodiffric functions of the first kind. However, one important property is different: The squares tessellate the plane, whereas the triangles do not. For the monodiffric functions of the second kind we can therefore state that the integral over a closed curve is zero. First we need to define the integral. If f is defined on a sequence $\Gamma = (a_0, a_1, \dots, a_m)$ of points, we define the function f_{aff} as the piecewise affine interpolation of f and then

$$\int_\Gamma f(z)dz = \int_\gamma f_{\text{aff}}(z)dz,$$

where γ is the polygon in the complex plane consisting of the segments $[a_j, a_{j+1}]$. The definition of f_{aff} means that it is given, if $a_{j+1} \neq a_j$, by

$$f_{\text{aff}}(z) = \frac{a_{j+1} - z}{a_{j+1} - a_j} f(a_j) + \frac{z - a_j}{a_{j+1} - a_j} f(a_{j+1}), \qquad z \in [a_j, a_{j+1}], \quad j = 0, \dots, m-1,$$

except for the finitely many points belonging to some other segment $[a_k, a_{k+1}]$. If $a_{j+1} = a_j$, the formula reduces to $f(z) = f(a_j) = f(a_{j+1})$.

We can now state the result that the integral of a monodiffric function of the second kind over a closed curve vanishes. We shall only use 4-curves, meaning that $a_{j+1} - a_j = \pm 1, \pm i$. The curve is closed if $a_m = a_0$.

Proposition 2.2. (Isaacs 1941:183, Lelong-Ferrand 1955:147–148.) *If $f \in \mathscr{O}_{\mathrm{F}}(\mathbf{Z}[i])$, then*

$$\int_\Gamma f(z)dz = 0$$

for every closed 4-curve Γ.

3. Cauchy–Riemann operators

A Cauchy–Riemann operator that corresponds to the definition of Ferrand is

$$\mathsf{CR}(f)(z) = f(z+1+i) - f(z) + if(z+i) - if(z+1), \qquad z \in \mathbf{Z}[i]. \tag{3.1}$$

Ferrand (1944:154) calls this quantity the *écart* of f on the mesh formed by $a = z$, $b = z+1$, $c = z+1+i$, $d = z+i$.

Remark 3.1. Actually the quantity $\mathsf{CR}(f)$ defined here should be associated not with a point but with the square $z + [0,1] + [0,i]$. And for reasons of symmetry it would be desirable to label this square with its center $z + \frac{1}{2} + \frac{1}{2}i$, i.e., to let $\mathsf{CR}(f)$ be defined on the grid $\left(\mathbf{Z} + \frac{1}{2}\right)[i]$ rather than on $\mathbf{Z}[i]$. However, in order to avoid introducing a second grid, we shall stick to this asymmetric definition. We should think of z just as an address of the square $z + [0,1] + [0,i]$.

A function $f\colon A \to \mathbf{C}$, where A is any subset of $\mathbf{Z}[i]$, is holomorphic in A in the sense of Ferrand (Definition 2.1) if and only if $\mathsf{CR}(f)(z) = 0$ at all points z such that z, $z+1+i$, $z+i$, $z+1 \in A$. This means that f solves a convolution equation $\mu * f = 0$ in $(A-1-i) \cap A \cap (A-i) \cap (A-1)$, where $\mu = \delta_{-1-i} - \delta_0 + i\delta_{-i} - i\delta_{-1}$.

Convolution is defined by

$$(f * g)(z) = \sum_{w\in \mathbf{Z}[i]} f(w)g(z-w), \qquad z \in \mathbf{Z}[i], \tag{3.2}$$

whenever the sum has a sense.

Example 3.2. To every function f we may associate its *dual* (Duffin 1956:338),

$$f^{\mathbf{d}}(z) = (-1)^{x+y}\overline{f(z)},$$

defined in the same set. A function is selfdual ($f^{\mathbf{d}} = f$) if and only if it is real at the pure points (those with real and imaginary parts of the same parity), and purely imaginary at the mixed points (those with real and imaginary parts of opposite parity). The operators $f \mapsto f^{\mathbf{d}}$ and CR commute, i.e., $\mathsf{CR}(f)^{\mathbf{d}} = \mathsf{CR}\big(f^{\mathbf{d}}\big)$ for all functions. In particular, $f^{\mathbf{d}}$ is monodiffric if and only if f is. We always have $|f^{\mathbf{d}}| = |f|$. Every function f can be uniquely written $f = f_1 + f_2$ with f_1 selfdual and f_2 antiselfdual ($f_2^{\mathbf{d}} = -f_2$).

Example 3.3. A polynomial in z of degree at most two is holomorphic in the sense of Ferrand. A polynomial of degree three is not monodiffric; indeed $\mathsf{CR}(z^3) = -1 + i$. In general $\mathsf{CR}(P)$ is of degree $m - 3$ if P is a polynomial in z of degree $m \geqslant 3$.

Example 3.4. An exponential function

$$F_{a,b}(z) = a^x b^y = e^{\alpha x+\beta y} = \exp\left(\tfrac{1}{2}(\alpha - i\beta)z + \tfrac{1}{2}(\alpha + i\beta)\bar{z}\right), \qquad z = x + iy \in \mathbf{C},$$

is holomorphic (in the classical sense) if and only if $\alpha + i\beta = 0$. Its restriction $G_{a,b} = F_{a,b}\big|_{\mathbf{Z}[i]}$ to the Gaussian integers is monodiffric if and only if $(a+i)(b-i) = 2$. Indeed, we find that

$$\mathsf{CR}(G_{a,b}) = \big((a + i)(b - i) - 2\big)G_{a,b}.$$

We get a one-parameter family of exponential functions $G_{a,b}$ by taking

$$b = \frac{1 + ia}{i + a}, \quad a \in \mathbf{C} \smallsetminus \{-i\}.$$

With this choice of b, $G_{a,b}$ is real-valued on the real axis, i.e., a is real, if and only if $|b| = 1$. When $G_{a,b}$ is bounded on the real axis, i.e., when $|a| = 1$, then and only then b is real.

Among the functions $F_{a,b}$ there is only one which is bounded and holomorphic: the constant $F_{1,1}$. By way of contrast, there are two bounded monodiffric functions $G_{a,b}$, viz., $G_{1,1}(z) = 1$ and its dual $G_{-1,-1}(z) = (-1)^{x+y}$. In particular, we note that Liouville's theorem does not hold in this setting. More generally, the dual to $G_{a,b}$ is $G_{-\bar{a},-\bar{b}}$ with absolute values $|G_{-\bar{a},-\bar{b}}| = |G_{a,b}| = G_{|a|,|b|}$.

Example 3.5. Let $f(z) = 1/z$, $z \in \mathbf{Z}[i] \smallsetminus \{0\}$. Then

$$\mathsf{CR}(f)(z) = \frac{1 - i}{(z^2 + z + iz)(z^2 + z + iz + i)}$$

where defined. Thus f is not monodiffric but nearly so for large $|z|$:
$z^4\mathsf{CR}(f)(z) \to 1 - i$ as $|z| \to +\infty$.

We can now generalize Proposition 2.2 to the following.

Proposition 3.6. (Duffin 1956:340.) *For all functions defined on* $\mathbf{Z}[i]$ *and all closed* 4-*curves* Γ *we have*

$$\int_\Gamma f(z)dz = \frac{i-1}{2} \sum_{z\in \operatorname{Int}\Gamma} \mathsf{CR}(f)(z),$$

where $\operatorname{Int}\Gamma$ *denotes the set of all lower lefthand corners in the squares surrounded by* Γ.

Cf. the corresponding formula in complex analysis,

$$\int_{\partial\Omega} f dz = \int_\Omega d(f dz) = \int_\Omega \overline{\partial} f \wedge dz = \int_\Omega \left(i\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) dx \wedge dy.$$

Primitive functions

Let us define the *primitive function* F to a given monodiffric function f by the formula

$$F(z) = \int_0^z f(t)dt = \int_0^z f_{\text{aff}}(t)dt, \qquad z \in \mathbf{Z}[i].$$

We have

$$\begin{aligned}\frac{F(z+1+i)-F(z)}{1+i} &= \frac{F(z+i)-F(z+1)}{i-1} \\ &= \tfrac{1}{2}f(z) + \tfrac{1}{4}(1+i)f(z+1) + \tfrac{1}{4}(1-i)f(z+i).\end{aligned}$$

Starting with 1, we get the following sequence of primitive functions:

$$F_0(z) = 1, \quad F_1(z) = z, \quad F_2(z) = \frac{z^2}{2}, \quad F_3(z) = \frac{z^3}{6} + \frac{\bar z}{12}, \quad F_4(z) = \frac{z^4}{24} + \cdots, \ \ldots$$

In general, F_m is a monodiffric polynomial of the form

$$F_m(z) = \frac{z^m}{m!} + \text{a polynomial in } z \text{ and } \bar z \text{ of degree } m-2.$$

We ask whether it is possible to expand an entire function in a series $\sum a_m F_m(z)$. Isaacs (1941:187) proved this for polynomials, thus when the sum is finite.

Richness of the class of monodiffric functions

The class of monodiffric functions is rich, perhaps surprisingly so. As an example, we note that there are non-constant entire functions with support in a half plane (cf. Duffin 1956:337).

Theorem 3.7. *Let* $g\colon \mathbf{N} \to \mathbf{C}$ *and* $\varphi\colon \mathbf{N} \to \mathbf{Z}$ *be arbitrary functions. Then there exists a unique entire function* $h \in \mathscr{O}_{\mathrm{F}}(\mathbf{Z}[i])$ *such that* $h(z) = 0$ *when* $\operatorname{Im} z \leqslant -1$ *and* $h(\varphi(y) + iy) = g(y)$ *for all* $y \in \mathbf{N}$.

Proof. We define $h(x+iy)=0$ for $y \leqslant -1$ and $h(\varphi(y)+iy)=g(y)$ for $y \geqslant 0$. Suppose that we have already defined $h(x+iy)$ for all $x+iy$ with $y<q$. This we have done for $q=0$. Then we go to the right by induction to successively define $h(x+iq)$, $x=\varphi(y)+1, \varphi(y)+2, \dots$ and to the left to define it for $x=\varphi(y)-1, \varphi(y)-2, \dots$ At every step, the function is defined at three points, and the Cauchy–Riemann equation is used to define it uniquely at the fourth point. (The argument works of course for any convolution operator with support equal to the four points $1+i, 0, i, 1$.) $\square$

A variant of the operator CR is

$$\mathsf{cr}(f)(z)=f(z+1)-f(z-1)+if(z+i)-if(z-i).$$

Also this operator was introduced by Isaacs (1941:179 (ii)). The companion operator to cr is

$$\mathsf{cr}^*(f)(z)=f(z+1)-f(z-1)-if(z+i)+if(z-i).$$

We note that $\mathsf{cr} \circ \mathsf{cr}^* = \Delta_{(2)}$, the two-step Laplacian, defined as

$$(\Delta_{(2)}f)(z)=f(z+2)+f(z-2)+f(z+2i)+f(z-2i)-4f(z), \qquad z \in \mathbf{Z}[i].$$

We refer to Kiselman (2005a) and the references mentioned there for more on the discrete Laplacian.

If G is a fundamental solution for the two-step Laplacian, then the formula $\mathsf{cr}(\mathsf{cr}^*G)=\Delta_{(2)}G=\delta$ shows that cr^*G is a fundamental solution for cr. The values of a fundamental solution for the Laplacian have been determined by Stöhr (1950:357); see also van der Pol (1959:251). In principle, these values can be used for finding values of a fundamental solution for the Cauchy–Riemann operator.

4. A fundamental solution with support in the first quadrant

Theorem 4.1. *There exists a solution E to the equation $\mathsf{CR}(E)=\delta_{-1-i}$ with support in the first quadrant. We have $E(x+iy)=i^{y-x}d_{x,y}$, $x,y \in \mathbf{Z}$, where $d_{x,y}$ are the Delannoy numbers. It is selfdual, $E^{\mathbf{d}}=E$.*

Proof. We have chosen to solve $\mathsf{CR}(E)=\delta_{-1-i}$ rather than $\mathsf{CR}(E)=\delta_0$ just to get notation in accordance with the Delannoy numbers. The latter are defined by the relations $d_{x,y}=0$ if x or y is negative, $d_{0,0}=1$, and

$$d_{x,y}=d_{x-1,y}+d_{x,y-1}+d_{x-1,y-1}, \qquad x,y \in \mathbf{N},\ (x,y) \neq (0,0).$$

Let us define a function $\widetilde{\mu}=\delta_{-1-i}-\delta_0-\delta_{-i}-\delta_{-1}$, giving rise to a convolution operator $f \mapsto \widetilde{\mu} * f$ related to the Cauchy–Riemann operator $\mathsf{CR}(f)=\mu * f$. It is easy to verify that $\widetilde{\mu} * \widetilde{f} = \widetilde{\mu * f}$, where in general we define $\widetilde{f}(z)=f(z)i^{\operatorname{Im} z - \operatorname{Re} z}$. It is clear that E can be inductively defined for $x+iy$, and that the relation between E and $d_{x,y}$ is as indicated. $\square$

The Delannoy numbers

The Delannoy numbers are named for Henri-Auguste Delannoy, 1833–1915. For a biography, see Schwer & Autebert (2006). He investigated the possible moves on a chessboard. The numbers under consideration here appear when one studies “la marche de la Reine.” They are explicitly given by

$$d_{x,y} = \sum_{j=0}^{x} \binom{x}{j}\binom{y}{j} 2^j = \sum_{j=0}^{x} \binom{y}{j}\binom{x+y-j}{y}, \qquad (x,y) \in \mathbf{N}^2,$$

(Delannoy 1895:77; Comtet 1974:81). They have a generating function

$$\sum_{x,y\in\mathbf{N}} d_{x,y} z^x w^y = \frac{1}{1-z-w-zw},$$

Comtet (1974:81).

The Delannoy numbers appear in many problems in mathematics; see Sulanke (2003), who lists 29 different examples. To mention just one, $d_{n,r} = d_{r,n}$ is the cardinality of the ball of radius r in $\mathbf{Z}^n$ equipped with the l^1 metric (also known as the hyperoctahedron),

$$\{t \in \mathbf{Z}^n;\ \|t\|_1 = |t_1| + \cdots + |t_n| \leqslant r\};$$

Vassilev & Atanasov (1994), quoted here from Sulanke (2003, note 18).

To Sulanke’s 29 examples I can now add a 30th: the number of Khalimsky-continuous functions $\mathbf{Z} \to \mathbf{Z}$ satisfying $f(0) = 0$ and $f(x+y) = x - y$, $|y| \leqslant x$. This number is equal to $d_{x,y}$. (The number of Khalimsky-continuous functions satisfying $f(0) = 0$ and $f(x+y+1) = x-y$ is the same as that of those satisfying $f(x+y) = x-y$.)

A lot is known about the *central Delannoy numbers* $d_{x,x}$; see Comtet (1974:81), Stanley (2001:185) and Sulanke (2003). Actually $d_{x,x} = P_x(3)$, where $P_x(t)$ is the Legendre polynomial (Comtet 1974:81). The sequence $(d_{x,x})$ has a generating function

$$\sum_{x\in\mathbf{N}} d_{x,x} t^x = \frac{1}{\sqrt{1-6t+t^2}}$$

(Comtet 1974:81, Stanley 2001:185). We have

$$d_{x,x} = 3\left(2 - \tfrac{1}{x}\right) d_{x-1,x-1} - \left(1 - \tfrac{1}{x}\right) d_{x-2,x-2} \sim \left(\sqrt{2}+1\right)^{2x} = \left(3+2\sqrt{2}\right)^{x}.$$

The number $\sqrt{2}+1$ is known as the Silver Ratio for its appearance in various contexts. We note that this asymptotic behavior shows that the fundamental solution is not temperate: it grows faster than any power of $|z|$.

Not so much is known about the general numbers $d_{x,y}$. I conjecture that

$$d_{x,y} \sim \left(\frac{r+y}{x}\right)^x \left(\frac{r+x}{y}\right)^y, \qquad r = \sqrt{x^2+y^2},$$

more precisely that

$$\lim_{\substack{t\in\mathbf{N}^* \\ t\to+\infty}} \frac{1}{t}\log d_{tx,ty} = x\log\big((r+y)/x\big) + y\log\big((r+x)/y\big), \qquad x,y\in\mathbf{N}^*.$$

The right-hand side is a concave function of $(x,y)\in\mathbf{R}^2$, $x,y>0$, and is positively homogeneous of degree one.

For a fixed y this implies that we have polynomial growth of degree y,

$$d_{x,y} \sim C_y(2x)^y, \qquad x\to+\infty.$$

The Delannoy numbers with $0\leqslant x,y\leqslant 6$ are:

6	1	13	85	377	1289	3653	8989
5	1	11	61	231	681	1683	3653
4	1	9	41	129	321	681	1289
3	1	7	25	63	129	231	377
2	1	5	13	25	41	61	85
1	1	3	5	7	9	11	13
0	1	1	1	1	1	1	1
	0	1	2	3	4	5	6

The values of $E(x,y)$ for $-1\leqslant x,y\leqslant 6$ are:

0	-1	$13i$	85	$-377i$	-1289	$3653i$	8989
0	i	11	$-61i$	-231	$681i$	1683	$-3653i$
0	1	$-9i$	-41	$129i$	321	$-681i$	-1289
0	$-i$	-7	$25i$	63	$-129i$	-231	$377i$
0	-1	$5i$	13	$-25i$	-41	$61i$	85
0	i	3	$-5i$	-7	$9i$	11	$-13i$
0	1	$-i$	-1	i	1	$-i$	-1
0	0	0	0	0	0	0	0

Hyperbolicity

The existence of a fundamental solution with support in a strict cone implies that one can solve the Cauchy–Riemann equation with an arbitrary right-hand side, just as for differential operators with constant coefficients, where the existence of a fundamental solution with support in a strict cone is equivalent to hyperbolicity. However, the estimates that can be obtained from this method are very bad because of the fast growth of E.

Theorem 4.2. (Isaacs 1941:197.) *Given any function f on $\mathbf{Z}[i]$, the equation $\mathsf{CR}(u) = f$ can be solved.*

Proof. Let E_1 be the fundamental solution with support in

$$\{x + iy \in \mathbf{Z}[i];\ x \geqslant 1, y \geqslant 1\},$$

and E_2 the fundamental solution with support in

$$\{x + iy;\ x \leqslant 0, y \leqslant 0\},$$

the existence of which is proved similarly. We denote by χ the characteristic function of the half plane $\{z;\ \operatorname{Re} z + \operatorname{Im} z \geqslant 0\}$. Then we can form the convolution products in the formula

$$u = E_1 * (\chi f) + E_2 * (1 - \chi)f \tag{4.1}$$

and apply the Cauchy–Riemann operator and then use the associative law as follows:

$$\begin{aligned} &\mathsf{CR}(u) = \mu * u = \mu * (E_1 * (\chi f)) + \mu * (E_2 * (1 - \chi)f) \\ &= (\mu * E_1) * (\chi f) + (\mu * E_2) * ((1 - \chi)f) = \chi f + (1 - \chi)f = f. \end{aligned} \tag{4.2}$$

□

Remark 4.3. We remark that convolution, defined by (3.2), is in general not an associative binary operation. A simple example can be found even in $\mathbf{Z}$: let $h = \chi_{\mathbf{N}}$ be the characteristic function of $\mathbf{N}$, let $g = \delta_0 - \delta_1$, and let $f = ah + b$ for some constants a, b. Then $(f * g) * h = a\delta_0 * h = ah$, while $f * (g * h) = f * \delta_0 = ah + b$. Associativity holds if and only if $b = 0$. Here $f * g$ and $g * h$ are well defined, but $f * h$ is not, except when $b = 0$. (Note that $f * h$ does not appear in the law of associativity and therefore does not seem to be needed.) However, in (4.1), the support of each fundamental solution is contained in a strict cone, and not only $\mu * E_1$ and $E_1 * (\chi f)$ but also $\mu * (\chi f)$ are well defined: the sum defining each convolution is finite, and the associative law holds.

Systems of equations

Theorem 4.4. *A system of equations* $\mathsf{CR}_1(u) = f_1$, $\mathsf{CR}_2(u) = f_2$, *where the* f_j *are given in* $\mathbf{Z}[i]^2$, *can be solved if and only if* $\mathsf{CR}_2(f_1) = \mathsf{CR}_1(f_2)$.

It seems that a convenient way to prove this theorem is to pass via a special case:

Proposition 4.5. *An equation* $\mathsf{CR}_2(v) = g$, *where* g *is given in* $\mathbf{Z}[i]^2$, *can be solved with* v *monodiffric in* z_1 *if and only if* g *is monodiffric in* z_1.

Proof. We construct for each fixed z_1 the solution in the variable z_2 as in (4.1): $v = E_1 * (\chi g) + E_2 * ((1 - \chi)g)$, where E_1, E_2 and χ are independent of z_1. We observe that v is monodiffric in z_1 if g is. Indeed, using (4.1),

$$\mathsf{CR}_1(v) = \mu_1 * v = E_1 * \big(\chi(\mu_1 * g)\big) + E_2 * \big((1 - \chi)(\mu_1 * g)\big) = 0.$$

The calculation is justified by the fact that μ_1 has its support in the plane $z_2 = 0$ and χ is a function of z_2, so that $\mu_1 * (\chi g) = \chi(\mu_1 * g)$. □

Proof of Theorem 4.4. We first solve $\mathsf{CR}_1(w) = f_1$ for each z_2. Now a new unknown function $v = u - w$ satisfies $\mathsf{CR}_1(v) = 0$ if and only if $\mathsf{CR}_1(u) = f_1$. And $\mathsf{CR}_2(v) = \mathsf{CR}_2(u) - \mathsf{CR}_2(w) = f_2 - \mathsf{CR}_2(w)$ if and only if $\mathsf{CR}_2(u) = f_2$. So we need to solve $\mathsf{CR}_1(v) = 0$ and $\mathsf{CR}_2(v) = f_2 - \mathsf{CR}_2(w) = g$. This we can do using the proposition, for g is monodiffric in z_1: $\mathsf{CR}_1(g) = \mathsf{CR}_1(f_2) - \mathsf{CR}_1(\mathsf{CR}_2(w)) = \mathsf{CR}_2(f_1 - \mathsf{CR}_1(w)) = 0$ in view of the condition $\mathsf{CR}_1(f_2) = \mathsf{CR}_2(f_1)$ and the choice of w. □

5. A bounded fundamental solution

We have seen that a fundamental solution with support in a quadrant grows too fast to be temperate. According to the general existence theorem of de Boor et al. (1989), there exists a temperate fundamental solution. We shall now study one such fundamental solution. It corresponds to the fundamental solution $(\pi z)^{-1}$ for the classical Cauchy–Riemann operator. Its Fourier transform is easy to find; however, it is difficult to find its values at individual points. Gürlebeck (1994:T 626) has calculated its values on the real and imaginary axes (to be precise, he used another indexing) using the values of the fundamental solution of the discrete Laplacian found by Stöhr (1950:357) and van der Pol (1959:251).

Our study will proceed in three steps. First we show that there exists a fundamental solution F tending to zero at infinity. Next we show that $zF(z)$ is bounded. Finally we study the asymptotic behavior of $F(z)$ as z tends to infinity.

Let us agree to define the Fourier transform $\mathscr{F}(U)$ of a function $U\colon \mathbf{Z}[i] \to \mathbf{C}$ as

$$\mathscr{F}(U)(\zeta) = \sum_{z\in\mathbf{Z}[i]} U(z)e^{-i\mathrm{Re}\,\zeta\bar z}, \quad \zeta \in \mathbf{C}. \tag{5.1}$$

It is defined on $\mathbf{C}$ and periodic in both the real and imaginary parts, thus actually defined on $(\mathbf{R} \bmod 2\pi)[i]$. The inverse Fourier transform of a complex-valued function V defined on $(\mathbf{R} \bmod 2\pi)[i]$ is

$$\mathscr{F}^{-1}(V)(z) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} V(\zeta)e^{i\mathrm{Re}\,\zeta\bar z}d\xi d\eta, \qquad z \in \mathbf{Z}[i]. \tag{5.2}$$

We have

$$\mathscr{F}(\mathsf{CR}(U))(\zeta) = \left[e^{i\xi+i\eta} - 1 + ie^{i\eta} - ie^{i\xi}\right]\mathscr{F}(U)(\zeta) = P(\zeta)\mathscr{F}(U)(\zeta);$$

the Cauchy–Riemann operator on the Latin side corresponds on the Greek side to multiplication by the trigonometric polynomial P.

Let us collect a few properties of P.

Lemma 5.1. *The trigonometric polynomial*

$$P(\zeta) = e^{i\xi+i\eta} - 1 + ie^{i\eta} - ie^{i\xi}, \qquad (\xi,\eta) \in \mathbf{R}^2, \quad \zeta = \xi + i\eta \in \mathbf{C}, \tag{5.3}$$

has zeros at the points $\zeta = 2\pi j + 2\pi i k$ and $\zeta = \pi + i\pi + 2\pi j + 2\pi i k$, $j, k \in \mathbf{Z}$, and only there. It possesses the following symmetry properties.

$$P(i\zeta) = ie^{-i\eta}P(\zeta); \tag{5.4}$$

$$P(-\zeta) = -e^{-i\xi - i\eta}P(\zeta); \tag{5.5}$$

$$P(\pi + i\pi + \zeta) = P(i\bar{\zeta}) = \overline{P(-\zeta)}. \tag{5.6}$$

Proof. A simple calculation gives $|P(\zeta)|^2 = 4 - 2\cos(\xi + \eta) - 2\cos(\xi - \eta)$, which shows that the zeros must satisfy $\cos(\xi + \eta) = \cos(\xi - \eta) = 1$, hence that $\xi, \eta \in \pi\mathbf{Z}$ and $\xi \equiv \eta \bmod 2\pi$. This proves the assertion about the zeros.

The symmetry rules (5.4) and (5.6) follow immediately from the definition of P; then (5.5) follows from using (5.4) twice. $\square$

A detailed study of the behavior of P near the origin reveals that it is close to the holomorphic function $(1 + i)\zeta$:

Lemma 5.2. *We have $|P(\zeta) - (1 + i)\zeta| \leqslant \frac{3}{2}|\zeta|^2$ for all $\zeta \in \mathbf{C}$, which implies that $|\zeta| \leqslant |P(\zeta)| \leqslant 2|\zeta|$ for $|\zeta| \leqslant \frac{2}{3}\big(\sqrt{2} - 1\big)$. Similarly, $|P(\pi + i\pi + \zeta) - (i - 1)\bar{\zeta}| \leqslant \frac{3}{2}|\zeta|^2$.*

Proof. We may write

$$P(\zeta) - (1 + i)\zeta = (e^{i\xi + i\eta} - 1 - i\xi - i\eta) - i(e^{i\xi} - 1 - i\xi) + i(e^{i\eta} - 1 - i\eta).$$

We now apply the inequality $|e^{i\xi} - 1 - i\xi| \leqslant \frac{1}{2}\xi^2$, $\xi \in \mathbf{R}$, to each of the three terms in this expression to obtain $|P(\zeta) - (1 + i)\zeta| \leqslant \frac{3}{2}|\zeta|^2$, from which the other estimate follows easily. The statement about the behavior near the point $\pi + i\pi$ follows from (5.6). $\square$

Theorem 5.3. *The inverse Fourier transform of $1/P$,*

$$\mathscr{F}^{-1}(1/P)(z) = F(z) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{e^{i\operatorname{Re}\zeta\bar{z}}}{P(\zeta)} d\xi d\eta, \qquad z \in \mathbf{Z}[i], \tag{5.7}$$

where P is the trigonometric polynomial (5.3), is a fundamental solution for CR and tends to zero at infinity. Under rotation by $90°$ it possesses the same symmetry as $\big(z - \frac{1}{2} - \frac{1}{2}i\big)^{-1}$, viz.,

$$F(1 + iz) = -iF(z), \quad F(1 + i - z) = -F(z), \qquad z \in \mathbf{Z}[i].$$

We have $F(0) = -\frac{1}{4}$, $F(1) = \frac{1}{4}i$, $F(1 + i) = \frac{1}{4}$, $F(i) = -\frac{1}{4}i$. Moreover,

$$F(i\bar{z}) = \overline{F(z)}, \qquad z \in \mathbf{Z}[i].$$

Finally, F is selfdual, $F^{\mathbf{d}} = F$.

Proof. That $\mathsf{CR}(F) = \delta_0$ follows from the inversion formula. We have seen that $|\zeta| \leqslant |P(\zeta)| \leqslant 2|\zeta|$ in a neighborhood of the origin. This implies that $1/P$ is integrable near the origin. The same is true in a neighborhood of the point $\pi + i\pi$ in view of (5.6). It is well known that the inverse Fourier transform of an L^1 function is bounded and that it tends to zero at infinity.

The first symmetry formula follows the definition of F if we use the property (5.4) of P. The second follows from a double use of the first.

The symmetry properties mean that F is symmetric not around the origin but around the point $\frac{1}{2} + \frac{1}{2}i$, just as can be understood from Remark 3.1. In particular, $F(1) = -iF(0)$, $F(1+i) = -F(0)$, $F(i) = iF(0)$. Together with the requirement that $\mathsf{CR}(F)(0)$ shall be equal to 1, this implies that $F(0) = -\frac{1}{4}$.

That $F(i\bar{z}) = \overline{F(z)}$ follows from a change of variables in the integral using the fact that $P(\eta + i\xi) = \overline{P(-\zeta)}$; cf. (5.6). In particular F has real values on the diagonal $x = y$.

Finally, the selfduality is a consequence of the formula $P(\pi + i\pi - \zeta) = \overline{P(\zeta)}$; cf. (5.6).

(The symmetry properties imply that F is determined by its values for $0 \leqslant y \leqslant x$, thus in an eighth of the plane.)

□

We would like to sharpen this result to say that $zF(z)$ is bounded. In general the Fourier transform of an L^1 function does not have this property; an example is the Bessel function J_0 of order zero, which we will encounter shortly.

In this study we shall need the Fourier transforms of functions $u\colon \mathbf{C} \to \mathbf{C}$,

$$\mathscr{F}_{\mathbf{C}}(u)(\zeta) = \int_{\mathbf{C}} u(z) e^{-i\operatorname{Re}\zeta\bar{z}} dx dy, \qquad \zeta \in \mathbf{C},$$

as well as the inverse transformation

$$\mathscr{F}_{\mathbf{C}}^{-1}(v)(z) = \frac{1}{4\pi^2} \int_{\mathbf{C}} v(\zeta) e^{i\operatorname{Re}\zeta\bar{z}} d\xi d\eta, \qquad z \in \mathbf{C}.$$

We thus have two inverse Fourier transformations, $\mathscr{F}^{-1}$, defined by (5.2), and $\mathscr{F}_{\mathbf{C}}^{-1}$. The relation between them is easy:

$$\mathscr{F}_{\mathbf{C}}^{-1}(v)\big|_{\mathbf{Z}[i]} = \mathscr{F}^{-1}(V), \quad \text{where } V(\zeta) = \sum_{j,k \in \mathbf{Z}} v(\zeta + 2\pi j + 2\pi i k) \tag{5.8}$$

is the doubly periodic function generated by v.

With this notation, the Fourier inversion formula becomes

$$\mathscr{F}_{\mathbf{C}}^{-1}(V) = \frac{1}{4\pi^2} \mathscr{F}_{\mathbf{C}}(\check{V}), \tag{5.9}$$

where $\check{V}(\zeta) = V(-\zeta)$.

By duality or by a density argument, the Fourier transformation can be extended to the space $\mathscr{S}'(\mathbf{C})$ of tempered distributions. In particular, the function $z \mapsto 1/|z|$, which is in $L^1_{\mathrm{loc}}(\mathbf{C}) \cap \mathscr{S}'(\mathbf{C})$ but not in $L^1(\mathbf{C})$, has the Fourier transform in the sense of $\mathscr{S}'(\mathbf{C})$,

$$\mathscr{F}_{\mathbf{C}}(1/z)(\zeta) = \frac{-2\pi i}{\zeta}. \tag{5.10}$$

If we cut off this function we get an L^1 function with compact support, the Fourier transform of which is

$$\mathscr{F}_{\mathbf{C}}(\chi_\tau/z)(\zeta) = \frac{2\pi i}{\zeta}\big(-1 + J_0(\tau|\zeta|)\big), \tag{5.11}$$

where χ_τ is the characteristic function of the disk $\{z \in \mathbf{C};\ |z| < \tau\}$, and J_0 is the Bessel function of order zero, defined as

$$J_0(\tau) = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{i\tau t}}{\sqrt{1-t^2}}dt, \qquad \tau \in \mathbf{R}.$$

It is the Fourier transform of a function in $L^1(\mathbf{R})$, and $J_0(\tau)$ tends to zero no faster than $|\tau|^{-1/2}$.

In view of (5.9) it follows that

$$\mathscr{F}_{\mathbf{C}}^{-1}(1/\zeta)(z) = \frac{1}{4\pi^2}\mathscr{F}_{\mathbf{C}}(-1/\zeta)(z) = \frac{i}{2\pi z}, \tag{5.12}$$

and

$$\mathscr{F}_{\mathbf{C}}^{-1}(\chi_\tau/\zeta)(z) = \frac{i}{2\pi z}\big(1 - J_0(\tau|z|)\big). \tag{5.13}$$

Theorem 5.4. *The fundamental solution F defined by (5.7) is such that $zF(z)$ is bounded in $\mathbf{Z}[i]$.*

Gürlebeck (1994: T 626) gives an estimate which implies that $zF(z)$ is bounded. Also, Theorem 7 in Gürlebeck & Hommel (2002) contains a statement similar to Theorem 5.4. However, no proof is given.

Proof. Let $v \in \mathscr{D}(\mathbf{C})$ be a radial test function which is equal to 1 in a neighborhood of the origin and has its support contained in the disk in $\mathbf{C}$ with center at the origin and of radius $\sqrt{2}\pi$. We shall prove in the following lemmas that the inverse Fourier transforms

$$F_1(z) = \mathscr{F}_{\mathbf{C}}^{-1}(v/\zeta)(z) \text{ and } F_2(z) = \mathscr{F}_{\mathbf{C}}^{-1}(v/P - v/((1+i)\zeta)(z)$$

are both bounded by a constant times $1/|z|$, more precisely that $zF_1(z)$ tends to $i/(2\pi)$ and that $zF_2(z)$ tends to 0 as $|z| \to +\infty$. It follows that v/P has this property.

Similary, also $F_3 = \mathscr{F}^{-1}(v_\pi/P)$, where $v_\pi(\zeta) = v(\zeta - \pi - i\pi)$, is bounded by a constant times $1/|z|$, Finally

$$\frac{1 - V - V_\pi}{P},$$

where V, V_π are the doubly periodic functions associated to v and v_π as in (5.8), is smooth in a neighborhood of the square $[0, 2\pi] + i[0, 2\pi]$, so its inverse Fourier transform F_4 defined by (5.2) is rapidly decreasing, thus $zF_4(z) \to 0$.

Summing up, we obtain that the inverse Fourier transform $\mathscr{F}^{-1}(1/P)$ of $1/P$ is bounded by a constant times $1/|z|$. $\square$

Lemma 5.5. *Let $v \in \mathscr{D}(\mathbf{C})$ be a radial test function which is equal to* 1 *in a neighborhood of the origin. Then*

$$F_1(z) = \mathscr{F}_{\mathbf{C}}^{-1}(v/\zeta)(z) = \frac{i}{2\pi z} + O(|z|^{-3/2}), \qquad z \in \mathbf{Z}[i], \ |z| \to +\infty.$$

Proof. Let as before χ_τ denote the characteristic function of the disk $\{z \in \mathbf{C};\ |z| < \tau\}$. It is known that

$$J_0(\tau) = \sqrt{\frac{2}{\pi\tau}}\big(\cos(\tau - \pi/4) + O(\tau^{-1})\big), \qquad \tau \to +\infty;$$

see for example Råde & Westergren (2000:271). Let $v_1 \colon \mathbf{R} \to \mathbf{C}$ be the function defined by $v_1(|\zeta|) = v(\zeta)$. We obtain

$$v(\zeta) = v_1(|\zeta|) = -\int_a^b \chi_\tau(|\zeta|) v_1'(\tau) d\tau, \qquad \zeta \in \mathbf{C},$$

where a and b are chosen so that $\operatorname{supp} v_1' \subset [a, b]$, $0 < a < b \leqslant 1$. Integration of (5.13) with respect to τ yields

$$\mathscr{F}_{\mathbf{C}}^{-1}(v/\zeta)(z) = \frac{i}{2\pi z} - \frac{i}{2\pi z}\int_a^b J_0(\tau|z|) v_1'(\tau) d\tau.$$

Here the integral tends to zero like $|z|^{-1/2}$. $\square$

Lemma 5.6. *Let v be as in Lemma 5.5 and with support contained in the disk of radius $\pi/\sqrt{2}$ and center at the origin. Then the gradient of $w(\zeta) = v/P - v/((1+i)\zeta)$ is in $L^1(\mathbf{C})$. Hence its inverse Fourier transform $F_2(z) = \mathscr{F}_{\mathbf{C}}^{-1}(w)(z)$ is bounded by a constant times $1/|z|$; it is even $o(|z|^{-1})$.*

Proof. In a punctured neighborhood of the origin, we have

$$\frac{\partial}{\partial\xi}\left(\frac{1}{P(\zeta)} - \frac{1}{(1+i)\zeta}\right) = \frac{-\partial P/\partial\xi}{P(\zeta)^2} + \frac{1}{(1+i)\zeta^2} = \frac{-(1+i)\zeta^2 \partial P/\partial\xi + P^2(\zeta)}{(1+i)\zeta^2 P(\zeta)^2}.$$

The absolute value of the denominator in the last expression is no smaller than $\sqrt{2}|\zeta|^4$ in view of Lemma 5.2. The numerator is the restriction to $\mathbf{R}^2$ of an entire function of two variables whose Maclaurin series does not contain any powers of degree less than three. Hence the numerator is $O(|\zeta|^3)$ and the quotient is $O(|\zeta|^{-1})$, implying that it is integrable in a neighborhood of the origin. A similar calculation can be made for $\partial\big(P^{-1} - ((1+i)\zeta)^{-1}\big)/\partial\eta$. $\square$

Finally we shall study the values of F in more detail:

Theorem 5.7. *The fundamental solution F defined by (5.7) satisfies*

$$F(z) = \frac{1+i}{4\pi}\left(\frac{1}{x+iy} + \frac{(-1)^{x+y}}{y+ix}\right) + o\left(\frac{1}{z}\right), \qquad |z| \to +\infty. \tag{5.14}$$

We see that F can be written $F = G + G^{\mathbf{d}} + o(1/z)$, where

$$G(z) = \frac{1+i}{4\pi z} \text{ and } G^{\mathbf{d}}(z) = \frac{(1-i)(-1)^{x+y}}{4\pi\overline{z}}.$$

The oscillation due to the factor $(-1)^{x+y}$ is damped if we form the mean of two adjacent points, e.g., z and $z+1$:

Corollary 5.8. *The fundamental solution F satisfies*

$$\tfrac{1}{2}F(z) + \tfrac{1}{2}F(z+1) = \frac{1+i}{4\pi z} + o(1/z), \qquad |z| \to +\infty. \tag{5.15}$$

Proof of Theorem 5.7. What needs to be done, in addition to the calculation already made, is a more careful study of the contribution of the singularity at $\zeta = \pi + i\pi$. We know that $P(\pi + i\pi + \zeta) = P(\eta + i\xi)$ is approximately equal to $(1+i)(\eta + i\xi) = (i-1)\overline{\zeta}$, and via a change of variable in the integral this shows that the function $F_3 = \mathscr{F}^{-1}(v_\pi/P)$ in the proof of Theorem 5.4 is

$$\frac{1+i}{4\pi} \cdot \frac{(-1)^{x+y}}{y+ix} + o(1/z).$$

This completes the proof. $\square$

6. Domains of holomorphy

Given two subsets A and B of $\mathbf{Z}[i]$ with $A \subset B$, we have a restriction operator

$$R_A^B \colon \mathscr{O}_{\mathrm{F}}(B) \to \mathscr{O}_{\mathrm{F}}(A).$$

It may well be that R_A^B is injective but not surjective, or surjective but not injective.

Example 6.1. Let $A = \{0, 1, i\}$, $B = A \cup \{1+i\}$. Then R_A^B is bijective.

Example 6.2. Let $A = \{0, i, 1+i, 2+i, 2\}$, $B = A \cup \{1\}$. Then R_A^B is easily seen to be injective, but it is not surjective, for a monodiffric function on B must satisfy

$$-if(1+i) + if(0) + f(i) = f(1) = f(2+i) + if(1+i) - if(2).$$

Its restriction to A must then satisfy

$$if(0) + f(i) - f(2+i) - 2if(1+i) + if(2) = 0.$$

But any function on A is monodiffric.

Example 6.3. Let $A = \{0, 1, 1+i, i\}$ and $B = A \cup \{2\}$. Then the restriction mapping is surjective but not injective.

In view of the properties of the restriction operator R_A^B is seems reasonable to propose the following definition.

Definition 6.4. A *domain of holomorphy in* $\mathbf{Z}[i]$ is a set A such that if $B \supset A$ and R_A^B is bijective, then $B = A$.

Remark 6.5. In analogy with the situation in $\mathbf{C}^n$, $n \geqslant 2$, it may be of interest to admit also Riemann domains over $\mathbf{Z}[i]$ and $\mathbf{Z}[i]^n$. The set

$$A = \{z;\ 1 \leqslant |z| \leqslant \sqrt{2}\},$$

consisting of eight points, is a domain of holomorphy in the sense of Definition 6.4, but not if we allow non-schlicht domains.

Let us call a set A *4-connected* if any two points in A can be joined by a path consisting of horizontal and vertical segments. We shall say that a subset of $\mathbf{Z}[i]$ is *horizontally convex* if it cuts every horizontal line in an interval. Similarly we define *vertically convex.*

To a bounded nonempty set A in $\mathbf{Z}[i]$ we associate the smallest rectangle $\mathbf{Rect}(A)$ of the form

$$[a_0, a_1] + i[b_0, b_1] = \{z \in \mathbf{Z}[i];\ a_0 \leqslant \operatorname{Re} z \leqslant a_1, b_0 \leqslant \operatorname{Im} z \leqslant b_1\} \tag{6.1}$$

which contains it. Here $a_0 \leqslant a_1$ and $b_0 \leqslant b_1$ are integers. From this we define also the smallest rectangle containing an unbounded set.

Theorem 6.6. *Let $A \subset \mathbf{Z}[i]$ be 4-connected and either horizontally or vertically convex. Then every $f \in \mathscr{O}_{\mathrm{F}}(A)$ can be uniquely extended to the smallest rectangle which contains A.*

Proof. The set $\mathbf{Rect}(A)$ is the union of four sets, viz. the four sets

$$(A \pm \mathbf{N}) \cap (A \pm i\mathbf{N}).$$

We shall prove that any monodiffric function on A can be uniquely extended to $(A - \mathbf{N}) \cap (A - i\mathbf{N})$. The proofs for the other three cases are of course similar. We shall define the extension step by step in the next lemma, assuming the set to be bounded. The unbounded case then follows easily since $\mathbf{Rect}(A) = \bigcup_k \mathbf{Rect}(A^k)$, where A^k is the bounded set $\{z \in A;\ |\operatorname{Re} z|, |\operatorname{Im} z| \leqslant k\}$, $k \in \mathbf{N}$. $\square$

Lemma 6.7. *Given two bounded* 4*-connected and vertically convex sets A and B with A a subset of B and B a proper subset of $(A - \mathbf{N}) \cap (A - i\mathbf{N})$ such that any monodiffric function on A can be uniquely continued to a monodiffric function on B, we can find a strictly larger set B' with the same properties.*

Proof. Suppose we have found a point $q \in (A - \mathbf{N}) \cap (A - i\mathbf{N}) \smallsetminus B$ such that all three points $q+1$, $q+1+i$ and $q+i$ belong to B. Then we can define $B' = B \cup \{q\}$ and extend any monodiffric function on B to B' in a unique way by using the equation $\mathsf{CR}(f)(q) = 0$. The set B' has all the properties required.

We shall now describe how to find such a point q. Since B is a proper subset of $(A - \mathbf{N}) \cap (A - i\mathbf{N})$, there exists a point $p \notin B$ but in $A - \mathbf{N}$ and $A - i\mathbf{N}$. There exists a point $b \in B$ such that $\operatorname{Im} b = \operatorname{Im} p$ and $\operatorname{Re} b > \operatorname{Re} p$. We take $\operatorname{Re} b$ minimal, so that $b - 1 \notin B$. Also, there exists a point $b' \in B$ such that $\operatorname{Re} b' = \operatorname{Re} p$ and $\operatorname{Im} b' > \operatorname{Im} p$. If $b - 1 + i \in B$, we choose $q = b - 1$, for then $q + 1 = b$ and $q + i = b - 1 + i$ belong to B and also $q + 1 + i = b + i$ must do so in view of the 4-connectedness and vertical convexity of B. If, on the other hand $b - 1 + i$ does not belong to B, then there is some point $c \in B$ with $\operatorname{Re} c = \operatorname{Re} b - 1$ and $\operatorname{Im} c > \operatorname{Im} b$. We take $\operatorname{Im} c$ minimal, so that $c - i \notin B$. We now choose $q = c - i$. Then $q + i = c$ belongs to B. Since b and b' are connected by a polygon with vertical and horizontal segments, there must exist a point d in B with $\operatorname{Re} d = \operatorname{Re} b$ and $\operatorname{Im} d \geqslant \operatorname{Im} c$. From the vertical convexity we conclude that the whole segment $[b, d]$ is contained in B; hence $q + 1$ and $q + 1 + i$ belong to B. Thus q has all properties required in the first paragraph of the proof. $\square$

Corollary 6.8. *A 4-connected horizontally or vertically convex subset of* $\mathbf{Z}[i]$ *is a domain of holomorphy if and only if* $A = \mathbf{Rect}(A)$.

Estimates

If Ω is a rectangle in $\mathbf{Z}[i]$ of the form (6.1), we define

$$\begin{aligned} \beta_0\Omega &= \{z \in \Omega;\ \operatorname{Re} z = a_0 \text{ or } \operatorname{Im} z = b_0\}, \\ \beta_1\Omega &= \{z \in \Omega;\ \operatorname{Re} z = a_1 \text{ or } \operatorname{Im} z = b_1\}, \text{ and} \\ \beta\Omega &= \beta_0\Omega \cup \beta_1\Omega. \end{aligned}$$

The set $\beta\Omega$ will serve as a kind of boundary of Ω in the following. Note that $\beta\Omega = \Omega$ if $a_0 \leqslant a_1 \leqslant a_0 + 1$ or $b_0 \leqslant b_1 \leqslant b_0 + 1$.

A monodiffric function on a rectangle Ω is determined by its restriction to $\beta_0\Omega$; likewise by its restriction to $\beta_1\Omega$. We have

$$h(z) = \sum_{s \in \beta_0\Omega} C_{\Omega,0}(z,s)h(s) = \sum_{s \in \beta_1\Omega} C_{\Omega,1}(z,s)h(s), \quad z \in \Omega, \quad h \in \mathscr{O}_{\mathrm{F}}(\Omega),$$

where $C_{\Omega,j}(\,\cdot\,, s) \in \mathscr{O}_{\mathrm{F}}(\Omega)$ for every $s \in \beta_j\Omega$, $j = 0, 1$.

These formulas lead to easy estimates:

$$|h(z)| \leqslant \sum_{s \in \beta_0\Omega} |C_{\Omega,0}(z,s)| \sup_{s \in \beta_0\Omega} |h(s)| \leqslant A \sup_{s \in \beta_0\Omega} |h(s)|, \tag{6.2}$$

where

$$A = \sup_{z \in \Omega} \sum_{s \in \beta_0\Omega} |C_{\Omega,0}(z,s)|$$

is a very bad constant (cf. the Delannoy numbers). It is known (Ferrand 1944:157) that the holomorphic functions in the sense of Ferrand satisfy a maximum principle, so that $|h(a)| \leqslant \sup_{z\in\beta\Omega} |h(z)|$ for all $a \in \Omega$. The passage from $\beta_0\Omega$ or $\beta_1\Omega$ to $\beta\Omega = \beta_0\Omega \cup \beta_1\Omega$ thus improves the constant dramatically. However, the formula with $\beta_0\Omega$ is still useful when proving the uniqueness of an extension.

7. The Hartogs phenomenon

In several complex variables it is known that a holomorphic function cannot have its singularities contained in a compact set: a holomorphic function defined on a neighborhood of the boundary of a bounded region can be extended holomorphically to the whole region. This is known as the Hartogs phenomenon or the Hartogs–Osgood theorem, or the Hartogs–Osgood–Brown theorem. For its history, see Range (2002).

There is a similar phenomenon in $\mathbf{Z}[i]^2$. We formulate it in the simplest case only. In a Cartesian product $\Omega_1 \times \Omega_2$, the set $(\beta\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \beta\Omega_2)$ serves as a kind of boundary, and its complement in Ω as a kind of interior.

Theorem 7.1. *Let $\Omega = \Omega_1 \times \Omega_2$, where Ω_j are two rectangles in $\mathbf{Z}[i]$. If h is holomorphic in the sense of Ferrand in $(\beta_0\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \beta_0\Omega_2)$, then there exists a function $H \in \mathscr{O}_{\mathrm{F}}(\Omega)$ which extends H.*

There are variants of this theorem which can be obtained by replacing $\beta_0\Omega_1$ by $\beta_1\Omega_1$ or $\beta\Omega_1$, and similarly for Ω_2.

Proof. Construct a function H by the formula

$$H(z_1, z_2) = \sum_{s_1\in\beta_0\Omega_1} C_{\Omega_1,0}(z_1, s_1)h(s_1, z_2), \qquad (z_1, z_2) \in \Omega.$$

Clearly H is monodiffric in z_1 (since $C_{\Omega_1,0}$ is), and also in z_2 (since h is).

First fix $z_2 \in \beta_0\Omega_2$. Then $H(z_1, z_2) = h(z_1, z_2)$ for all $z_1 \in \Omega_1$; the formula just reproduces the function $h(\,\cdot\,, z_2)$.

Next fix $z_1 \in \beta_0\Omega_1$. We know that $H(z_1, z_2) = h(z_1, z_2)$ when $z_2 \in \beta_0\Omega_2$. In view of the uniqueness of monodiffric continuation, we must have $H(z_1, z_2) = h(z_1, z_2)$ for all $z_2 \in \Omega_2$.

Summing up, we have proved that H is holomorphic in the sense of Ferrand in Ω and that $H = h$ in the domain of h. □

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