

- 1.1. Calculate $(1 + i\sqrt{3})^6$ (a) using polar representation; (b) using the binomial formula... if you do not get too tired, that is.
- 1.2. Solve completely the equation $z^7 = 7$, $z \in \mathbf{C}$.
- 1.3. Find the dimension of the space of all polynomials of one complex variable which are of degree at most 17 and vanish at the points $z = 0$, $z = 1$, $z = 2$.
- 1.4. Is the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalizable?
- 1.5. Let V be the space of all polynomials of one real variable of degree at most m . Is the mapping $T: V \ni f \mapsto f' \in V$ injective? Is it surjective? Determine the dimensions of $\text{im } T$ and $\text{ker } T$. Determine the eigenvalues of T as well as all eigenvectors. Is T diagonalizable?
- 1.6. We compare convolution on \mathbf{Z} and on $\mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$, $N \geq 2$. Let $a = \delta_{-1} - \delta_0$, where we define $\delta_k(j)$ to be zero for $j \neq k$ and to be one for $j = k$. Prove that on \mathbf{Z} the equation $a * z = b$ has a solution for any given b , and that the space of solutions to $a * z = 0$ has dimension 1. By way of contrast, show that on \mathbf{Z}_N , the equation $a * z = b$ can be solved only if b satisfies a certain condition—and find this condition. For the homogeneous equation, the situation for \mathbf{Z}_N is the same as for \mathbf{Z} . (The lesson is that, on \mathbf{Z} , the equation $a * z = b$ with periodic b can have nonperiodic solutions z .)
- 1.7. Define a mapping $T: l^2(\mathbf{Z}_4) \rightarrow l^2(\mathbf{Z}_4)$ by
- $$T(z) = (2z(0) - z(1), iz(1) + 2z(2), z(1), 0).$$
- Is T linear? Is T translation invariant? Calculate $T(R_1 z)$ and $R_1(Tz)$ for $z = (1, 0, -2, i)^T$.
- 1.8. Let S, T be two translation-invariant linear mappings of $l^2(\mathbf{Z}_N)$ into itself. Prove that they commute, i.e., that $T \circ S = S \circ T$. Deduce that circulant matrices always commute—try also to prove this fact by a direct calculation using the definition of circulant matrices.
- 1.9. Prove that $T(z * w) = T(z) * w = z * T(w)$ if T is a translation-invariant linear mapping of $l^2(\mathbf{Z}_N)$ into itself.
- 1.10. Prove that a linear mapping T of $l^2(\mathbf{Z}_N)$ into itself is translation invariant if and only if it is a polynomial in the translation operator R_1 .