UPPSALA UNIVERSITET
Matematiska institutionen
Christer Kiselman, 0708-870708

Prov i matematik
Teknisk fysik, B
Transformer för beräkningar
2001 05 21

Skrivtid: 9:00–14:00.
Tillåtna hjälpmedel: Skrivdon. Ordlista 2001 04 05. Formelsamling 2001 05 06. Räknare. Ej mobiltelefon.
Svara på svenska eller annat språk.

1. Let a and b be two vectors in $l^2(\mathbf{Z}_3)$, a being given by its Fourier transform $\hat{a} = (7, 0, 8)$, and $b = (\omega^2, \omega, 1)$, where $\omega = e^{-2\pi i/3} = -\frac{1}{2} - \frac{i}{2}\sqrt{3}$.

(a) Calculate the Fourier transform $\hat{b}$. (3)

(b) Calculate the convolution product $a * b$. (3)

2. We consider two black boxes B and C, defined by vectors $b, c \in l^2(\mathbf{Z}_4)$ and the relations $u = b * z$ and $v = c * z$ between the input signal z and the output signals u and v. We take $b = (1, 0, 1, 0)$ and $c = (1, 2, 2, 1)$.

(a) Prove that knowledge of u alone is not sufficient to recover z; prove also that knowledge of v alone is not sufficient to recover z. (3)

(b) Prove that knowledge of both u and v is indeed enough to reconstruct the input signal z. Give an explicit formula for $\hat{z}$ in terms of $\hat{u}$ and $\hat{v}$. (3)

3. Let $u = (7, 0, 8)$, $v = (0, 1, 2)$, and $z = (7, 0, 0, 1, 8, 2)$. Compute $\hat{u}$, $\hat{v}$ and then $\hat{z}$ using the fast Fourier transformation. (5)

4. Let $u, v \in l^2(\mathbf{Z}_8)$ be the vectors

$$u = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0, 0, 0, 0), \qquad v = (1/\sqrt{2}, -1/\sqrt{2}, 0, 0, 0, 0, 0, 0).$$

(The first-stage Haar basis.)

(a) Prove that the vectors $R_{2k}u, R_{2k}v$, $k = 0, 1, 2, 3$, form an orthonormal system in $l^2(\mathbf{Z}_8)$. (2)

(b) Define

$$P(z) = \sum_{k=0}^{3} \langle z, R_{2k}u \rangle R_{2k}u, \qquad Q(z) = \sum_{k=0}^{3} \langle z, R_{2k}v \rangle R_{2k}v.$$

Calculate $P(z)$ and $Q(z)$ when $z = (1, 3, 5, 3, 3, 7, 9, 1)$. (3)

5. We consider a convolution equation of the second degree, $z * z = z$ for vectors $z = (z(j))_{j \in \mathbf{Z}}$.

(a) Find all solutions of this equation in $l^1(\mathbf{Z})$. (3)

(b) Prove that the equation has infinitely many solutions in $l^2(\mathbf{Z})$. (It would be nice if you could define infinitely many solutions explicitly.) (3)

6. Consider the Poisson equation with boundary conditions zero

$$\begin{aligned} &u'' = f, \qquad 0 < x < 1, \\ &u(0) = u(1) = 0, \end{aligned}$$

and the following finite difference approximation

$$\begin{aligned} &u_{k+1} - 2u_k + u_{k-1} = h^2 f_k, \qquad k = 1, ..., N-1, \\ &u_0 = u_N = 0. \end{aligned}$$

Here $f_k = f(x_k)$, $x_k = kh$, and $h = 1/N$ for some positive integer N.

(a) Use the familiar relation

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

to prove that

$$\mathrm{DST}\big[(u_{k+1} - 2u_k + u_{k-1})_k\big](j) = 2\Big(\cos\frac{j\pi}{N} - 1\Big)\mathrm{DST}[u](j),$$

where DST denotes the discrete sine transform, viz.

$$\mathrm{DST}[z](j) = \sum_{k=1}^{N-1} z(k) \sin\frac{jk\pi}{N}, \qquad j = 1, ..., N-1.$$

The inverse transform is given by

$$z(k) = \frac{2}{N} \sum_{j=1}^{N-1} \mathrm{DST}[z](j) \sin\frac{jk\pi}{N}, \qquad k = 0, ..., N.$$

(3)

(b) Explain how this relation can be used to implement a fast Poisson solver. (3)

7. Define $f\colon \mathbf{R}^2 \to \mathbf{R}$ by $f(x) = 1$ when $|x_1| < 1$ and $|x_2| < 1$; $f(x) = 0$ otherwise. Calculate the Radon transform $\varphi(\omega, p) = \mathcal{R}f(\omega, p)$ of f for arbitrary $p \in \mathbf{R}$ and a particular choice of ω, viz. $\omega = (1/\sqrt{2}, 1/\sqrt{2}) \in S^1$. (6)

Svar till tentamen i Transformer för beräkningar 2001 05 21

1. We find that $\hat{b} = (0, 3\omega^2, 0) = (0, -3 - 3\omega, 0) = (0, -\frac{3}{2} + \frac{3}{2}\sqrt{3}\,i, 0)$, so that $\hat{a}\hat{b} = 0$. Therefore also $a * b = 0$, since its Fourier transform is $\hat{a}\hat{b}$.

2. We find $\hat{b} = (2, 0, 2, 0)$ and $\hat{c} = (6, -1 - i, 0, -1 + i)$, so that $u = b * z = 0$ as soon as $\hat{z}(0) = \hat{z}(2) = 0$ ($\hat{z}(1)$ and $\hat{z}(3)$ being arbitrary), while $v = c * z = 0$ as soon as $\hat{z}(0) = \hat{z}(1) = \hat{z}(3) = 0$ ($\hat{z}(2)$ being arbitrary). The reconstruction of z follows from the formulas $\hat{z}(0) = \frac{1}{2}\hat{u}(0)$, $\hat{z}(1) = \hat{v}(1)/(-1 - i) = \frac{1}{2}(-1 + i)\hat{v}(1)$, $\hat{z}(2) = \frac{1}{2}\hat{u}(2)$, $\hat{z}(3) = \hat{v}(3)/(-1 + i) = \frac{1}{2}(-1 - i)\hat{v}(3)$.

3. We find $\hat{u} = (15, 7+8\omega^2, 7+8\omega) = (15, -1-8\omega, 7+8\omega)$ and $\hat{v} = (3, \omega+2\omega^2, 2\omega+\omega^2) = (3, -2-\omega, -1+\omega)$, where $\omega = e^{-2\pi i/3} = -\frac{1}{2} - \frac{i}{2}\sqrt{3}$. With the fast Fourier transformation we the obtain

$$\begin{aligned}
\hat{z}(0) &= \hat{u}(0) + \hat{v}(0) = 18,\\
\hat{z}(1) &= \hat{u}(1) + \theta\hat{v}(1) = -2 - 10\omega = 3 + 5\sqrt{3}\,i,\\
\hat{z}(2) &= \hat{u}(2) + \theta^2\hat{v}(2) = 6 + 6\omega = 3 - \sqrt{3}\,i,\\
\hat{z}(3) &= \hat{u}(0) - \hat{v}(0) = 12,\\
\hat{z}(4) &= \hat{u}(1) - \theta\hat{v}(1) = -6\omega = 3 + 3\sqrt{3}\,i,\\
\hat{z}(5) &= \hat{u}(2) - \theta^2\hat{v}(2) = 8 + 10\omega = 3 - 5\sqrt{3}\,i,
\end{aligned}$$

where $\theta = e^{-\pi i/3} = -\omega^2 = 1 + \omega$, $\theta^2 = \omega$, $\theta^3 = -1$, $\theta^4 = \omega^2 = -1 - \omega$, $\theta^5 = -\omega$. So $\hat{z} = (18, -2-10\omega, 6+6\omega, 12, -6\omega, 8+10\omega) = (18, 3+5\sqrt{3}\,i, 3-\sqrt{3}\,i, 12, 3+3\sqrt{3}\,i, 3-5\sqrt{3}\,i)$. (Of course it is also possible to calculate $\hat{z}$ directly.)

4. To prove (a) is routine. To prove (b): we know that (or just calculate) that $P(z)$ is given by the mean values of the numbers taken two and two, so that $P(z) = (2, 2, 4, 4, 5, 5, 5, 5)$, while $Q(z)$ is the information needed to pass from $P(z)$ to z, i.e., $Q(z) = z - P(z) = (-1, 1, 1, -1, -2, 2, 4, -4)$.

5. (a) Taking the Fourier transform we see that $\hat{z}(t)^2 = \hat{z}(t)$ so that $\hat{z}(t) = 0$ or $\hat{z}(t) = 1$ for each $t \in [0, 2\pi[$. Since $\hat{z}$ is continuous, the only solutions are $\hat{z}(t) = 0$ for all t and $\hat{z}(t) = 1$ for all t, corresponding to $z = 0$ and $z = \delta$.

(b) When z is allowed to lie in the larger space $l^2(\mathbf{Z})$, we can for instance take f as the characteristic function of an interval I, $f = \chi_I$. Then $z = \check{f}$ should satisfy $z * z = z$. Since $\widehat{z * z}$ is problematic, we can argue as follows. Take $f_j \in C^2$ of period 2π such that $f_j = 1$ wherever $f = 1$ and such that $f_j \to f$ in L^2. Then $f_j f = f$ so that $z_j = \check{f}_j$ satisfies $z_j * z = z$. Now $z_j \in l^1(\mathbf{Z})$ and $z_j \to z$ in $l^2(\mathbf{Z})$, so the convolution $z_j * z$ is well-defined in l^2 and $z_j * z \to z * z$ in $l^\infty(\mathbf{Z})$. So $z = z_j * z$ tends to $z * z$, which proves that $z = z * z$. Since different intervals give rise to different z, there are infinitely many solutions z. Explicitly, we may take $f(t) = 1$ for $-a < t < a$ and $f(t) = 0$ for $-\pi \leqslant t \leqslant -a$ and $a \leqslant t < \pi$, where a is any number in $[0, \pi]$. These functions are all in $L^2([-\pi, \pi[)$ and define infinitely many solutions $z = \check{f} \in l^2(\mathbf{Z})$. The inverse Fourier transform of f is $\check{f}(n) = z(n) = \sin(na)/(\pi n)$, $n \neq 0$; $z(0) = a/\pi$. For $a = 0$ we get $z = 0$; for $a = \pi$ we get $z = \delta$. Any a between 0 and π gives a solution in $l^p(\mathbf{Z})$, $p > 1$.

[The following was not required.

(c) Slightly more generally, we may take $z \in l^2(\mathbf{Z})$ such that $\hat{z}$ takes only the values 0 and 1. Then z is a solution. To prove this, take $z_j \in l^1(\mathbf{Z})$ such that $z_j \to z$ in $l^2(\mathbf{Z})$. Then $z_j * z \to z * z$ in $l^\infty(\mathbf{Z})$. Moreover $\widehat{z_j * z} = \hat{z}_j \hat{z} \to \hat{z}^2 = \hat{z}$ in $L^1(]-\pi, \pi])$, which implies that $z_j * z \to z$ in $l^\infty(\mathbf{Z})$. Thus we get even more solutions than those found in (b).

(d) Are there other solutions than those found in (c)? No. Suppose that $z \in l^2(\mathbf{Z})$ is such that $z * z = z$ as sequences in $l^\infty(\mathbf{Z})$. We can then show that $\hat{z}$ takes the values 0 and 1 only. Take again $z_j \in l^1(\mathbf{Z})$ such that $z_j \to z$ in $l^2(\mathbf{Z})$. Then $z_j * z \to z * z = z$ in $l^\infty(\mathbf{Z})$, which implies that $\hat{z}_j \hat{z} \to \hat{z}$ in $\mathcal{D}'(\mathbf{R})$ (the space of distributions). Moreover $\hat{z}_j \to \hat{z}$ in $L^2(]-\pi, \pi])$, which implies that $\hat{z}_j \hat{z} \to \hat{z}^2$ in L^1, hence also in $\mathcal{D}'$. So $\hat{z}^2 = \hat{z}$ as distributions. Since they are functions in L^1, they must be equal as elements of that space, which means that they are equal at almost every point. Hence $\hat{z}^2 = \hat{z}$ almost everywhere: the solutions we found in (c) are all solutions.]

6. (a) The derivation of the symbol is straightforward.

(b) The sine transform of the solution is

$$\mathrm{DST}[u](j) = \frac{h^2 \mathrm{DST}[f](j)}{2(\cos(j\pi/N) - 1)}, \qquad j = 1..., N-1.$$

Thus the solution can be computed by transforming the right hand side $h^2 f$, dividing it by $2(\cos(j\pi/N) - 1)$, and computing the inverse sine transform of the result. Each transform can be computed in $O(N \log N)$ a.o.

7. We see that $\varphi(\omega, p) = 0$ when $|p| \geqslant \sqrt{2}$, for then the line $\omega \cdot x = p$ does not cut the square $|x_j| < 1$. Moreover, it is clear that $\varphi(\omega, 0) = 2\sqrt{2}$, that $\varphi(\omega, \pm\sqrt{2}) = 0$, and that $\varphi(\omega, p)$ is affine in the intervals $[-\sqrt{2}, 0]$, $[0, \sqrt{2}]$. Thus $\varphi(\omega, p) = 2(\sqrt{2} - |p|)$ when $|p| < \sqrt{2}$, and $\mathcal{R}f(\omega, p) = \max(0, 2(\sqrt{2} - |p|))$ for this particular ω. (For any $\omega \in S^1$, $p \mapsto \mathcal{R}f(\omega, p)$ is piecewise affine and with a little more work we can determine it exactly.)