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The Tau-Laws of Fusion

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Abstract

We present complete axiomatizations of weak hypercongruence in the finite fragment of the fusion calculus, an extension and simplification of the π-calculus. We treat both the full fusion calculus and the subcalculus without mismatch operators. The axiomatizations are obtained from the laws for hyperequivalence and adding so called tau-laws. These are similar to the well known tau-laws for CCS and the π-calculus, but there is an interesting difference which highlights an aspect of the higher expressive power of the fusion calculus.

1 Introduction

The fusion calculus [PV98] is an extension of the π-calculus [MPW92], allowing actions with a special kind of side effect. The idea is that these so called *fusion* actions make names identical, and that this fact can be tested by all agents within the scope of the names. Its theory of strong bisimulations (i.e., bisimulations that make no special provisions for an internal action to be "unobservable") has been completely axiomatized in [PV98]. In this paper we shall provide the extra axioms for *weak* bisimulation, and prove completeness for a few different varieties.

As we have demonstrated in our previous papers the fusion calculus gains not only in expressiveness but also in simplicity over π. There is only one scoping operator, and there is a complete duality between input and output actions, neither of which needs to bind names. A typical example is

$$xy\,.\,P \mid \overline{x}z\,.\,Q \mid R \xrightarrow{\{y=z\}} P \mid Q \mid R$$

Here the action prefix xy can be thought of as "receive y along x" and $\overline{x}z$ as "send z along x"; their interaction results in a fusion $\{y = z\}$ affecting all agents in the scope of y and z. In particular, if R is within this scope it can use a match construct $[y = z]$ to test if y and z are equal. For example,

$$(y)(z)(xy\,.\,P \mid \overline{x}z\,.\,Q \mid [y=z]R) \xrightarrow{\mathbf{1}} (y)((P \mid Q \mid [y=z]R)\{y/z\})$$

In the agent to the left R cannot execute since y and z are not the same. The interaction results in the internal action $\mathbf{1}$, and in that y and z are fused (formally, that means they are substituted by the same name). Assume that y and z are *only* used in the match $[y = z]$, then

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the agent to the right can be written $P \mid Q \mid [y = y]R$, and $[y = y]R$ has exactly the actions of R. Thus the fusion affects the rightmost parallel component $[y = z]R$ even though it does not take part in the interaction. In this respect the fusion caclulus is similar to the Chi-calculus [Fu97].

In our previous paper [PV98] we have explored the algebraic theory of bisimulation congruence. A bisimulation here is a binary relation on agents such that if two agents are related and one has an action, then the other has the same action so that the derivatives are again related. The example above makes clear that for an equivalence to be a congruence it must be closed under substitution of names (since an environment of an agent can accomplish a substitution without the agent taking part). Although this is true also for the standard equivalences in the π-calculus the effect is more dramatic in the fusion calculus, where it turns out that this substitution closure is required after every transition. In other words, the bisimulation congruence (called *hyperequivalence*) comes out as the largest bisimulation which is closed under arbitrary substitutions.

The effect on the algebraic theory is perhaps most clearly demonstrated through an example. Consider the law

$$[x \neq y]\alpha \,.\, P \quad = \quad [x \neq y]\alpha \,.\, [x \neq y]P \tag{$*$}$$

This law holds in the π-calculus equivalences: If $x \neq y$ then these names will continue to be distinct within the agent, so inserting an extra test for inequality is harmless. In the fusion calculus this law is invalid since the agent may be a component in a parallel composition where another agent may fuse x and y; so even if $x \neq y$ holds initially it may not hold after α.

In this paper we shall study the theory of *weak* bisimulation equivalence. The main idea is that the internal action $\mathbf{1}$ need not be simulated. This can be expressed formally in different ways, leading to different weak equivalences. One characteristic of the weak hyperequivalence defined in [VP98] is that fusion actions cannot be "observed" as such, although their effects on other agents may be observable. However, the effect of two fusions $\{x = y\}$ and $\{u = v\}$ in sequence is exactly the same as the effect of one polyadic fusion $\{x = y, u = v\}$. Therefore it will hold that

$$\{x = y\} \,.\, \{u = v\} \,.\, P \quad \approx \quad \{x = y\} \,.\, \{u = v\} \,.\, P + \{x = y, u = v\} \,.\, P \tag{$**$}$$

since the effect of the extra summand in the right hand side is simulated by the left hand side performing two fusions in sequence.

Algebraic laws for observation equivalence were first presented in [Mil80], and the first completeness proof for weak bisimulation equivalence is by Hennessy and Milner [HM85]. For the π-calculus weak early and late bisimulation have been axiomatized by Lin [Lin95a]. Axiomatizations are usually formulated in a set of so called tau-laws, τ being the name of the unobservable action in CCS. We will in this paper keep the by now well established epithet "tau-law" even though the unobservable action in the fusion calculus is denoted $\mathbf{1}$. Milner's original tau-laws can thus be written

$$\begin{array}{lrcl}
\mathbf{T1} & \alpha \,.\, \mathbf{1} \,.\, P & = & \alpha \,.\, P \\
\mathbf{T2} & P + \mathbf{1} \,.\, P & = & \mathbf{1} \,.\, P \\
\mathbf{T3} & \alpha \,.\, (P + \mathbf{1} \,.\, Q) & = & \alpha \,.\, (P + \mathbf{1} \,.\, Q) + \alpha \,.\, Q
\end{array}$$

The main result in this paper is to give a complete axiomatization of weak hypercongruence in the fusion calculus. It might be expected that adding the three laws **T1**-**T3** to an axiomatization of hyperequivalence would be enough, just as in CCS and in the π-calculus. However this turns out not to be the case, for two independent reasons.

The first reason has to do with the mismatch operator $[x \neq y]P$. In the fusion calculus we have fewer laws for it since $(*)$ above does not hold. So it turns out that we actually need a stronger version of **T3**, involving an arbitrary sequence $\tilde{M}$ of mismatches

$$\textbf{T3a} \qquad \alpha\,.\,(P + \tilde{M}\mathbf{1}\,.\,Q) \quad = \quad \alpha\,.\,(P + \tilde{M}\mathbf{1}\,.\,Q) + \tilde{M}\alpha\,.\,Q$$

In the π-calculus all instances of **T3a** are derivable from the other axioms including **T3**; this is not the case in the fusion calculus.

The second reason is that weak hyperequivalence allows a fusion to be simulated by several smaller fusions with the same combined effect, as in $(**)$ above. None of **T1**-**T3** caters for this. So we need an additional law. As in **T3a** it needs a sequence of mismatches.

$$\textbf{T3b} \qquad \varphi\,.\,(P + \tilde{M}\psi\,.\,Q) \quad = \quad \varphi\,.\,(P + \tilde{M}\psi\,.\,Q) + \tilde{M}(\varphi \wedge \psi)\,.\,Q$$

Here φ and ψ are fusion actions, $\varphi \wedge \psi$ is a fusion with the same effect as φ and ψ combined, and a side condition says that if $\tilde{M} \Rightarrow x \neq y$ then x and y may not be fused by φ (otherwise the law would be unsound).

The rest of the paper is organized as follows. In Section 2 we recapitulate the syntax and semantics of the fusion calculus, and in Section 3 the definition of hyperequivalence and its algebraic theory. The paper is formally self contained but a reader is referred to our previous papers [PV98, VP98] for explanations and motivations. In Section 4 we recall the definition of weak hyperbisimulation, and there the original contribution of the present paper starts. We define weak hypercongruence and prove it is the largest congruence in weak hyperequivalence (this is analogous to observation congruence being the largest congruence in observation equivalence). We then show that **T1**, **T2**, **T3a** and **T3b** yield a complete axiomatization. In Section 5 we consider the subcalculus without mismatch and show that simpler versions of **T3a** and **T3b**, without the mismatch sequences, suffice for completeness. Finally in Section 6 we characterize the equivalence obtained by omitting **T3b**. Although this turns out to be finer than the weak barbed congruence it may hold interest since its algebraic theory is closer to the weak equivalences in the π-calculus.

2 Syntax and Semantics

We assume an infinite set $\mathcal{N}$ of *names* ranged over by $u, v, \ldots, z$. We write $\tilde{x}$ for a (possibly empty) finite sequence $x_1 \cdots x_n$ of names. φ ranges over total equivalence relations over $\mathcal{N}$ (i.e. equivalence relations with $\mathrm{dom}(\varphi) = \mathcal{N}$) with only finitely many non-singular equivalence classes. If x and y are related by φ we write $x\ \varphi\ y$. We write $\{\tilde{x} = \tilde{y}\}$ to mean the smallest such equivalence relation relating each x_i with y_i, and write $\mathbf{1}$ for the identity relation. As a consequence, a fusion written $\{x = x\}$ is the same as $\{y = y\}$, namely $\mathbf{1}$, and $\{x = y, x = z\}$ is the same as $\{x = y, y = z\}$.

Definition 1 *The* free actions, *ranged over by* α, *and the* agents, *ranged over by* $P, Q, \ldots$, *are defined by*

$$\begin{array}{llllll} \alpha ::= & u\tilde{x} & \text{(Input)} & P ::= & \mathbf{0} & \text{(Inaction)} \\ & \overline{u}\tilde{x} & \text{(Output)} & & \alpha \,.\, Q & \text{(Prefix)} \\ & \varphi & \text{(Fusion)} & & Q + R & \text{(Summation)} \\ & & & & Q \mid R & \text{(Composition)} \\ & & & & (x)Q & \text{(Scope)} \\ & & & & [x = y]Q & \text{(Match)} \\ & & & & [x \neq y]Q & \text{(Mismatch)} \end{array}$$

Input and output actions are collectively called *communication* actions. In these, the names $\tilde{x}$ are the *objects* of the action, and the name u is the *subject*. We write a to stand for either u or $\overline{u}$, thus $a\tilde{x}$ is the general form of a communication action. Fusion actions have neither subject nor objects.

We often omit a trailing $\mathbf{0}$ and write α for $\alpha \,.\, \mathbf{0}$ if no confusion can arise. The name x is said to be *bound* in $(x)P$. We write $(\tilde{x})P$ for $(x_1)\cdots(x_n)P$. The *free names* in P, denoted $\mathrm{fn}(P)$, are the names in P with a non-bound occurrence; here the names occurring in the fusion φ are defined to be the names in the non-singular equivalence classes, i.e., in the relation $\varphi - \mathbf{1}$. Notice that since the syntax $\{x = x\}$ means $\mathbf{1}$, the free names of that fusion do not include x, but is the empty set. As usual we will not distinguish between alpha-variants of agents, i.e., agents differing only in the choice of bound names. We use M, N to stand for a match or a mismatch operator, and write "match sequence" for a sequence of match *and* mismatch operators, ranged over by $\tilde{M}$, $\tilde{N}$, and we say that $\tilde{M}$ *implies* $\tilde{N}$, written $\tilde{M} \Rightarrow \tilde{N}$, if the conjunction of all matches and mismatches in $\tilde{M}$ logically implies all elements in $\tilde{N}$, and that $\tilde{M} \Leftrightarrow \tilde{N}$ if $\tilde{M}$ and $\tilde{N}$ imply each other. We write $\sum_{i \in I} P_i$ for finite general summation, $P_1 + \cdots + P_n$.

The action of a transition may be free or bound:

Definition 2 *The* actions*, ranged over by γ, consist of the fusion actions and of communication actions of the form $(z_1)\cdots(z_n)a\tilde{x}$ (written $(\tilde{z})a\tilde{x}$), where $n \geq 0$ and all names in $\tilde{z}$ are also in $\tilde{x}$. If $n > 0$ we say it is a* bound *action.*

In the bound actions above, $\tilde{z}$ are the *bound objects* and the names in $\tilde{x}$ that are not in $\tilde{z}$ are the *free objects*. Free actions have no bound objects. We further write $\mathrm{n}(\gamma)$ to mean all names occurring in γ (i.e., also including the subject of communication actions and the names in non-singular equivalence classes in fusion actions).

For convenience we define $\varphi \backslash z$ to mean $\varphi \cap (\mathcal{N} - \{z\})^2 \cup \{(z, z)\}$, i.e., the equivalence relation φ with all references to z removed (except for the identity). For example, $\{x = z, z = y\} \backslash z = \{x = y\}$, and $\{x = y\} \backslash y = \mathbf{1}$.

We now define a structural congruence which equates all agents we will never want to distinguish for any semantic reason, and then use this when giving the transitional semantics.

Definition 3 *The* structural congruence*, $\equiv$, between agents is the least congruence satisfying the abelian monoid laws for Summation and Composition (associativity, commutativity and $\mathbf{0}$ as identity), and the scope laws*

$$(x)\mathbf{0} \equiv \mathbf{0}, \quad (x)(y)P \equiv (y)(x)P, \quad (x)(P + Q) \equiv (x)P + (x)Q$$
$$(x)MP \equiv M(x)P, \quad \textit{if } x \notin \mathrm{n}(M)$$

and also the scope extension *law $P \mid (z)Q \equiv (z)(P \mid Q)$ where $z \notin \mathrm{fn}(P)$.*

PREF $\dfrac{-}{\alpha\,.\,P \xrightarrow{\alpha} P}$	SUM $\dfrac{P \xrightarrow{\alpha} P'}{P+Q \xrightarrow{\alpha} P'}$	PAR $\dfrac{P \xrightarrow{\alpha} P'}{P \mid Q \xrightarrow{\alpha} P' \mid Q}$
MATCH $\dfrac{P \xrightarrow{\alpha} P'}{[x=x]P \xrightarrow{\alpha} P'}$	MISMATCH $\dfrac{P \xrightarrow{\alpha} P',\ x \neq y}{[x \neq y]P \xrightarrow{\alpha} P'}$	
COM $\dfrac{P \xrightarrow{u\tilde{x}} P',\ \ Q \xrightarrow{\overline{u}\tilde{y}} Q',\ \ \lvert\tilde{x}\rvert = \lvert\tilde{y}\rvert}{P \mid Q \xrightarrow{\{\tilde{x}=\tilde{y}\}} P' \mid Q'}$	SCOPE $\dfrac{P \xrightarrow{\varphi} P',\ z\,\varphi\,x,\ z \neq x}{(z)P \xrightarrow{\varphi\backslash z} P'\{x/z\}}$	
PASS $\dfrac{P \xrightarrow{\alpha} P',\ z \notin \mathrm{n}(\alpha)}{(z)P \xrightarrow{\alpha} (z)P'}$	OPEN $\dfrac{P \xrightarrow{(\tilde{y})a\,\tilde{x}} P',\ z \in \tilde{x} - \tilde{y},\ a \notin \{z, \overline{z}\}}{(z)P \xrightarrow{(z\tilde{y})a\,\tilde{x}} P'}$	

Table 1: Laws of action.

Definition 4 *The family of* transitions $P \xrightarrow{\gamma} Q$ *is the least family satisfying the laws in Table 1. In this definition structurally equivalent agents are considered the same, i.e., if* $P \equiv P'$ *and* $Q \equiv Q'$ *and* $P \xrightarrow{\gamma} Q$ *then also* $P' \xrightarrow{\gamma} Q'$.

3 Hyperequivalence

This section recalls pertinent definitions and results from [PV98].

Definition 5 *A substitution* σ *is* idempotent *if* $\sigma\sigma = \sigma$.

Definition 6 *A substitution* σ agrees with *the fusion* φ *if* $\forall x, y : x\,\varphi\,y \Leftrightarrow \sigma(x) = \sigma(y)$. *A* substitutive effect *of a fusion* φ *is an idempotent substitution* σ *agreeing with* φ *(i.e.,* σ *sends all members of each equivalence class of* φ *to one representative in the class). The only substitutive effect of a communication action is the identity substitution.*

For example, the substitutive effects of $\{x = y\}$ are $\{x/y\}$ and $\{y/x\}$. Note that not all substitutions which agree with φ are substitutive effects of φ. So, e.g., any injective substitution agrees with $\mathbf{1}$, but the only substitutive effect of $\mathbf{1}$ is the identity substitution.

Definition 7 *A* bisimulation *is a binary symmetric relation* $\mathcal{S}$ *between agents such that* $P\ \mathcal{S}\ Q$ *implies:*

If $P \xrightarrow{\gamma} P'$ *with* $\mathrm{bn}(\gamma) \cap \mathrm{fn}(Q) = \emptyset$ *then*
$Q \xrightarrow{\gamma} Q'$ *and* $P'\sigma_\gamma\ \mathcal{S}\ Q'\sigma_\gamma$ *for some substitutive effect* σ_γ *of* γ.

A hyperbisimulation *is a substitution closed bisimulation, i.e., a bisimulation* $\mathcal{S}$ *with the property that* $P\ \mathcal{S}\ Q$ *implies* $P\sigma\ \mathcal{S}\ Q\sigma$ *for any substitution* σ. *Two agents* P *and* Q *are* hyperequivalent, *written* $P \sim Q$, *if they are related by a hyperbisimulation.*

Bisimilarity is closed under injective substitutions, and any substitution can be defined as a composition of an idempotent substitution and an injective substitution. Therefore if only

Summation			
S1	$P + \mathbf{0} = P$		
S2	$P + Q = Q + P$		
S3	$P + (Q + R) = (P + Q) + R$		
Scope			
R0	$(x)\mathbf{0} = \mathbf{0}$		
R1	$(x)(y)P = (y)(x)P$		
R2	$(x)(P + Q) = (x)P + (x)Q$		
Match and Scope			
RM1	$(x)[y = z]P = [y = z](x)P$	if $x \neq y, x \neq z$	

Table 2: Axioms from structural congruence.

idempotent substitutions are considered in Definition 7, the resulting hyperequivalence will be the same. We will often use this fact implicitly in proofs to come.

For the axiomatization of hyperequivalence we subsume the fact that the equivalence is a congruence. We also use some of the laws for structural congruence (see Table 2). The axioms are given in Table 3, and in Table 4 we present some derived rules (whose names start with **D**).

Definition 8 *A substitution σ* agrees with *a match sequence $\tilde{M}$, and $\tilde{M}$ agrees with σ, if for all x, y which appear in $\tilde{M}$ it holds that $\sigma(x) = \sigma(y)$ iff $\tilde{M} \Rightarrow [x = y]$.*

Definition 9 *The* depth *of an agent P, $\mathrm{d}(P)$, is defined inductively as follows:*
$\mathrm{d}(\mathbf{0}) = 0$, $\mathrm{d}(\alpha . P) = 1 + \mathrm{d}(P)$, $\mathrm{d}((\tilde{x})P) = \mathrm{d}(MP) = \mathrm{d}(P)$, $\mathrm{d}(P \mid Q) = \mathrm{d}(P) + \mathrm{d}(Q)$, $\mathrm{d}(P + Q) = \max(\mathrm{d}(P), \mathrm{d}(Q))$.

Definition 10 *A match sequence $\tilde{M}$ is* complete on *a set of names V if for some equivalence relation $\mathcal{R}$ on V, called the equivalence relation* corresponding to *$\tilde{M}$, it holds that $\tilde{M} \Rightarrow [x = y]$ iff $x \mathcal{R} y$, and $\tilde{M} \Rightarrow [x \neq y]$ iff $\neg(x \mathcal{R} y)$*

Lemma 1 [PS95] *Let V be a set of names and let $\tilde{M}$ be complete on V.*

1. *If $\tilde{N}$ is another match sequence with names in V, then either $\tilde{M}\tilde{N}$ is unsatisfiable or $\tilde{M}\tilde{N} \Leftrightarrow \tilde{M}$.*
2. *If $\tilde{N}$ is another match sequence complete on V such that $\tilde{M}$ and $\tilde{N}$ both agree with the same substitution σ, then $\tilde{M} \Leftrightarrow \tilde{N}$.*

Definition 11 *An agent P is in* head normal form (HNF) on *V (a finite set of names) if P is of the form*

$$\sum_{i \in I} \tilde{M}_i(\tilde{x}_i)\alpha_i . P_i$$

where for all i, $\tilde{x}_i \cap V = \emptyset$, $\tilde{x}_i \subseteq \mathrm{obj}(\alpha_i)$ and $\tilde{M}_i$ is complete on V.

For the sake of brevity we will use the derived bound prefix $((\tilde{x})a\tilde{z}) . P$ to mean $(\tilde{x})a\tilde{z} . P$ when $\tilde{x} \subseteq \tilde{z}$ and $a \notin \tilde{x}$, and let γ range over any (free or bound) prefix. A HNF can then be written as a sum of terms of type $\tilde{M}\gamma . P$.

Summation		
S4	$P + P = P$	
Match		
M1	$\tilde{M}P = \tilde{N}P$	if $\tilde{M} \Leftrightarrow \tilde{N}$
M2	$[x = y]P = [x = y](P\{x/y\})$	
M3	$MP + MQ = M(P + Q)$	
M4	$[x \neq x]P = \mathbf{0}$	
M5	$P = [x = y]P + [x \neq y]P$	
Scope		
R3	$(x)\alpha\,.\,P = \alpha\,.\,(x)P$	if $x \notin \mathrm{n}(\alpha)$
R4	$(x)\alpha\,.\,P = \mathbf{0}$	if x is the subject of α
Match and Scope		
RM2	$(x)[x = y]P = \mathbf{0}$	if $x \neq y$
Fusion		
F1	$\varphi\,.\,P = \varphi\,.\,[x = y]P$	if $x\,\varphi\,y$
F2	$(z)\varphi\,.\,P = \varphi\backslash z\,.\,P$	if $z \notin \mathrm{fn}(P)$
Expansion		
E	for $P \equiv \Sigma_i M_i(\tilde{x}_i)\alpha_i.P_i$, $Q \equiv \Sigma_j N_j(\tilde{y}_j)\beta_j.Q_j$, $$P \mid Q \;=\; \sum_i M_i(\tilde{x}_i)\alpha_i\,.\,(P_i \mid Q) + \sum_j N_j(\tilde{y}_j)\beta_j\,.\,(P \mid Q_j) + \sum_{\alpha_i \mathrm{opp} \beta_j} M_i N_j(\tilde{x}_i\,\tilde{y}_j)[u_i = v_j]\{\tilde{z}_i = \tilde{w}_j\}\,.\,(P_i \mid Q_j)$$ where $\alpha_i \mathrm{opp} \beta_j$ means $\alpha_i \equiv \overline{u_i}\tilde{z}_i$ and $\beta_j \equiv v_j\tilde{w}_j$ or vice versa, and $\lvert\tilde{z}_i\rvert = \lvert\tilde{w}_j\rvert$.	

Table 3: Axioms.

Match		
DM1	$[x = x]P = P$	
DM2	$[x = y]\alpha\,.\,P = [x = y]\alpha\,.\,[x = y]P$	
DM3	$\tilde{M}P = \tilde{M}(P\sigma)$	for σ agreeing with $\tilde{M}$
DM4	$M\mathbf{0} = \mathbf{0}$	
DM5	$MP + P = P$	
Match and Scope		
DRM1	$(x)[y \neq z]P = [y \neq z](x)P$	if $x \neq y, x \neq z$
DRM2	$(x)[x \neq y]P = (x)P$	if $x \neq y$
Fusion		
DF1	$\varphi\,.\,P = \varphi\,.\,(P\sigma)$	where σ agrees with φ
DF2	$(z)\varphi\,.\,P = \varphi\backslash z\,.\,(P\{w/z\})$	if $z\varphi w$ and $z \neq w$

Table 4: Derived rules.

Lemma 2 [PV98] *For all agents P and finite V such that $\mathrm{fn}(P) \subseteq V$, there is an agent H such that $\mathrm{d}(H) \leq \mathrm{d}(P)$, H is in HNF on V, and $\vdash P = H$ from the axioms of tables 2 and 3.*

Theorem 3 [PV98] *$P \sim Q$ iff $\vdash P = Q$ from the axioms of tables 2 and 3.*

4 Weak hypercongruence

The definition of weak hyperequivalence is from [VP98]. The original contribution in this paper begins with the definition of weak hypercongruence and its axiomatization.

4.1 Definitions

Definition 12 *Define the* composition *of two transitions, $\circ$, by $P(\xrightarrow{\gamma} \circ \xrightarrow{\gamma'})Q$ iff there exists an agent P' such that $P \xrightarrow{\gamma} P'$ and $P'\sigma_\gamma \xrightarrow{\gamma'} Q$, where σ_γ is a substitutive effect of γ. Define the* conjunction *of two fusions φ and ψ, written $\varphi \wedge \psi$, to be the least equivalence relation containing φ and ψ, i.e., $(\varphi \cup \psi)^*$. Define the* weak transition $\stackrel{\gamma}{\Longrightarrow}$ *by the following: $P \stackrel{\gamma}{\Longrightarrow} Q$ means that for some $n \geq 0$, $P \xrightarrow{\gamma_1} \circ \cdots \circ \xrightarrow{\gamma_n} Q$ and either of*

1. *γ is a communication and $\gamma = \gamma_i$ for some i and $\gamma_j = \mathbf{1}$ for all $j \neq i$, or*
2. *γ and all γ_i are fusions and $\gamma = \gamma_1 \wedge \cdots \wedge \gamma_n$. Here we allow $n = 0$ where the empty conjunction is $\mathbf{1}$, in other words $P \stackrel{\mathbf{1}}{\Longrightarrow} P$ holds for all P.*

Definition 13 *A* weak bisimulation *is a binary symmetric relation $\mathcal{S}$ between agents such that $P \,\mathcal{S}\, Q$ implies:*

If $P \xrightarrow{\gamma} P'$ with $\mathrm{bn}(\gamma) \cap \mathrm{fn}(Q) = \emptyset$ then
$Q \stackrel{\gamma}{\Longrightarrow} Q'$ and $P'\sigma_\gamma \,\mathcal{S}\, Q'\sigma_\gamma$ for some substitutive effect σ_γ of γ.

A weak hyperbisimulation *is a substitution closed weak bisimulation. Two agents P and Q are* weakly hyperequivalent, *written $P \approx Q$, if they are related by a weak hyperbisimulation.*

Definition 14 *Define $P \stackrel{\gamma}{\Longrightarrow}_+ Q$ to mean $P \stackrel{\gamma}{\Longrightarrow} Q$ if $\gamma \neq \mathbf{1}$, and $P \xrightarrow{\mathbf{1}}\stackrel{\gamma}{\Longrightarrow} Q$ if $\gamma = \mathbf{1}$. Two agents P and Q are* weakly hypercongruent, *written $P \approx_+ Q$, iff for any substitution σ*

$$P\sigma \xrightarrow{\gamma} P' \text{ and } \mathrm{bn}(\gamma) \cap \mathrm{fn}(Q) = \emptyset \text{ implies } Q\sigma \stackrel{\gamma}{\Longrightarrow}_+ Q' \text{ and } P'\sigma_\gamma \approx Q'\sigma_\gamma$$

where σ_γ is a substitutive effect of γ, and vice versa.

As for the strong hyperequivalence it suffices to consider idempotent substitutions in Definition 13 and Definition 14.

Proposition 4 *Weak hypercongruence is the largest congruence in weak hyperequivalence.*
PROOF: Very much as for the corresponding result in [Mil89], pages 153–154. □

T1	$\alpha\,.\,\mathbf{1}\,.\,P = \alpha\,.\,P$	
T2	$P + \mathbf{1}\,.\,P = \mathbf{1}\,.\,P$	
T3a	$\alpha\,.\,(P + \tilde{M}\mathbf{1}\,.\,Q) = \alpha\,.\,(P + \tilde{M}\mathbf{1}\,.\,Q) + \tilde{M}\alpha\,.\,Q$	if α is a communication
T3b	$\varphi\,.\,(P + \tilde{M}\psi\,.\,Q) = \varphi\,.\,(P + \tilde{M}\psi\,.\,Q) + \tilde{M}(\varphi \wedge \psi)\,.\,Q$	if $\forall x, y : \tilde{M} \Rightarrow x \neq y$ implies $\neg x\varphi y$

Table 5: Axioms for weak hypercongruence

4.2 Axiomatization

Table 5 contains the additional axioms for weak hypercongruence. Let $\mathcal{W}$ be the axioms in tables 2,3 and 5. We write $\vdash_{\mathcal{W}} P = Q$ if P and Q can be proven equal from $\mathcal{W}$.

Axioms **T1** and **T2** are direct counterparts of the familiar two first "tau-laws" from Milner. The third law, which in Milner reads $\alpha\,.\,(P + \tau\,.\,Q) = \alpha\,.\,(P + \tau\,.\,Q) + \alpha\,.\,Q$, needs more care. Here, in **T3a** and **T3b** we need to distinguish between communication and fusion actions, and in both these cases a sequence $\tilde{M}$ appears. Note the condition in **T3b** which forbids e.g. $\varphi = \{x = y\}$ and $\tilde{M} = [x \neq y]$. Without this condition the law would be invalid.

T3a generalizes to bound prefixes. For any free or bound prefix γ with $\mathrm{bn}(\gamma) \cap \mathrm{fn}(\tilde{M}) = \emptyset$ we can use **R2, RM1, DRM1** to derive the more general form of **T3a** where a bound communication prefix replaces α.

Proposition 5 *The axioms in Table 5 are sound for* $\approx_+$.
PROOF: Directly from the definition of $\approx_+$. For **T3b**, note that if $\tilde{M}\sigma$ is true then so is $\tilde{M}\sigma\sigma_\varphi$, for σ_φ a substitutive effect of φ, because of the side condition. □

The completeness proof stretches over several lemmas. In the following, σ will range over idempotent substitutions.

Lemma 6 *If* $P\sigma \xrightarrow{\gamma} P'$ *then* $P' \equiv P'\sigma$.
PROOF: By alpha-conversion we can assume γ does not bind names in $\mathrm{dom}(\sigma)$. By induction over transitions it is easy to establish that $\mathrm{fn}(P') \subseteq \mathrm{fn}(P\sigma) \cup \mathrm{bn}(\gamma)$. So $\mathrm{fn}(P') \subseteq \mathrm{ran}(\sigma) \cup \mathrm{bn}(\gamma)$ and therefore $\sigma(x) = x$ for all names in $\mathrm{fn}(P')$ and the result follows. □

Lemma 7 *Let P be in HNF on V, where* $\mathrm{fn}(P) \subseteq V$. *If* $P\sigma \xrightarrow{\gamma} P'$ *then* $\vdash_{\mathcal{W}} P = P + \tilde{M}\gamma\,.\,P'$ *where $\tilde{M}$ agrees with σ and is complete on V.*
PROOF: Since P is in HNF it has a summand $\tilde{N}\gamma'\,.\,Q$ such that $(\tilde{N}\gamma'\,.\,Q)\sigma \xrightarrow{\gamma} P'$, for $\tilde{N}$ complete on V and agreeing with σ. So $\gamma = \gamma'\sigma$ and $P' = Q\sigma$. So by **DM3**, $\vdash_{\mathcal{W}} \tilde{N}\gamma'\,.\,Q = \tilde{N}(\gamma'\,.\,Q)\sigma \equiv \tilde{N}\gamma\,.\,P'$. Now $\tilde{M}$ and $\tilde{N}$ are complete on V and agree with σ, so $\tilde{M} \Leftrightarrow \tilde{N}$. So by **M1**, $\vdash_{\mathcal{W}} \tilde{N}\gamma\,.\,P' = \tilde{M}\gamma\,.\,P'$. So by **S4**, $\vdash_{\mathcal{W}} P = P + \tilde{M}\gamma\,.\,P$. □

Lemma 8 (Saturation lemma) *Let P be in HNF on V, where* $\mathrm{fn}(P) \subseteq V$. *If* $P\sigma \overset{\gamma}{\Longrightarrow}_+ P'$ *then* $\vdash_{\mathcal{W}} P = P + \tilde{M}\gamma\,.\,P'$ *where $\tilde{M}$ agrees with σ and is complete on V.*
PROOF: By induction on the depth of P. There are four cases for $P\sigma \overset{\gamma}{\Longrightarrow}_+ P'$, the first of which also covers the base of the induction.

Case 1 $P\sigma \xrightarrow{\gamma} P'$. The result is immediate from Lemma 7.

Case 2 $P\sigma \xrightarrow{\gamma} Q$ and $Q \overset{\mathbf{1}}{\Longrightarrow}_+ P'$, where γ is a communication (γ a fusion is handled by Case 4 below). Let $\tilde{M}$ agree with σ and be complete on V. By Lemma 7 we get $\vdash_{\mathcal{W}} P = P + \tilde{M}\gamma . Q$. By alpha-conversion we can assume $\mathrm{fn}(\tilde{M}) \cap \mathrm{bn}(\gamma) = \emptyset$. By Lemma 6 $Q \equiv Q\sigma$. So $Q\sigma \overset{\mathbf{1}}{\Longrightarrow}_+ P'$. By induction then $\vdash_{\mathcal{W}} Q = Q + \tilde{N}\mathbf{1} . P'$. Since $\tilde{M}$ and $\tilde{N}$ are complete and agree with σ we have $\tilde{M} \Leftrightarrow \tilde{N}$. So by **M1**, $\vdash_{\mathcal{W}} Q = Q + \tilde{M}\mathbf{1} . P'$. In summary,

$$\begin{array}{rcll}
\vdash_{\mathcal{W}} P & = & P + \tilde{M}\gamma . (Q + \tilde{M}\mathbf{1} . P') & (*) \\
& \overset{\mathbf{T3a}}{=} & P + \tilde{M}(\gamma . (Q + \tilde{M}\mathbf{1} . P') + \tilde{M}\gamma . P') & \\
& \overset{\mathbf{M3}}{=} & P + \tilde{M}\gamma . (Q + \tilde{M}\mathbf{1} . P') + \tilde{M}\tilde{M}\gamma . P' & \\
& \overset{\mathbf{M1}}{=} & P + \tilde{M}\gamma . (Q + \tilde{M}\mathbf{1} . P') + \tilde{M}\gamma . P' & \\
& \overset{(*)}{=} & P + \tilde{M}\gamma . P' &
\end{array}$$

as required.

Case 3 $P\sigma \xrightarrow{\mathbf{1}} Q$ and $Q \overset{\gamma}{\Longrightarrow}_+ P'$, where γ is a communication (γ a fusion is handled by Case 4 below). Let $\tilde{M}$ agree with σ and be complete on V. By Lemma 7 we get $\vdash_{\mathcal{W}} P = P + \tilde{M}\mathbf{1} . Q$. By Lemma 6 $Q \equiv Q\sigma$. So $Q\sigma \overset{\gamma}{\Longrightarrow}_+ P'$. By induction then $\vdash_{\mathcal{W}} Q = Q + \tilde{N}\gamma . P'$. Since $\tilde{M}$ and $\tilde{N}$ are complete and agree with σ we have $\tilde{M} \Leftrightarrow \tilde{N}$. So by **M1**, $\vdash_{\mathcal{W}} Q = Q + \tilde{M}\gamma . P'$. In summary,

$$\begin{array}{rcl}
\vdash_{\mathcal{W}} P & = & P + \tilde{M}\mathbf{1} . (Q + \tilde{M}\gamma . P') \\
& \overset{\mathbf{T2}}{=} & P + \tilde{M}(\mathbf{1} . (Q + \tilde{M}\gamma . P') + Q + \tilde{M}\gamma . P') \\
& \overset{\mathbf{S4}}{=} & P + \tilde{M}(\mathbf{1} . (Q + \tilde{M}\gamma . P') + Q + \tilde{M}\gamma . P' + \tilde{M}\gamma . P') \\
& \overset{\mathbf{T2}}{=} & P + \tilde{M}(\mathbf{1} . (Q + \tilde{M}\gamma . P') + \tilde{M}\gamma . P') \\
& \overset{\mathbf{M3}}{=} & P + \tilde{M}\mathbf{1} . (Q + \tilde{M}\gamma . P') + \tilde{M}\tilde{M}\gamma . P' \\
& \overset{\mathbf{M1}}{=} & P + \tilde{M}\mathbf{1} . (Q + \tilde{M}\gamma . P') + \tilde{M}\gamma . P' \\
& = & P + \tilde{M}\gamma . P'
\end{array}$$

as required.

Case 4 $P\sigma \xrightarrow{\varphi} Q$, and $Q\sigma_\varphi \overset{\psi}{\Longrightarrow}_+ P'$ where σ_φ agrees with φ, and γ is a fusion with $\gamma = \varphi \wedge \psi$. Let $\tilde{M}$ agree with σ and be complete on V. By Lemma 7 we get $\vdash_{\mathcal{W}} P = P + \tilde{M}\varphi . Q$. By Lemma 6 $Q \equiv Q\sigma$. So $Q\sigma\sigma_\varphi \overset{\psi}{\Longrightarrow}_+ P'$. By induction then $Q \vdash_{\mathcal{W}} Q + \tilde{N}\psi . P'$ where $\tilde{N}$ agrees with $\sigma\sigma_\varphi$ and is complete on V. So,

$$\begin{array}{rcl}
\vdash_{\mathcal{W}} P & = & P + \tilde{M}\varphi . (Q + \tilde{N}\psi . P') \\
& \overset{(*)}{=} & P + \tilde{M}\varphi . (Q + \tilde{N}\psi . P') + \tilde{M}(\varphi \wedge \psi) . P' \\
& = & P + \tilde{M}(\varphi \wedge \psi) . P' \\
& = & P + \tilde{M}\gamma . P'
\end{array}$$

as required, where there remains to prove $(*)$. Let F be a sequence of matches corresponding to the fusion φ, i.e. $F \Rightarrow [x = y]$ iff $x \,\varphi\, y$. Let $\tilde{M}'$ be formed from $\tilde{M}$ by removing every mismatch $[x \neq y]$ where $x \,\varphi\, y$. It follows that $\tilde{M}'F \Leftrightarrow \tilde{N}$ since both are complete on V and agree with $\sigma\sigma_\varphi$. Now,

$$\begin{array}{rcl}
\vdash_{\mathcal{W}} \varphi\,.\,(Q+\tilde{N}\psi\,.\,P') & \stackrel{\mathbf{M1}}{=} & \varphi\,.\,(Q+\tilde{M}'F\psi\,.\,P') \\
& \stackrel{\mathbf{DF1,DM1}}{=} & \varphi\,.\,(Q+\tilde{M}'\psi\,.\,P') \\
& \stackrel{\mathbf{T3b}}{=} & \varphi\,.\,(Q+\tilde{M}'\psi\,.\,P')+\tilde{M}'(\varphi\wedge\psi)\,.\,P' \\
& = & \varphi\,.\,(Q+\tilde{N}\psi\,.\,P')+\tilde{M}'(\varphi\wedge\psi)\,.\,P'
\end{array}$$

Note that the side condition in **T3b** is fulfilled by construction of $\tilde{M}'$. Therefore,

$$\begin{array}{rcl}
\vdash_{\mathcal{W}} \tilde{M}\varphi\,.\,(Q+\tilde{N}\psi\,.\,P') & = & \tilde{M}(\varphi\,.\,(Q+\tilde{N}\psi\,.\,P')+\tilde{M}'(\varphi\wedge\psi)\,.\,P') \\
& \stackrel{\mathbf{M3}}{=} & \tilde{M}\varphi\,.\,(Q+\tilde{N}\psi\,.\,P')+\tilde{M}\tilde{M}'(\varphi\wedge\psi)\,.\,P' \\
& \stackrel{\mathbf{M1}}{=} & \tilde{M}\varphi\,.\,(Q+\tilde{N}\psi\,.\,P')+\tilde{M}(\varphi\wedge\psi)\,.\,P'
\end{array}$$

as required, where the last step follows since all matches/mismatches in $\tilde{M}'$ are also in $\tilde{M}$.

This completes Case 4 and the proof of the lemma. □

Definition 15 *A HNF P on V is called a* full *HNF on V if it has all summands implied by Lemma 8.*

Proposition 9 *For any HNF on V there is a provably equivalent full HNF on V of no greater depth.*
Proof: Just apply Lemma 8 repeatedly. Eventually the HNF becomes a full HNF since there are only a finite number of derivatives and a finite number of substitutions on V to consider. □

Lemma 10 *$P \approx Q$ iff ($P \approx_+ Q$ or $P \approx_+ \mathbf{1}\,.\,Q$ or $\mathbf{1}\,.\,P \approx_+ Q$).*
Proof: Precisely as in [Mil89], Proposition 11 on page 156–157. □

Theorem 11 *$P \approx_+ Q$ iff $\vdash_{\mathcal{W}} P = Q$.*
Proof: That $\vdash_{\mathcal{W}} P = Q$ implies $P \approx_+ Q$ is easily verified, and we concentrate on the other implication. The proof is by induction on the sum of the depths of P and Q. We can assume that P and Q are full HNFs on V, for $\mathrm{fn}(P)\cup\mathrm{fn}(Q)\subseteq V$, by Proposition 9. The base case $P\equiv Q\equiv\mathbf{0}$ is trivial. For the inductive step assume $P\approx_+ Q$. Let $\tilde{M}\gamma\,.\,P'$ be a summand of P. By alpha-conversion we can assume that γ does not bind any name in $\tilde{M}$. We shall prove that Q has a provably equivalent summand.

Let σ agree with $\tilde{M}$ and be complete on V. Then

$$P\sigma \xrightarrow{\gamma\sigma} P'\sigma$$

From $P\approx_+ Q$ we get that

$$Q\sigma \stackrel{\gamma\sigma}{\Longrightarrow}_+ Q''$$

where $P\sigma\sigma_\gamma \approx Q''\sigma_\gamma$ for a substitutive effect σ_γ of γ. Since Q is a full HNF, Q must have a summand

$$\tilde{N}\gamma'\,.\,Q'$$

such that $Q'\sigma\sigma_\gamma \equiv Q''\sigma_\gamma$ and $\gamma'\sigma = \gamma\sigma$ and σ agrees with $\tilde{N}$ and $\tilde{N}$ is complete on V. So $\tilde{M} \Leftrightarrow \tilde{N}$ and $P'\sigma\sigma_\gamma \approx Q'\sigma\sigma_\gamma$. Therefore,

$$\vdash_{\mathcal{W}} \tilde{M}\gamma\,.\,P' \overset{\mathbf{DM3}}{=} \tilde{M}(\gamma\,.\,P')\sigma = \tilde{M}(\gamma\,.\,P'\sigma)\sigma \overset{\mathbf{DF1}}{=} \tilde{M}(\gamma\,.\,P'\sigma\sigma_\gamma)\sigma$$

(where the last step is void if γ is a communication). We cannot immediately apply induction to $P'\sigma\sigma_\gamma$ since we only know $P'\sigma\sigma_\gamma \approx Q'\sigma\sigma_\gamma$, and not that they are hypercongruent. So we use Lemma 10 to consider three cases.

Case 1 $P'\sigma\sigma_\gamma \approx_+ Q'\sigma\sigma_\gamma$. Then by induction they are provably equal, so

$$\begin{aligned}\vdash_{\mathcal{W}} \tilde{M}(\gamma\,.\,P'\sigma\sigma_\gamma)\sigma &= \tilde{M}(\gamma\,.\,Q'\sigma\sigma_\gamma)\sigma = \tilde{M}(\gamma\sigma\,.\,Q'\sigma\sigma_\gamma)\sigma \\ &= \tilde{M}(\gamma'\sigma\,.\,Q'\sigma\sigma_\gamma)\sigma \overset{\mathbf{M1}}{=} \tilde{N}(\gamma'\sigma\,.\,Q'\sigma\sigma_\gamma)\sigma = \tilde{N}\gamma'\,.\,Q'\end{aligned}$$

Case 2 $P'\sigma\sigma_\gamma \approx_+ \mathbf{1}\,.\,Q'\sigma\sigma_\gamma$. We can now apply induction because the sum of the depths is one less than that for P and Q. So

$$\vdash_{\mathcal{W}} \tilde{M}(\gamma\,.\,P'\sigma\sigma_\gamma)\sigma = \tilde{M}(\gamma\,.\,\mathbf{1}\,.\,Q'\sigma\sigma_\gamma)\sigma \overset{\mathbf{T1}}{=} \tilde{M}(\gamma\,.\,Q'\sigma\sigma_\gamma)\sigma = \tilde{N}\gamma'\,.\,Q'$$

where the last equality is similar to Case 1 above.

Case 3 $\mathbf{1}\,.\,P'\sigma\sigma_\gamma \approx_+ Q'\sigma\sigma_\gamma$. This is symmetric to Case 2.

We have proved that each summand in P has a provably equal summand in Q. The converse is symmetric. **S1-S4** thus completes the induction and proof of the theorem, giving us $\vdash_{\mathcal{W}} P = Q$. □

5 The subcalculus without mismatch

In this section we consider the calculus without Mismatch, and let $\tilde{M}$ etc. range over sequences of Match operators. From [PV98] we recall that by dropping axioms **M4** and **M5**, and promoting **DM5** to an axiom, we get an axiomatization of hyperequivalence without mismatch. Call the new set of axioms $\mathcal{M}$.

Definition 16 *An agent P is in* mismatch-free head normal form (mHNF) *if P is on the form $\sum_{i\in I} \tilde{M}_i(\tilde{x}_i)\alpha_i\,.\,P_i$, where*

1. *$\forall i : \tilde{x}_i \cap \mathrm{fn}(P) = \emptyset$, and $\tilde{x}_i \subseteq \mathrm{obj}(\alpha_i)$*
2. *if $i \neq j$ then $\tilde{M}_i(\tilde{x}_i)\alpha_i\,.\,P_i \not\succ \tilde{M}_i(\tilde{x}_i)\alpha_i\,.\,P_i + \tilde{M}_j(\tilde{x}_j)\alpha_j\,.\,P_j$*

As in the previous section we use the derived bound prefix, so the terms in a mHNF are on the form $\tilde{M}\gamma\,.\,P$.

Lemma 12 [PV98] *For all agents P there is an agent H such that $\mathrm{d}(H) \leq \mathrm{d}(P)$, H is in mHNF, and $\vdash_{\mathcal{M}} P = H$.*

Theorem 13 [PV98] *If P and Q contain no mismatch operators, then $P \sim Q$ iff $\vdash_{\mathcal{M}} P = Q$.*

Tm3a	$\alpha . (P + \mathbf{1} . Q) = \alpha . (P + \mathbf{1} . Q) + \alpha . Q$
Tm3b	$\varphi . (P + \psi . Q) = \varphi . (P + \psi . Q) + (\varphi \wedge \psi) . Q$

Table 6: **T3**-laws for the calculus without mismatch.

For the weak hypercongruence it turns out that simpler versions of **T3a** and **T3b** suffice: there is no longer a need for the match sequences to be part of the axioms. The simpler versions are called **Tm3a** and **Tm3b** and are given in Table 6.

Let $\mathcal{MW}$ be the axioms **T1**, **T2**, **Tm3a** and **Tm3b** plus the axioms in $\mathcal{M}$. Then $\mathcal{MW}$ is easily seen to be sound for $\approx_+$. Note that α in **Tm3a** can be a fusion (that would be equivalent to **Tm3b** with $\psi = \mathbf{1}$). Again the completeness proof stretches over several lemmas.

Lemma 14 *If $P\sigma \xrightarrow{\gamma} P'$ then $\vdash_{\mathcal{MW}} P = P + \tilde{M}\gamma . P'$ where $\tilde{M}$ agrees with σ.*
PROOF: By Lemma 12 we can assume that P is in mHNF. Suppose $P\sigma \xrightarrow{\gamma} P'$. Let $\tilde{M}$ agree with σ. Then P has a summand $\tilde{N}\gamma' . Q'$ such that $\tilde{M} \Rightarrow \tilde{N}$, this means that $\tilde{M} \Leftrightarrow \tilde{L}\tilde{N}$ for some $\tilde{L}$, and further $\gamma'\sigma = \gamma$ and $Q'\sigma \equiv P'$. So we have:

$$\begin{aligned} \vdash_{\mathcal{MW}} P &\overset{\mathbf{S4}}{=} P + \tilde{N}\gamma' . Q' \overset{\mathbf{DM5}}{=} P + \tilde{N}\gamma' . Q' + \tilde{L}\tilde{N}\gamma' . Q' \\ &\overset{\mathbf{M1}}{=} P + \tilde{N}\gamma' . Q' + \tilde{M}\gamma' . Q' \overset{\mathbf{S4}}{=} P + \tilde{M}\gamma' . Q' \\ &\overset{\mathbf{DM3}}{=} P + \tilde{M}\gamma'\sigma . Q'\sigma = P + \tilde{M}\gamma . P' \end{aligned}$$

□

The proof of the saturation lemma (Lemma 8) does not carry over immediately since it relies on complete match sequences. Without mismatches we instead prove it as follows.

Lemma 15 (Saturation lemma without mismatch) *If $P\sigma \overset{\gamma}{\Longrightarrow}_+ P'$ then $\vdash_{\mathcal{MW}} P = P + \tilde{M}\gamma . P'$ where $\tilde{M}$ agrees with σ.*
PROOF: By induction on the depth of P. There are four cases for $P\sigma \overset{\gamma}{\Longrightarrow}_+ P'$, the first of which also covers the base of the induction.

Case 1 $P\sigma \xrightarrow{\gamma} P'$. The result is immediate from Lemma 14.

Case 2 $P\sigma \xrightarrow{\gamma} Q$ and $Q \overset{\mathbf{1}}{\Longrightarrow}_+ P'$, where γ is a communication (γ a fusion is handled by Case 4 below). Let $\tilde{M}$ agree with σ. By Lemma 14 we get that $\vdash_{\mathcal{MW}} P = P + \tilde{M}\gamma . Q$. By induction and **DM1**, $\vdash_{\mathcal{MW}} Q = Q + \mathbf{1} . P'$. So,

$$\begin{aligned} \vdash_{\mathcal{MW}} P &= P + \tilde{M}\gamma . (Q + \mathbf{1} . P') \overset{\mathbf{Tm3a}}{=} P + \tilde{M}(\gamma . (Q + \mathbf{1} . P') + \gamma . P') \\ &\overset{\mathbf{M3}}{=} P + \tilde{M}\gamma . (Q + \mathbf{1} . P') + \tilde{M}\gamma . P' = P + \tilde{M}\gamma . P' \end{aligned}$$

Case 3 $P\sigma \xrightarrow{\mathbf{1}} Q$ and $Q \overset{\gamma}{\Longrightarrow}_+ P'$, where γ is a communication (γ a fusion is handled by Case 4 below). Let $\tilde{M}$ agree with σ. By Lemma 14 we get that $\vdash_{\mathcal{MW}} P = P + \tilde{M}\mathbf{1} . Q$. By induction, $\vdash_{\mathcal{MW}} Q = Q + \gamma . P'$. So,

$$\begin{aligned} \vdash_{\mathcal{MW}} P &= P + \tilde{M}\mathbf{1} . (Q + \gamma . P') \overset{\mathbf{T2}}{=} P + \tilde{M}(\mathbf{1} . (Q + \gamma . P') + Q + \gamma . P') \\ &\overset{\mathbf{S4,T2}}{=} P + \tilde{M}(\mathbf{1} . (Q + \gamma . P') + \gamma . P') \overset{\mathbf{M3}}{=} P + \tilde{M}\mathbf{1} . (Q + \gamma . P') + \tilde{M}\gamma . P') \\ &= P + \tilde{M}\gamma . P' \end{aligned}$$

Case 4 $P\sigma \xrightarrow{\varphi} Q$, and $Q\sigma_\varphi \overset{\psi}{\Longrightarrow}_+ P'$ where σ_φ agrees with φ, and γ is a fusion with $\gamma = \varphi \wedge \psi$. Let $\tilde{M}$ agree with σ. By Lemma 14 we get

$$\vdash_{\mathcal{MW}} P = P + \tilde{M}\varphi \,.\, Q \overset{\mathbf{DF1}}{=} P + \tilde{M}\varphi \,.\, Q\sigma_\varphi$$

Induction gives $\vdash_{\mathcal{MW}} Q\sigma_\varphi = Q\sigma_\varphi + \psi \,.\, P'$. So,

$$\begin{array}{rcl}
\vdash_{\mathcal{MW}} P & = & P + \tilde{M}\varphi \,.\, (Q\sigma_\varphi + \psi \,.\, P') \\
 & \overset{\mathbf{Tm3b}}{=} & P + \tilde{M}(\varphi \,.\, (Q\sigma_\varphi + \psi \,.\, P') + (\varphi \wedge \psi) \,.\, P') \\
 & \overset{\mathbf{M3}}{=} & P + \tilde{M}\varphi \,.\, (Q\sigma_\varphi + \psi \,.\, P') + \tilde{M}(\varphi \wedge \psi) \,.\, P' \\
 & = & P + \tilde{M}(\varphi \wedge \psi) \,.\, P' \\
 & = & P + \tilde{M}\gamma \,.\, P'
\end{array}$$

This completes the proof of the lemma. □

For the completeness proof we need a variant of mHNF which uses weak hypercongruence:

Definition 17 *An agent P is in* mismatch-free weak head normal form (mwHNF) *if P is on the form $\sum_{i \in I} \tilde{M}_i(\tilde{x}_i)\alpha_i \,.\, P_i$, where*

1. $\forall i : \tilde{x}_i \cap \mathrm{fn}(P) = \emptyset$, *and* $\tilde{x}_i \subseteq \mathrm{obj}(\alpha_i)$
2. *if* $i \neq j$ *then* $\tilde{M}_i(\tilde{x}_i)\alpha_i \,.\, P_i \not\approx_+ \tilde{M}_i(\tilde{x}_i)\alpha_i \,.\, P_i + \tilde{M}_j(\tilde{x}_j)\alpha_j \,.\, P_j$

Again we use the derived bound prefixes, so the terms in a mwHNF are on the form $\tilde{M}\gamma \,.\, P$.

Definition 18 *A substitution σ* satisfies *a match/mismatch sequence M, written $\sigma \models M$, if for all x, y, $M \Rightarrow x = y$ implies $\sigma(x) = \sigma(y)$ and $M \Rightarrow x \neq y$ implies $\sigma(x) \neq \sigma(y)$.*

Definition 19 *A* full mwHNF *is a mwHNF with the property that if $P\sigma \overset{\gamma}{\Longrightarrow}_+ P'$ then P has a summand $\tilde{M}\gamma' \,.\, Q'$ such that $\sigma \models \tilde{M}$, $\gamma'\sigma = \gamma$ and $Q'\sigma\sigma_\gamma \approx_+ P'\sigma_\gamma$.*

The following lemma and theorem are proved by a simultaneous induction on depth:

Lemma 16 *For all agents P there is an agent H such that $\mathrm{d}(H) \leq \mathrm{d}(P)$, H is in full mwHNF, and $\vdash_{\mathcal{MW}} P = H$.*

Theorem 17 *$P \approx_+ Q$ iff $\vdash_{\mathcal{MW}} P = Q$.*
Proof: Soundness is easily established, and we turn to completeness. We here only outline how the proof differs from previous completeness proofs. It uses induction on the depth of the agents involved. Lemma 16 uses Theorem 17 for agents of strictly smaller depth than the agents in the lemma. Theorem 17 uses Lemma 16 for agents of equal or smaller depth than the agents in the theorem.

For Lemma 16 first apply Lemma 15 repeatedly to P; there are only a finite number of substitutions that matter (those affecting $\mathrm{fn}(P)$) and a finite number of derivatives. The resulting agent P' has all terms implied by Lemma 15. But it may not be in full mwHNF because of condition 2 in the definition of mwHNF. Let Q and R be two summands in P' such that $Q \approx_+ Q + R$. We can then prove $\vdash_{\mathcal{MW}} Q = Q + R$ much as in the proof of Lemma 17 in [PV98]. Thus we can repeatedly remove such summands R from P' until we gain a mwHNF.

Obviously removing summands in that way will preserve the condition in the definition of a full mHNF.

Also the proof of Theorem 17 follows the proof of Theorem 18 in [PV98] closely. The differences here (and also in the proof that $\vdash_{\mathcal{MW}} Q = Q + R$ mentioned above) are the following. Obviously $\approx_+$ replaces $\sim$, and therefore simulating transitions are $\stackrel{\gamma}{\Longrightarrow}_+$, but by saturation we obtain that there are simulating transitions $\stackrel{\gamma}{\longrightarrow}$. Resulting derivatives are related by $\approx$ (rather than $\sim$) and we then use Lemma 10 to consider three cases. Any actions $\mathbf{1}$ introduced by that lemma will disappear by **T1**, just as in the proof of Theorem 3. $\square$

6 A more traditional T3

The law **T3b** (and similarly **Tm3b**) captures the circumstance when two fusions can be simulated by one. In the same way, in CCS and in the π-calculus, the tau-laws say when two actions can be simulated by one. In those calculi one of the actions has to be the unobservable action τ. It is therefore natural to ask what the effect would be if **T3b** were restricted to the case $\psi = \mathbf{1}$. Call this restricted form **T3b$'$**:

$$\textbf{T3b}' \quad \varphi\,.\,(P + \tilde{M}\mathbf{1}\,.\,Q) = \varphi\,.\,(P + \tilde{M}\mathbf{1}\,.\,Q) + \tilde{M}\varphi\,.\,Q \quad \text{if } \forall x, y : \tilde{M} \Rightarrow x \neq y \text{ implies } \neg x\,\varphi\,y$$

This is structurally more similar to **T3a**. In fact, by defining "$x\,\alpha\,y$" to never hold when α is a communication, **T3a** and **T3b$'$** can now easily be formulated as one law:

$$\textbf{T3}' \quad \alpha\,.\,(P + \tilde{M}\mathbf{1}\,.\,Q) = \alpha\,.\,(P + \tilde{M}\mathbf{1}\,.\,Q) + \tilde{M}\alpha\,.\,Q \quad \text{if } \forall x, y : \tilde{M} \Rightarrow x \neq y \text{ implies } \neg x\,\alpha\,y$$

Given the tau-laws of CCS and the π-calculus, the laws **T1**, **T2** and **T3$'$** might be thought of as the "traditional" laws of the fusion calculus. Let $\mathcal{W}'$ consist of these axioms plus the axioms for hyperequivalence in tables 2 and 3. We will here characterize the equivalence generated by $\mathcal{W}'$ coinductively, and demonstrate that it is strictly finer than weak hypercongruence.

For the purpose of this section, make the following changes in the previous definitions: In Definition 12, redefine the *weak transition* $\stackrel{\gamma}{\Longrightarrow}$ by the following:

> $P \stackrel{\gamma}{\Longrightarrow} Q$ means that for some $n \geq 0$, $P \stackrel{\gamma_1}{\longrightarrow} \circ \cdots \circ \stackrel{\gamma_n}{\longrightarrow} Q$ and $\gamma = \gamma_i$ for some i and $\gamma_j = \mathbf{1}$ for all $j \neq i$. We allow $n = 0$ and $\gamma = \mathbf{1}$, in other words $P \stackrel{\mathbf{1}}{\Longrightarrow} P$ holds for all P.

Note that the special case when γ is a fusion has disappeared. This has consequences for the definitions of weak hyperequivalence and weak hypercongruence, which with the amended definition are denoted $\approx'$ and $\approx'_+$. That these are smaller than $\approx$ and $\approx_+$ can be seen with a simple instance of **T3b**:

$$\varphi\,.\,\psi\,.\,P \not\approx' \varphi\,.\,\psi\,.\,P + (\varphi \wedge \psi)\,.\,P$$

since RHS $\stackrel{\varphi\wedge\psi}{\longrightarrow} P$ can no longer be simulated by LHS. However, fusions still exhibit some absorptions that communications do not. For example,

$$\varphi\,.\,\varphi\,.\,P \approx' \varphi\,.\,P$$

holds by **DF2** and **T1**.

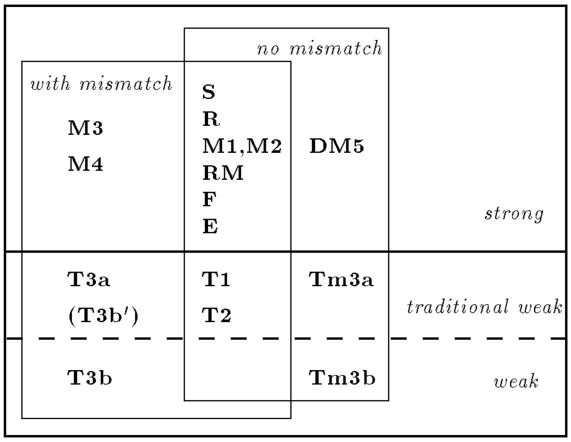

Figure 1: Overview of the axiom systems.

Theorem 18 *$P \approx'_+ Q$ iff $\vdash_{\mathcal{W}'} P = Q$.*
PROOF: Soundness is easily established. For completeness we only indicate where the proofs of Theorem 11 and its supporting lemmas change. The only significant change is in the proof of Lemma 8. Here Case 4 disappears because of the amended definition of $\stackrel{\gamma}{\Longrightarrow}$. On the other hand Case 2 and 3 must now also consider the case that γ is a fusion. For Case 3 this represents no problem, the demonstrated derivation works also for γ a fusion. For Case 2, if γ is a fusion we need to apply **T3b′**. This is proved exactly as in Case 4 with the additional requirement that $\psi = \mathbf{1}$ (since **T3b′** is just the special case of **T3b** where $\psi = \mathbf{1}$). The rest of the completeness proof is unchanged. □

In the subcalculus without mismatch a similar effect is obtained by requiring $\psi = \mathbf{1}$ in **Tm3b**. Since **Tm3a** admits α to be a fusion this means that **Tm3b** can be dropped altogether. Let $\mathcal{MW}'$ be the axioms of $\mathcal{M}$ plus **T1**, **T2** and **Tm3b**. Note that these three laws correspond exactly to the three tau-laws of CCS!

Theorem 19 *In the subcalculus without mismatch, $P \approx'_+ Q$ iff $\vdash_{\mathcal{MW}'} P = Q$.*
PROOF: The proof is completely analogous to the proof of Theorem 18. Only Lemma 15 changes in that in case 4, $\psi = 1$ is now sufficient. So **Tm3a** suffices for this case. Cases 2 and 3 need to be strengthened to consider that γ is a fusion, but that is trivial. □

7 Conclusion

We have given complete axiomatizations of weak hypercongruence in the fusion calculus, both with and without mismatch. An overview is shown in Figure 1.

There are several obvious avenues of further work. The axiomatization can presumably be extended to cover the finite-control fragment of the fusion calculus, following ideas from Lin [Lin95b]. A variant for weak open equivalence in the π-calculus is also probably straightforward by extending Victor's proofs for strong open equivalence in [Vic98].

Proving the axioms independent is probably less straightforward. Indeed, formal independence proofs have attracted very little attention so far. For example, we are not aware of a formal proof that the three tau-laws of CCS are independent (though such a proof is probably not very hard). Of the laws presented in this paper we conjecture that **T3a**, with its sequences of matches and mismatches, cannot be derived from **Tm3a** (without these sequences) and the other laws. On the other hand it is not difficult to show that a version of **T3a** with only *mis*matches is sufficient.

There is a wide spectrum of behavioural equivalences which do not discriminate on the basis of internal actions, for an overview see e.g. [Gla93]. It is not at this point clear how interesting they are for the fusion calculus and its applications, though our work on concurrent constraints in the fusion calculus [VP98] indicates that relevant equivalences should in some ways respect divergence.

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