Uppsala University
Department of Information Technology
Systems and Control
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Final exam: Control Design (Reglerteknisk design, 1TT492)

Date: April 19, 2007

Responsible examiner: Torsten Söderström

Preliminary grades: 3 = 23–32p, 4 = 33–42p, 5 = 43–50p.

Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 1b is an alternative to the homework assignment. (In case you choose to hand in a solution to **Problem 1b** you will be accounted for the best performance of the homework assignments and **Problem 1b**.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as 'Reglerteori – flervariabla och olinjära metoder', 'Reglerteknik – Grundläggande teori', and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators. Note that the following are **not allowed**: Exempelsamling med lösningar, copies of OH transparencies.

Good luck!

Problem 1

Consider a system with three inputs and two outputs, having a transfer function

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} & \frac{3}{s+2} \\ \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

(a) Determine the poles and the zeros of the system. **3 points**

(b) The system can be represented in state space form as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and B a suitable matrix of dimension 3×3. Determine the matrix B. **6 points**

Problem 2

When Internal Model Control (IMC) is applied, one can in a general case for a minimum-phase system use

$$Q(s) = \frac{1}{A(s)} G^{-1}(s)$$

where $A(s)$ is a polynomial of degree k. When λ-tuning is applied one makes the specific choice $A(s) = (1 + \lambda s)^k$.

(a) What condition on $A(s)$ has to be applied for the controller to work? What additional condition should $A(s)$ satisfy in order to guarantee that the sensitivity function fulfils $S(0) = 0$? **3 points**

(b) Consider the SISO case. Can one choose $A(s)$ so that the stationary error vanishes when the reference signal is a ramp, that is the error coefficient e_1 satisfies $e_1 = 0$? **4 points**

Problem 3

Consider LQ control of the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{aligned}$$

The criterion to be minimized is

$$V = \int [\alpha^2 y^2(t) + u^2(t)] dt, \qquad (\alpha > 0)$$

and hence

$$Q_1 = \alpha^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad Q_2 = 1$$

(a) Determine the optimal feedback gain vector L. Determine also the loop gain $L(pI - A)^{-1}B$. **5 points**

Hint. In this example, the nondiagonal elements of the solution to the Riccati equation will be positive.

(b) Show that the transfer function from u to y for the given system is in fact of first order. Use this fact, to derive the loop gain in a simpler way than in part (a). **3 points**

Problem 4

Consider a simple feedback system where the nominal model is $G(s) = 1/s$, the feedback is a proportional regulator $F(s) = K$ and the true system is

$$G_o(s) = \frac{1}{s(1 + sT)}$$

Both K and T can be assumed to be positive.

(a) Determine the relative model error $\Delta_G(s)$. **2 points**

(b) Assume that the criterion

$$\| \Delta_G \|_\infty \| T \|_\infty < 1$$

is used to examine for which values of K the closed loop system can be guaranteed to be stable. What is the result? **2 points**

(c) Assume that the criterion

$$\| \Delta_G T \|_\infty < 1$$

is used to examine for which values of K the closed loop system can be guaranteed to be stable. What is the result? **3 points**

(d) Determine the poles of the closed loop system. Find out when the closed loop system is asymptotically stable. **2 points**

Problem 5

Consider a system with a zero on the imaginary axis, so $G(i\omega_z) = 0$. For the design, use a weighting

$$W_S(s) = \frac{s + \omega_0}{S_0 s}, \qquad S_0 = 2$$

Determine what the design condition

$$|W_S(i\omega_z)| \leq 1$$

implies for the bandwidth ω_0? **4 points**

Problem 6

Consider the scalar system

$$\begin{aligned} x(t+1) &= x(t) + u(t) + v(t) \\ y(t) &= x(t) + e(t) \end{aligned}$$

where the process noise $v(t)$ and the measurement noise $e(t)$ have variances $r_v = R_1 = 1$ and $r_e = R_2 = 2$, respectively.

(a) Assume that the state $x(t)$ is estimated using a standard observer

$$\hat{x}(t+1) = \hat{x}(t) + u(t) + K(y(t) - \hat{x}(t))$$

with a constant gain K. Determine the stationary variance, say V, of the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$ as a function of K. **2 points**

(b) Determine what value of the observer gain that minimizes V. Let K^* denote this value of the gain. What is the minimum value of V? **4 points**

(c) What is the solution to the associated Riccati equation? **2 points**

(d) Assume next that the gain K^* is used, but that the observation process is improved by using a much more accurate sensor, so that the measurement noise has no variance. Hence, $R_2 = r = 0$. What is then the variance of the estimation error? **2 points**

(e) How much lower value of V can be obtained by re-optimizing the observer gain for the case treated in part (d) if $r = 0$? **3 points**

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Control Design, April 19, 2007 — Answers and brief solutions

Problem 1

(a) Determine first the pole polynomial. The 1×1 minors are the matrix elements. It is enough to consider

$$\frac{2}{s+1}, \quad \frac{3}{s+2}$$

There are 3 different 2×2 minors (each obtained by deleting one column of $G(s)$ when forming the determinant). These minors are

$$\begin{aligned}
\frac{2}{s+1} \times \frac{1}{s+1} - \frac{3}{s+2} \times \frac{1}{s+1} &= \frac{2(s+2) - 3(s+1)}{(s+1)^2(s+2)} = \frac{(-s+1)}{(s+1)^2(s+2)}, \\
\frac{2}{s+1} \times \frac{1}{s+1} - \frac{3}{s+2} \times \frac{1}{s+1} &= \frac{2(s+2) - 3(s+1)}{(s+1)^2(s+2)} = \frac{(-s+1)}{(s+1)^2(s+2)}, \\
\frac{3}{s+2} \times \frac{1}{s+1} - \frac{3}{s+2} \times \frac{1}{s+1} &= 0
\end{aligned}$$

The least common denominator for all the minors, that is the pole polynomial, is hence

$$(s+1)^2(s+2)$$

The system has a double pole in $s = -1$ and a single pole in $s = -2$.

To find the zeros of the system, consider the numerators of the 2×2 minors. These minors have already the pole polynomial as denominator. The zero polynomial is therefore $-s+1$, and the system has one zero in $s = 1$.

(b) Set

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

We then get the transfer function

$$\begin{aligned}
G(s) &= C(sI - A)^{-1}B \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s+1 & 0 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{s+1} & 0 & \frac{1}{s+2} \\ 0 & \frac{1}{s+1} & 0 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \frac{b_{11}}{s+1} + \frac{b_{31}}{s+2} & \frac{b_{12}}{s+1} + \frac{b_{32}}{s+2} & \frac{b_{13}}{s+1} + \frac{b_{33}}{s+2} \\ \\ \frac{b_{21}}{s+1} & \frac{b_{22}}{s+1} & \frac{b_{23}}{s+1} \end{pmatrix}$$

Comparing with the given expression for $G(s)$ we find that

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 3 \end{pmatrix}$$

Problem 2

(a) It holds

$$S(s) = I - Q(s)G(s) = \left(1 - \frac{1}{A(s)}\right) I = \frac{A(s) - 1}{A(s)} I$$

The conditions to impose on $A(s)$ are

- $A(s)$ must have all zeros in the left half plan.
- $A(s)$ must have sufficient degree so that $Q(s)$ is proper.
- $A(0) = 1$.

(b) Write the polynomial $A(s)$ as

$$A(s) = a_o s^k + a_1 s^{k-1} + \ldots + a_k$$

Now, $A(0) = 1 \Rightarrow a_k = 1$, and

$$e_1 = \frac{dS}{ds}_{|s=0} = \frac{\frac{dA}{ds}A - (A-1)\frac{dA}{ds}}{A^2}_{|s=0} = \frac{dA}{ds}_{|s=0} = a_{k-1}$$

As the polynomial $A(s)$ must have all zeros inside the left half plan it is necessary that $a_{k-1} > 0$, so it is not possible to achieve $e_1 = 0$.

Problem 3

One has to solve the Riccati equation

$$0 = A^T S + SA + Q_1 - SBQ_2^{-1}B^T S, \qquad L = Q_2^{-1}B^T S$$

If $L = \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix}$, the loop gain $H(p)$ will be

$$H(p) = L(pI - A)^{-1}B = \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} \begin{pmatrix} p+1 & 0 \\ -1 & p \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\ell_1 p + \ell_2}{p(p+1)}$$

(a) The Riccati equation becomes

$$\begin{aligned} 0 \;=\; & \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} + \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ & - \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \end{aligned}$$

Written elementwise, this becomes

$$\begin{aligned} 0 &= -2s_{11} + 2s_{12} + \alpha^2 - s_{11}^2 \\ 0 &= -s_{12} + s_{22} + \alpha^2 - s_{11}s_{12} \\ 0 &= 0 + \alpha^2 - s_{12}^2 \end{aligned}$$

The last equation gives

$$s_{12} = \pm\alpha$$

The first equation then gives

$$s_{11}^2 + 2s_{11} \mp 2\alpha - \alpha^2 = 0 \Rightarrow s_{11} = -1 \pm \left[1 \pm 2\alpha + \alpha^2\right]^{1/2} = -1 \pm (1 \pm \alpha)$$

There are two possibilities to get s_{11} positive.

$$\begin{aligned} I: &\quad s_{12} = \alpha, \quad s_{11} = \alpha \\ II: &\quad s_{12} = -\alpha, \quad s_{11} = -2 + \alpha \text{ (requires } \alpha > 2) \end{aligned}$$

The middle equation gives

$$s_{22} = s_{12}(1 + s_{11}) - \alpha^2$$

This gives the two cases

$$\begin{aligned} I: &\quad s_{22} = \alpha(1+\alpha) - \alpha^2 = \alpha, \quad \text{and } S \text{ will be singular and} \\ &\quad\quad \text{positive semidefinite for all } \alpha > 0 \\ II: &\quad s_{22} = -\alpha(-1+\alpha) - \alpha^2 = \alpha - 2\alpha^2, \end{aligned}$$

In case II, we need to examine whether or not the determinant of S is non-negative definite. (It should hardly be so as the solution in case I gives a positive semidefinite solution). In case II it holds that

$$\begin{aligned} \det S &= (-2+\alpha)\alpha(1-2\alpha) - \alpha^2 \\ &= \alpha\left(-2 + 4\alpha - 2\alpha^2\right) \\ &= -2\alpha(1-\alpha)^2 < 0 \end{aligned}$$

As the determinant is negative, S will be indefinite in this case, and thus case I applies.

The feedback vector L is easily obtained as

$$L = \begin{pmatrix} s_{11} & s_{12} \end{pmatrix} = \begin{pmatrix} \alpha & \alpha \end{pmatrix}$$

The loop gain becomes

$$L(pI - A)^{-1}B = \begin{pmatrix} \alpha & \alpha \end{pmatrix} \begin{pmatrix} p+1 & 0 \\ -1 & p \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\alpha(p+1)}{p(p+1)} = \frac{\alpha}{p}$$

(b) The system has a common pole and zero in $s = -1$, and its transfer function can be simplified to $G(s) = 1/s$. Treating the system as a first order system one would get

$$A = 0,\ B = 1,\ C = 1,\ Q_2 = 1, Q_1 = \alpha^2$$

The Riccati equation becomes

$$0 = \alpha^2 - S^2 \Rightarrow S = \alpha \Rightarrow L = \alpha$$

and the loop gain is $G(p)L = \alpha/p$.

Problem 4

(a) As $G_o = G(1 + \Delta_G)$ holds, we find that

$$\Delta_G(s) = \frac{G_o(s) - G(s)}{G(s)} = \frac{\frac{1}{s}\frac{1}{1+sT} - \frac{1}{s}}{\frac{1}{s}} = -\frac{sT}{1+sT}$$

(b) We find easily

$$\| \Delta_G \|_\infty = \sup_\omega |\Delta_G(i\omega)| = \sup_\omega \left| \frac{i\omega T}{1 + i\omega T} \right| = \sup_\omega \frac{\omega T}{\sqrt{1 + \omega^2 T^2}} = 1$$

Furthermore,

$$T(s) = \frac{G(s)F(s)}{1 + G(s)F(s)} = \frac{K/s}{1 + K/s} = \frac{K}{s + K} \Rightarrow \| T \|_\infty = 1$$

Hence, the stated sufficient stability condition is not satisfied for any value of K.

(c) In this case we need to examine

$$\| \Delta_G(s)T(s) \|_\infty = \| \frac{-sKT}{(s+K)(1+sT)} \|_\infty$$

Here we have

$$|\Delta_G(i\omega)T(i\omega)|^2 = \frac{\omega^2 K^2 T^2}{(K - \omega^2 T)^2 + \omega^2 (1 + KT)^2}$$

Seek maximum with respect to ω^2! This leeds to

$$K^2T^2 \left[\omega^4 T^2 + \omega^2(1 + K^2T^2) + K^2\right] - \omega^2 K^2 T^2 \left[2\omega^2 T^2 + (1 + K^2T^2)\right] = 0$$
$$\Rightarrow -K^2T^4\omega^4 + K^4T^2 = 0 \Rightarrow \omega^2 = K/T$$
$$\| \Delta_G T \|_\infty^2 = \frac{K^3 T}{K/T(1+KT)^2} = \frac{K^2T^2}{(1+KT)^2} < 1$$

Hence, stability is guaranteed for all positive values of K.

(d) The closed loop system becomes

$$G_c(s) = \frac{G_o(s)K}{1 + G_o(s)K} = \frac{K}{s(1+sT) + K}$$

which apparently has both poles in the left half plan for all $K > 0$.

Problem 5

$$\begin{aligned} |W_S(i\omega_z)| &= \left|\frac{i\omega_z + \omega_0}{i2\omega_z}\right| = \frac{\sqrt{\omega_o^2 + \omega_z^2}}{2\omega_z} \leq 1 \\ &\Rightarrow \ \omega_0^2 + \omega_z^2 \leq 4\omega_z^2 \qquad \omega_0 \leq \sqrt{3}\omega_z \end{aligned}$$

Problem 6

To treat the general case, let the measurement noise have variance r. The estimation error $\tilde{x}(t)$ satisfies

$$\tilde{x}(t+1) = (1-K)\tilde{x}(t) - Ke(t) + v(t)$$

Applying the Lyapunov equation then gives easily

$$P = (1-K)^2 P + 1 + rK^2$$

leading to

$$V(K) = \frac{1 + K^2 r}{K(2-K)}$$

The expression is valid when $0 < K < 2$ (which is precisely when $A - KC$ has its eigenvalue(s) inside the unit circle.

(a) Setting $r_e = 2$ gives

$$V(K) = \frac{2K^2 + 1}{K(2-K)}$$

(b) The minimizing element of $V(K)$ is found to be $K^* = 0.5$. Further, the minimal value turns out to be $V(K^*) = 2$.

(c) The associated Riccati equation is

$$P = P + 1 - P^2/(P+2)$$

which leads to

$$P^2 - P - 2 = 0$$

Since P must be positive, the unique solution is $P = 2$, as $V(K^*)$ in part (b). The Kalman gain becomes $K = P/(P+2) = 0.5 = K^*$, also as in part (b).

(d) The variance using the fixed observer gain K^* for the noise variance $R_2 = r$ becomes

$$V = \frac{1 + r(K^*)^2}{K^*(2 - K^*)} = \frac{1 + r/4}{3/4} = \frac{4 + r}{3}.$$

When r is decreasing from $r = 2$ towards $r = 0$, the variance V decreases from $V = 2$ to $V = 4/3$.

(e) The minimal variance of the estimation error, when $r = 0$ is obtained by solving the Riccati equation

$$P = P + 1 - P^2/(P + 0)$$

which gives $P = 1$.