Final exam: Control Design (Reglertechnisk design, 1TT492)

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Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 1b is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1b you will be accounted for the best performance of the homework assignments and Problem 1b.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as ‘Reglerteori – flervariabla och olinjära metoder’, ‘Reglerteknik – Grundläggande teori’, and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators. Note that the following are not allowed: Exempelsamling med lösningar, copies of OH transparencies.

Good luck!
Problem 1

Consider a system with three inputs and two outputs, having a transfer function
\[ G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix} \]

(a) Determine the poles and the zeros of the system. \hspace{1cm} 3 points

(b) The system can be represented in state space form as
\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]
with
\[ A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \]
and \( B \) a suitable matrix of dimension \( 3 \times 3 \). Determine the matrix \( B \). \hspace{1cm} 6 points

Problem 2

When Internal Model Control (IMC) is applied, one can in a general case for a minimum-phase system use
\[ Q(s) = \frac{1}{A(s)}G^{-1}(s) \]
where \( A(s) \) is a polynomial of degree \( k \). When \( \lambda \)-tuning is applied one makes the specific choice \( A(s) = (1 + \lambda s)^k \).

(a) What condition on \( A(s) \) has to be applied for the controller to work? What additional condition should \( A(s) \) satisfy in order to guarantee that the sensitivity function fulfills \( S(0) = 0 \)? \hspace{1cm} 3 points

(b) Consider the SISO case. Can one choose \( A(s) \) so that the stationary error vanishes when the reference signal is a ramp, that is the error coefficient \( e_1 \) satisfies \( e_1 = 0 \)? \hspace{1cm} 4 points

Problem 3

Consider LQ control of the system
\[ \dot{x} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}u \]
\[ y = \begin{pmatrix} 1 & 1 \end{pmatrix}x \]

The criterion to be minimized is
\[ V = \int [\alpha^2 y^2(t) + u^2(t)]dt, \quad (\alpha > 0) \]
and hence
\[ Q_1 = \alpha^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad Q_2 = 1 \]

(a) Determine the optimal feedback gain vector \( L \). Determine also the loop gain
\( L(pI - A)^{-1}B \).  

*Hint.* In this example, the nondiagonal elements of the solution to the Riccati equation will be positive.

(b) Show that the transfer function from \( u \) to \( y \) for the given system is in fact of first order. Use this fact, to derive the loop gain in a simpler way than in part (a).

3 points

**Problem 4**

Consider a simple feedback system where the nominal model is \( G(s) = 1/s \), the feedback is a proportional regulator \( F(s) = K \) and the true system is
\[ G_o(s) = \frac{1}{s(1 + sT)} \]

Both \( K \) and \( T \) can be assumed to be positive.

(a) Determine the relative model error \( \Delta G(s) \).  

2 points

(b) Assume that the criterion
\[ \| \Delta G \|_\infty \| T \|_\infty < 1 \]

is used to examine for which values of \( K \) the closed loop system can be guaranteed to be stable. What is the result?  

2 points

(c) Assume that the criterion
\[ \| \Delta G T \|_\infty < 1 \]

is used to examine for which values of \( K \) the closed loop system can be guaranteed to be stable. What is the result?  

3 points

(d) Determine the poles of the closed loop system. Find out when the closed loop system is asymptotically stable.  

2 points

**Problem 5**

Consider a system with a zero on the imaginary axis, so \( G(i\omega_z) = 0 \). For the design, use a weighting
\[ W_S(s) = \frac{s + \omega_0}{S_0s}, \quad S_0 = 2 \]

Determine what the design condition
\[ |W_S(i\omega_z)| \leq 1 \]

3
implies for the bandwidth $\omega_0$?

Problem 6

Consider the scalar system

$$
\begin{align*}
\dot{x}(t+1) &= x(t) + u(t) + v(t) \\
y(t) &= x(t) + e(t)
\end{align*}
$$

where the process noise $v(t)$ and the measurement noise $e(t)$ have variances $r_v = R_1 = 1$ and $r_e = R_2 = 2$, respectively.

(a) Assume that the state $x(t)$ is estimated using a standard observer

$$
\dot{\hat{x}}(t+1) = \hat{x}(t) + u(t) + K(y(t) - \hat{x}(t))
$$

with a constant gain $K$. Determine the stationary variance, say $V$, of the estimation error $\hat{x}(t) = x(t) - \hat{x}(t)$ as a function of $K$.  

(b) Determine what value of the observer gain that minimizes $V$. Let $K^*$ denote this value of the gain. What is the minimum value of $V$? 

(c) What is the solution to the associated Riccati equation? 

(d) Assume next that the gain $K^*$ is used, but that the observation process is improved by using a much more accurate sensor, so that the measurement noise has no variance. Hence, $R_2 = r = 0$. What is then the variance of the estimation error? 

(e) How much lower value of $V$ can be obtained by re-optimizing the observer gain for the case treated in part (d) if $r = 0$?
Problem 1

(a) Determine first the pole polynomial. The $1 \times 1$ minors are the matrix elements. It is enough to consider

\[
\begin{align*}
\frac{2}{s + 1}, & \quad \frac{3}{s + 2} \\
\end{align*}
\]

There are 3 different $2 \times 2$ minors (each obtained by deleting one column of $G(s)$ when forming the determinant). These minors are

\[
\begin{align*}
&\frac{2}{s + 1} \times \frac{1}{s + 1} - \frac{3}{s + 2} \times \frac{1}{s + 1} = \frac{2(s + 2) - 3(s + 1)}{(s + 1)^2(s + 2)} = \frac{(-s + 1)}{(s + 1)^2(s + 2)}, \\
&\frac{2}{s + 1} \times \frac{1}{s + 1} - \frac{3}{s + 2} \times \frac{1}{s + 1} = \frac{2(s + 2) - 3(s + 1)}{(s + 1)^2(s + 2)} = \frac{(-s + 1)}{(s + 1)^2(s + 2)}, \\
&\frac{3}{s + 2} \times \frac{1}{s + 1} - \frac{3}{s + 2} \times \frac{1}{s + 1} = 0
\end{align*}
\]

The least common denominator for all the minors, that is the pole polynomial, is hence

\[(s + 1)^2(s + 2)\]

The system has a double pole in $s = -1$ and a single pole in $s = -2$.

To find the zeros of the system, consider the numerators of the $2 \times 2$ minors. These minors have already the pole polynomial as denominator. The zero polynomial is therefore $-s + 1$, and the system has one zero in $s = 1$.

(b) Set

\[
B = \begin{pmatrix}
  b_{11} & b_{12} & b_{13} \\
  b_{21} & b_{22} & b_{23} \\
  b_{31} & b_{32} & b_{33}
\end{pmatrix}
\]

We then get the transfer function

\[
G(s) = C(sI - A)^{-1}B
\]

\[
= \begin{pmatrix}
  1 & 0 & 1 \\
  0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
  s+1 & 0 & 0 \\
  0 & s+1 & 0 \\
  0 & 0 & s+2
\end{pmatrix}
\begin{pmatrix}
  b_{11} & b_{12} & b_{13} \\
  b_{21} & b_{22} & b_{23} \\
  b_{31} & b_{32} & b_{33}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  \frac{1}{s+1} & 0 & \frac{1}{s+2} \\
  0 & \frac{1}{s+1} & 0
\end{pmatrix}
\begin{pmatrix}
  b_{11} & b_{12} & b_{13} \\
  b_{21} & b_{22} & b_{23} \\
  b_{31} & b_{32} & b_{33}
\end{pmatrix}
\]

\[
5
\]
Comparing with the given expression for $G(s)$ we find that

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 3 \end{pmatrix}$$

**Problem 2**

(a) It holds

$$S(s) = I - Q(s)G(s) = \left(1 - \frac{1}{A(s)}\right)I = \frac{A(s) - 1}{A(s)}I$$

The conditions to impose on $A(s)$ are

- $A(s)$ must have all zeros in the left half plane.
- $A(s)$ must have sufficient degree so that $Q(s)$ is proper.
- $A(0) = 1$.

(b) Write the polynomial $A(s)$ as

$$A(s) = a_0 s^k + a_1 s^{k-1} + \ldots + a_k$$

Now, $A(0) = 1 \Rightarrow a_k = 1$, and

$$\epsilon_1 = \frac{dS}{ds}_{s=0} = \frac{dA}{ds} A - (A - 1) \frac{dA}{ds}_{s=0} = \frac{dA}{ds}_{s=0} = a_{k-1}$$

As the polynomial $A(s)$ must have all zeros inside the left half plane it is necessary that $a_{k-1} > 0$, so it is not possible to achieve $\epsilon_1 = 0$.

**Problem 3**

One has to solve the Riccati equation

$$0 = A^T S + SA + Q_1 - SBQ_2^{-1} B^T S, \quad L = Q_2^{-1} B^T S$$

If $L = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}$, the loop gain $H(p)$ will be

$$H(p) = L(pI - A)^{-1} B = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \begin{pmatrix} p + 1 & 0 \\ -1 & p \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\ell_1 p + \ell_2}{p(p + 1)}$$

(a) The Riccati equation becomes

$$0 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} + \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
Written elementwise, this becomes

\[ \begin{align*}
0 &= -2s_{11} + 2s_{12} + \alpha^2 - s_{11}^2 \\
0 &= -s_{12} + s_{22} + \alpha^2 - s_{11}s_{12} \\
0 &= 0 + \alpha^2 - s_{12}^2
\end{align*} \]

The last equation gives

\[ s_{12} = \pm \alpha \]

The first equation then gives

\[ s_{11}^2 + 2s_{11} \mp 2\alpha - \alpha^2 = 0 \Rightarrow s_{11} = -1 \pm \left(1 \pm 2\alpha + \alpha^2\right)^{1/2} = -1 \pm (1 \pm \alpha) \]

There are two possibilities to get \( s_{11} \) positive.

\[
I : \quad s_{12} = \alpha, \quad s_{11} = \alpha \\
II : \quad s_{12} = -\alpha, \quad s_{11} = -2 + \alpha \text{ (requires } \alpha > 2) \]

The middle equation gives

\[ s_{22} = s_{12}(1 + s_{11}) - \alpha^2 \]

This gives the two cases

\[
I : \quad s_{22} = \alpha(1 + \alpha) - \alpha^2 = \alpha, \quad \text{and } S \text{ will be singular and positive semidefinite for all } \alpha > 0 \\
II : \quad s_{22} = -\alpha(-1 + \alpha) - \alpha^2 = \alpha - 2\alpha^2, \]

In case II, we need to examine whether or not the determinant of \( S \) is non-negative definite. (It should hardly be so as the solution in case I gives a positive semidefinite solution). In case II it holds that

\[
\begin{align*}
\det S &= (-2 + \alpha)\alpha(1 - 2\alpha) - \alpha^2 \\
&= \alpha(-2 + 4\alpha - 2\alpha^2) \\
&= -2\alpha(1 - \alpha)^2 < 0
\end{align*}
\]

As the determinant is negative, \( S \) will be indefinite in this case, and thus case I applies.

The feedback vector \( L \) is easily obtained as

\[ L = \begin{pmatrix} s_{11} & s_{12} \end{pmatrix} = \begin{pmatrix} \alpha & \alpha \end{pmatrix} \]

The loop gain becomes

\[ L(pI - A)^{-1}B = \begin{pmatrix} \alpha & \alpha \end{pmatrix} \begin{pmatrix} p + 1 & 0 \\ -1 & p \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\alpha(p + 1)}{p(p + 1)} = \frac{\alpha}{p} \]
(b) The system has a common pole and zero in \( s = -1 \), and its transfer function can be simplified to \( G(s) = 1/s \). Treating the system as a first order system one would get

\[
A = 0, \ B = 1, \ C = 1, \ Q_2 = 1, \ Q_1 = \alpha^2
\]

The Riccati equation becomes

\[
0 = \alpha^2 - S^2 \Rightarrow S = \alpha \Rightarrow L = \alpha
\]

and the loop gain is \( G(p)L = \alpha/p \).

**Problem 4**

(a) As \( G_o = G(1 + \Delta_G) \) holds, we find that

\[
\Delta_G(s) = \frac{G_o(s) - G(s)}{G(s)} = \frac{1 - 1}{s^{1.5 + sT} - \frac{1}{s}} = -\frac{sT}{1 + sT}
\]

(b) We find easily

\[
\| \Delta_G \|_\infty = \sup_{\omega} |\Delta_G(i\omega)| = \sup_{\omega} \left| \frac{i\omega T}{1 + i\omega T} \right| = \sup_{\omega} \frac{\omega T}{\sqrt{1 + \omega^2 T^2}} = 1
\]

Furthermore,

\[
T(s) = \frac{G(s)F(s)}{1 + G(s)F(s)} = \frac{K/s}{1 + K/s} = \frac{K}{s + K} \Rightarrow \| T \|_\infty = 1
\]

Hence, the stated sufficient stability condition is not satisfied for any value of \( K \).

(c) In this case we need to examine

\[
\| \Delta_G(s)T(s) \|_\infty = \| -sKT \|_{(s + K)(1 + sT)} \|_\infty
\]

Here we have

\[
|\Delta_G(i\omega)T(i\omega)|^2 = \frac{\omega^2 K^2 T^2}{(K - \omega^2 T^2)^2 + \omega^2 (1 + KT)^2}
\]

Seek maximum with respect to \( \omega^2 \). This leads to

\[
K^2 T^2 \left[ \omega^4 T^2 + \omega^2 (1 + K^2 T^2) + K^2 \right] - \omega^2 K^2 T^2 \left[ 2 \omega^2 T^2 + (1 + K^2 T^2) \right] = 0
\]

\[
\Rightarrow -K^2 T^4 \omega^4 + K^4 T^2 = 0 \Rightarrow \omega^2 = K/T
\]

\[
\| \Delta_GT \|_\infty^2 = \frac{K^3 T}{K/T(1 + KT)^2} = \frac{K^2 T^2}{(1 + KT)^2} < 1
\]

Hence, stability is guaranteed for all positive values of \( K \).
(d) The closed loop system becomes

\[
G_e(s) = \frac{G_o(s)K}{1 + G_o(s)K} = \frac{K}{s(1 + sT) + K}
\]

which apparently has both poles in the left half plane for all \( K > 0 \).

**Problem 5**

\[
|W_s(i\omega_z)| = \left| \frac{i\omega_z + \omega_0}{i2\omega_z} \right| = \frac{\sqrt{\omega_0^2 + \omega_z^2}}{2\omega_z} \leq 1
\]

\[
\Rightarrow \omega_0^2 + \omega_z^2 \leq 4\omega_z^2 \quad \omega_0 \leq \sqrt{3}\omega_z
\]

**Problem 6**

To treat the general case, let the measurement noise have variance \( r \). The estimation error \( \tilde{x}(t) \) satisfies

\[
\tilde{x}(t + 1) = (1 - K)\tilde{x}(t) - K e(t) + v(t)
\]

Applying the Lyapunov equation then gives easily

\[
P = (1 - K)^2 P + 1 + rK^2
\]

leading to

\[
V(K) = \frac{1 + K^2r}{K(2 - K)}
\]

The expression is valid when \( 0 < K < 2 \) (which is precisely when \( A - KC \) has its eigenvalue(s) inside the unit circle.

(a) Setting \( r = 2 \) gives

\[
V(K) = \frac{2K^2 + 1}{K(2 - K)}
\]

(b) The minimizing element of \( V(K) \) is found to be \( K^* = 0.5 \). Further, the minimal value turns out to be \( V(K^*) = 2 \).

(c) The associated Riccati equation is

\[
P = P + 1 - P^2/(P + 2)
\]

which leads to

\[
P^2 - P - 2 = 0
\]

Since \( P \) must be positive, the unique solution is \( P = 2 \), as \( V(K^*) \) in part (b). The Kalman gain becomes \( K = P/(P + 2) = 0.5 = K^* \), also as in part (b).
(d) The variance using the fixed observer gain $K^*$ for the noise variance $R_2 = r$ becomes

$$V = \frac{1 + r(K^*)^2}{K^*(2 - K^*)} = \frac{1 + r/4}{3/4} = \frac{4 + r}{3}.$$  

When $r$ is decreasing from $r = 2$ towards $r = 0$, the variance $V$ decreases from $V = 2$ to $V = 4/3$.

(e) The minimal variance of the estimation error, when $r = 0$ is obtained by solving the Riccati equation

$$P = P + 1 - P^2/(P + 0)$$

which gives $P = 1$. 