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Introduction to recursive identification algorithms

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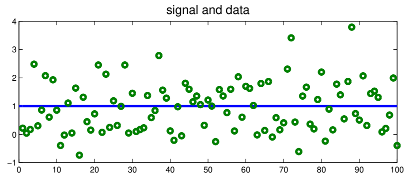
Introduction

Consider the problem of estimating a constant signal from noisy measurements.
Model

$$y(t) = s + e(t)$$

Natural estimate

$$\hat{s}(t) = \frac{1}{t} \sum_{s=1}^{t} y(s)$$



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Making the estimate recursive

Aim: Rewrite the estimate so that $\hat{s}(t)$ is updated from $\hat{s}(t-1)$ when the new measurement $y(t)$ is available.
Solution:

$$\begin{aligned}
\hat{s}(t) &= \frac{1}{t}\left[y(t) + \sum_{s=1}^{t-1} y(s)\right] \\
&= \frac{1}{t}\left[y(t) + (t-1)\hat{s}(t-1)\right] \\
&= \hat{s}(t-1) + \frac{1}{t}\left[y(t) - \hat{s}(t-1)\right]
\end{aligned}$$

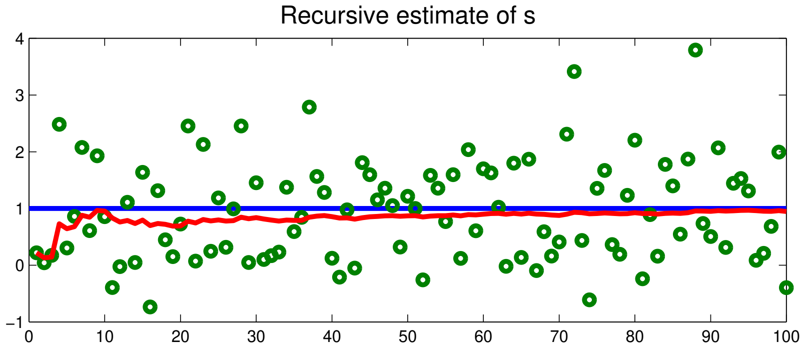
This equation captures the essence of recursive algorithms!

Interpretation

$$\hat{s}(t) = \hat{s}(t-1) + \frac{1}{t}\left[y(t) - \hat{s}(t-1)\right]$$

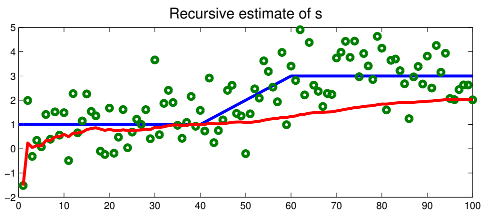
$\hat{s}(t)$	new estimate
$\hat{s}(t-1)$	old estimate
$1/t$	gain sequence
$y(t)$	new measurement
$\hat{s}(t-1)$	prediction of $y(t)$ based on available data at time $t-1$ and model parameter $\hat{s}(t-1)$
$y(t) - \hat{s}(t-1)$	prediction error

Simulations



Time-varying signals

Assume $s(t)$ changes with time.



After a change, give less weight to old data.
How?

Approaches for time-varying case

- Time-invariant case
  $V_t(\theta) = \sum_{s=1}^{t}[y(s) - \theta]^2$
- Exponential forgetting
  $V_t^{(e)}(\theta) = \sum_{s=1}^{t} \lambda^{t-s}[y(s) - \theta]^2$
- Rectangular forgetting
  $V_t^{(r)}(\theta) = \sum_{s=t-L+1}^{t}[y(s) - \theta]^2$
- Observer
  $\hat{s}(t+1) = \hat{s}(t) + K[y(t) - \hat{s}(t)]$

Exponential forgetting

Criterion

$$\begin{aligned} V_t^{(e)}(\theta) &= \sum_{s=1}^{t} \lambda^{t-s}[y(s)-\theta]^2 \\ &= \theta^2 \sum_{s=1}^{t} \lambda^{t-s} - 2\theta \sum_{s=1}^{t} \lambda^{t-s} y(s) + \text{const.} \end{aligned}$$

Estimate

$$\begin{aligned} \hat{\theta}(t) &= \frac{\sum_{s=1}^{t} \lambda^{t-s} y(s)}{\sum_{s=1}^{t} \lambda^{t-s}} = \dots \\ &= \hat{\theta}(t-1) + \frac{1-\lambda}{1-\lambda^t}[y(t)-\hat{\theta}(t-1)] \end{aligned}$$

Rectangular forgetting

Criterion

$$\begin{aligned} V_t^{(r)}(\theta) &= \sum_{s=t-L+1}^{t} [y(s)-\theta]^2 \\ &= \theta^2 \times L - 2\theta \sum_{s=t-L+1}^{t} y(s) + \text{const.} \end{aligned}$$

Estimate

$$\begin{aligned} \hat{\theta}(t) &= \frac{1}{L} \sum_{s=t-L+1}^{t} y(s) \\ &= \hat{\theta}(t-1) + \frac{1}{L}[y(t)-y(t-L)] \end{aligned}$$

Observer-based estimator

Model

$$\begin{aligned} s(t+1) &= s(t) \\ y(t) &= s(t) + e(t) \end{aligned}$$

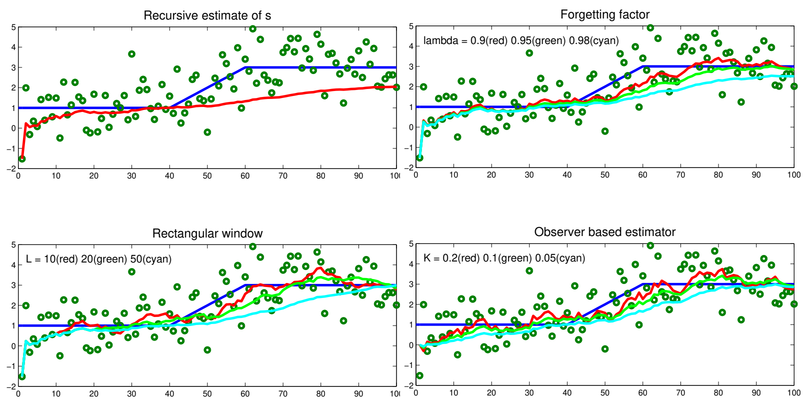
Observer

$$\hat{s}(t+1) = \hat{s}(t) + K[y(t) - \hat{s}(t)]$$

Observer error: $\tilde{s}(t) = s(t) - \hat{s}(t)$

$$\tilde{s}(t+1) = (1-K)\tilde{s}(t) + Ke(t)$$

Simulations



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Noise sensitivity

1. Time-invariant case: $E\tilde{s}^2(t) = \sigma^2/t$
2. Time-varying case (when $t \to \infty$):
   - Exponential forgetting: $E\tilde{s}^2(t) = \sigma^2 \frac{1-\lambda}{1+\lambda}$
   - Rectangular forgetting: $E\tilde{s}^2(t) = \sigma^2/L$
   - Observer: $E\tilde{s}^2(t) = \sigma^2 \frac{K}{2-K}$

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Inherent tradeoff

- Decrease noise sensitivity
  $[\lambda \approx 1$; L large; $K \approx 0]$

- Fast tracking capability
  $[\lambda \ll 1$; L small; $K \approx 1]$

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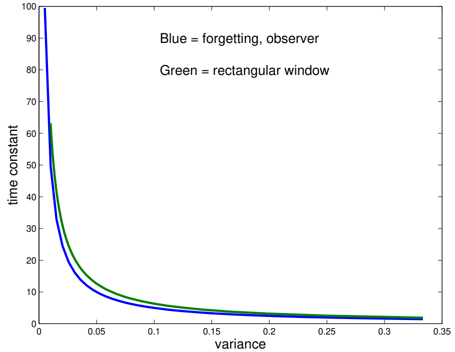
Inherent tradeoff, cont'd

Time constant for step change in $s(t)$ vs estimation error variance



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Approaches for recursive identification

- Modification of off-line methods
- Nonlinear filtering
- Stochastic approximation
- Pseudolinear regression and model reference techniques

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Approach 1: Modify off-line algorithms

Least squares method, linear regression model

$$y(t) = \varphi^T(t)\theta + v(t)$$

The regressor vector $\varphi(t)$ depends on data available at time t.

Least squares estimate at time t:

$$\left[\sum_{s=1}^{t} \varphi(s)\varphi^T(s)\right]\hat{\theta}(t) = \left[\sum_{s=1}^{t} \varphi(s)y(s)\right]$$

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Derivation of RLS

Set

$$R(t) = \sum_{s=1}^{t} \varphi(s)\varphi^T(s), \quad P(t) = R^{-1}(t)$$

$$\begin{aligned} R(t) &= R(t-1) + \varphi(t)\varphi^T(t) \\ &\Rightarrow \text{[matrix inversion lemma]} \\ P(t) &= P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{1+\varphi^T(t)P(t-1)\varphi(t)} \end{aligned}$$

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Updating the parameter estimates

$$\begin{aligned} \hat{\theta}(t) &= P(t)\left[\varphi(t)y(t) + \sum_{s=1}^{t-1} \varphi(s)y(s)\right] \\ &= P(t)\left[\varphi(t)y(t) + \left(R(t) - \varphi(t)\varphi^T(t)\right)\hat{\theta}(t-1)\right] \\ &= \hat{\theta}(t-1) + P(t)\varphi(t)\left[y(t) - \varphi^T(t)\hat{\theta}(t-1)\right] \\ &= \hat{\theta}(t-1) + K(t)\varepsilon(t) \end{aligned}$$

Prediction error

$$\varepsilon(t) = y(t) - \varphi^T(t)\hat{\theta}(t-1)$$

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Estimator gain

$$\begin{aligned} K(t) &= P(t)\varphi(t) \\ &= P(t-1)\varphi(t) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{1+\varphi^T(t)P(t-1)\varphi(t)}\varphi(t) \\ &= P(t-1)\varphi(t)\left(1 - \frac{\varphi^T(t)P(t-1)\varphi(t)}{1+\varphi^T(t)P(t-1)\varphi(t)}\right) \\ &= P(t-1)\varphi(t)\frac{1}{1+\varphi^T(t)P(t-1)\varphi(t)} \end{aligned}$$

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Comments on RLS

- No approximations here!
- An instrumental variable estimator can be written in a similar way [Substitute $\varphi(t)$ by $z(t)$; keep $\varphi^T(t)$].
- Recursive (or on-line) algorithms RLS and RIV have the same convergence properties ($t \to \infty$) as the batch (or off-line) algorithms.
- How to set initial values $P(0),\ \hat{\theta}(0)$?

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Recursive prediction error method

Prediction error criterion

$$V_t(\theta) = \frac{1}{2}\sum_{s=1}^{t} \varepsilon^2(s,\theta)$$

The model structure specifies how the predition error $\varepsilon(t,\theta)$ is to be computed from available input and output data.
An exact recursive minimizer of $V_t(\theta)$ does not exist in general.
Important exception: LS, for linear regression model.
In the general case: Make approximations, that hold without errors for the LS case.

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RPEM, cont'd

Taylor series expansion

$$\begin{aligned} V_t(\theta) &= V_t(\hat{\theta}(t-1)) + V_t'(\hat{\theta}(t-1))(\theta - \hat{\theta}(t-1)) \\ &+ \frac{1}{2}(\theta - \hat{\theta}(t-1))^T V_t''(\hat{\theta}(t-1))(\theta - \hat{\theta}(t-1)) \end{aligned}$$

Minimize!

$$\hat{\theta}(t) = \hat{\theta}(t-1) - \left[V_t''(\hat{\theta}(t-1))\right]^{-1}\left[V_t'(\hat{\theta}(t-1))\right]^T$$

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RPEM, cont'd

Negative gradient

$$\psi(t,\theta) = \left[-\frac{\partial}{\partial\theta}\varepsilon(t,\theta)\right]^T$$

Recursion of V_t implies

$$\begin{aligned} V_t'(\theta) &= -\sum \varepsilon(s,\theta)\psi(s,\theta) \\ &= V_{t-1}'(\theta) - \varepsilon(t,\theta)\psi(t,\theta) \\ V_t''(\theta) &= \sum \psi(s,\theta)\psi^T(s,\theta) + \sum \varepsilon(s,\theta)\varepsilon''(s,\theta) \\ &= V_{t-1}''(\theta) + \psi(t,\theta)\psi^T(t,\theta) + \varepsilon(t,\theta)\varepsilon''(t,\theta) \end{aligned}$$

RPEM cont'd

Approximations

- Taylor series of $V_t(\theta)$ up to second order terms
- $V'_{t-1}(\hat{\theta}(t-1)) = 0$
- $\varepsilon\varepsilon''$ neglectable compared to $\psi\psi^T$
- $V''_t(\hat{\theta}(t)) = V''_t(\hat{\theta}(t-1))$

RPEM derivation, cont'd

Set

$$\begin{aligned} R(t) &= \sum_{s=1}^{t} \psi(s,\hat{\theta}(t-1))\psi^T(s,\hat{\theta}(t-1)) \\ &= R(t-1) + \psi(t,\hat{\theta}(t-1))\psi^T(t,\hat{\theta}(t-1)) \\ \hat{\theta}(t) &= \hat{\theta}(t-1) + R^{-1}(t)\psi(t,\hat{\theta}(t-1))\varepsilon(t,\hat{\theta}(t-1)) \end{aligned}$$

How to compute
$\psi(s,\hat{\theta}(t-1)),\quad \varepsilon(s,\hat{\theta}(t-1)),\quad s \le t$?

Computing prediction errors

Find approximation (depends on the model parameterization)

$$\varepsilon(t,\hat{\theta}(t-1)) \sim \varepsilon(t), \quad \psi(t,\hat{\theta}(t-1)) \sim \psi(t)$$

Set $P(t) = R^{-1}(t)$. Algorithm

$$\begin{aligned} \hat{\theta}(t) &= \hat{\theta}(t-1) + K(t)\varepsilon(t) \\ P(t) &= P(t-1) - \frac{P(t-1)\psi(t)\psi^T(t)P(t-1)}{1+\psi^T(t)P(t-1)\psi(t)} \\ K(t) &= P(t)\psi(t) = \frac{P(t-1)\psi(t)}{1+\psi^T(t)P(t-1)\psi(t)} \end{aligned}$$

Approximate prediction errors

Principle: Use recent parameter estimates
Example: MA(1) process

$$\begin{aligned} y(t) &= e(t) + ce(t-1), \quad \theta = c \\ \varepsilon(t,\theta) &= \frac{1}{1+cq^{-1}} y(t) \\ \psi(t,\theta) &= \frac{-1}{(1+cq^{-1})^2} y(t-1) = \frac{-1}{1+cq^{-1}} \varepsilon(t-1,\theta) \end{aligned}$$

Approximation

$$\begin{aligned} \varepsilon(t) &= -\hat{c}(t-1)\varepsilon(t-1) + y(t) \\ \psi(t) &= -\hat{c}(t-1)\psi(t-1) - \varepsilon(t-1) \end{aligned}$$

Approach 2: Nonlinear filtering

Regard θ as a random variable.

Find the conditional pdf $p(\theta|Z^t)$,
$Z^t = \{y(1), u(1), \dots y(t), u(t)\}$.

Choose

$$\hat{\theta}(t) = E(\theta|Z^t)$$

Nonlinear filtering, linear regresssion

Model

$$y(t) = \varphi^T(t)\theta + e(t), \quad e(t) \sim N(0, r_2)$$

Here, $\varphi(t)$ depends on Z^{t-1}.
A priori distribution $\theta \sim N(\theta_o, P_o)$.

Use Bayes' formula

$$\begin{aligned} p(\theta|Y^t) &= p(\theta|Y^{t-1}, y(t)) = \frac{p(y(t)|\theta, Y^{t-1})p(\theta|Y^{t-1})}{p(y(t)|Y^{t-1})} \\ p(\theta|Y^0) &\sim N(\theta_o, P_o) \end{aligned}$$

Nonlinear filtering, cont'd

$$p(y(t)|\theta, Y^{t-1}) \sim N(\varphi^T(t)\theta, r_2(t))$$

Assume

$$p(\theta|Y^{t-1}) \sim N(\hat{\theta}(t-1)|P(t-1))$$

Use induction to show

$$p(\theta|Y^t) \sim N(\hat{\theta}(t)|P(t))$$

How are $\hat{\theta}(t)$ and $P(t)$ updated?

Nonlinear filtering algorithm

$$\begin{aligned} \hat{\theta}(t) &= \hat{\theta}(t-1) + K(t)[y(t) - \varphi^T(t)\hat{\theta}(t-1)] \\ K(t) &= \frac{P(t-1)\varphi(t)}{r_2 + \varphi^T(t)P(t-1)\varphi(t)} \\ P(t) &= P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{r_2 + \varphi^T(t)P(t-1)\varphi(t)} \end{aligned}$$

Kalman filter interpretation

Linear regression model

$$\begin{aligned}\theta(t+1) &= \theta(t) \quad [= x] \\ y(t) &= \varphi^T(t)\theta + e(t)\end{aligned}$$

Find

$$\hat{x}(t+1) = E[x(t+1)|Y^t] = \hat{x}(t+1|t)$$

Kalman filter, cont'd

$$\begin{aligned}\hat{x}(t+1) &= \hat{x}(t) + K(t)[y(t) - \varphi^T(t)\hat{x}(t-1)] \\ K(t) &= \frac{\overline{P}(t)\varphi(t)}{r_2 + \varphi^T(t)\overline{P}(t)\varphi(t)} \\ \overline{P}(t) &= \overline{P}(t-1) + R_1 - \frac{\overline{P}(t)\varphi(t)\varphi^T(t)\overline{P}(t)}{r_2 + \varphi^T(t)\overline{P}(t)\varphi(t)}\end{aligned}$$

(Set $\overline{P}(t) = P(t-1)$.)

Note: Interpretation of $\hat{\theta}(0), P(0)$.

Nonlinear filtering, general case

State-space model

$$\begin{aligned}x(t+1) &= F(\theta)x(t) + G(\theta)u(t) + w(t) \\ y(t) &= H(\theta)x(t) + e(t) \\ &\quad \operatorname{cov}(w(t)) = R_1(\theta),\ \operatorname{cov}(e(t)) = R_2(\theta)\end{aligned}$$

Form an extended state space model using

$$\overline{x}(t) = \begin{pmatrix} x(t) \\ \theta \end{pmatrix}$$

and use, say, an extended Kalman filter for estimating $\overline{x}(t)$.

Approach 3: Stochastic approximation

Problem: Solve

$$0 = f(x) = EQ(x, e(t))$$

w r t x; $e(t)$ is unknown; $Q(x, e)$ can be observed for arbitrary x.

Example: Minimize

$$V(\theta) = \frac{1}{2}E[y(t) - \varphi^T(t)\theta]^2$$

$$0 = E\varphi(t)[y(t) - \varphi^T(t)\theta]$$

Here, $x = \theta, \quad e(t) = \begin{pmatrix} y(t) & \varphi(t) \end{pmatrix}$.

Stochastic approximation algorithm

$$\hat{x}(t) = \hat{x}(t-1) + \gamma(t)Q(\hat{x}(t-1), e(t))$$

Steplength $\gamma(t)$ satisfies

$$\gamma(t) > 0, \quad \gamma(t) \to 0, t \to \infty$$

Usual conditions

$$\sum \gamma(t) = \infty, \quad \sum \gamma^2(t) < \infty$$

Example: $\gamma(t) = \frac{1}{t}$

Stochastic approximation example

Solve

$$E[e(t) - x] = 0$$

[Hence, $x = Ee(t)$.] Algorithm with $\gamma(t) = 1/t$:

$$\begin{aligned}
\hat{x}(t) &= \hat{x}(t-1) + \frac{1}{t}[e(t) - \hat{x}(t-1)] \\
&= \frac{t-1}{t}\hat{x}(t-1) + \frac{1}{t}e(t) \\
t\hat{x}(t) &= (t-1)\hat{x}(t-1) + e(t) \\
t\hat{x}(t) &= \sum_{s=1}^{t} e(s) \Rightarrow \hat{x}(t) = \frac{1}{t}\sum_{s=1}^{t} e(s)
\end{aligned}$$

SA algorithm, linear regresion

$$\hat{\theta}(t) = \hat{\theta}(t-1) + \gamma(t)\varphi(t)[y(t) - \varphi^T(t)\hat{\theta}(t-1)]$$

LMS (least mean squares)

Some gradient algorithms:

- $\gamma(t) = \gamma_o \quad (\text{fix})$
- $\gamma(t) = \frac{\gamma_o}{|\varphi(t)|^2} \quad (\text{normed})$
- $\gamma(t) = \frac{\gamma_o}{\sum_{s=1}^{t} |\varphi(s)|^2}$

SA algorithm, Newton direction

Set

$$V'' = E\varphi(t)\varphi^T(t)$$

$$\hat{\theta}(t) = \hat{\theta}(t-1) + \gamma(t)\widehat{V''}^{-1}Q(\hat{\theta}(t-1), e(t))$$

Update $V'' \Rightarrow$
Algorithm

$$\begin{aligned}
\hat{\theta}(t) &= \hat{\theta}(t-1) + \gamma(t)R^{-1}(t)\varphi(t)[y(t) - \varphi^T(t)\hat{\theta}(t-1)] \\
R(t) &= R(t-1) + \gamma(t)[\varphi(t)\varphi^T(t) - R(t-1)]
\end{aligned}$$

Approach 4: Pseudolinear regression

Idea: Treat $\varepsilon(t)$ as a second input, known at times $s \geq t+1$. Apply standard RLS.

The idea is applicable to the general linear model

$$A(q^{-1})y(t) = \frac{B(q^{-1})}{F(q^{-1})}u(t) + \frac{C(q^{-1})}{D(q^{-1})}e(t)$$

(and special cases thereof).

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PLR for ARMA models

Model

$$A(q^{-1})y(t) = C(q^{-1})e(t)$$

$$\begin{aligned} y(t) &= \varphi_o^T(t)\theta + e(t) \\ \theta &= \begin{pmatrix} a_1 & \dots & c_n \end{pmatrix}^T \\ \varphi_o^T(t) &= \begin{pmatrix} -y(t-1) & \dots & e(t-n) \end{pmatrix} \end{aligned}$$

Define model regressor

$$\varphi(t) = \begin{pmatrix} -y(t-1) & \dots & \varepsilon(t-n) \end{pmatrix}$$

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PLR algorithm

Note $\varphi(t)$ is formed from Y^{t-1} and $\hat{\theta}(t-1),\ \hat{\theta}(t-2), \dots$

$$\begin{aligned} \hat{\theta}(t) &= \hat{\theta}(t-1) + K(t)\varepsilon(t) \\ \varepsilon(t) &= y(t) - \varphi^T(t)\hat{\theta}(t-1) \\ K(t) &= \frac{P(t-1)\varphi(t)}{1+\varphi^T(t)P(t-1)\varphi(t)} \\ P(t) &= P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{1+\varphi^T(t)P(t-1)\varphi(t)} \end{aligned}$$

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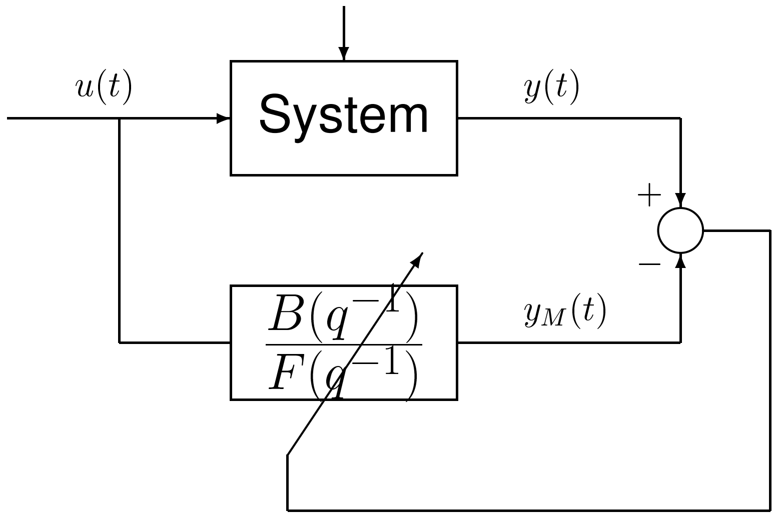
Model reference technique



Model reference technique, cont'd

Find unknown output $\hat{y}_M(t)$.

$$y(t) \approx \frac{B(q^{-1})}{F(q^{-1})} u(t)$$

Rewrite model

$$\begin{aligned} \hat{y}_M(t) &= \varphi^T(t)\hat{\theta}(t-1) \\ &= \begin{pmatrix} -\hat{y}_M(t-1) & \ldots & u(t-n) \end{pmatrix} \begin{pmatrix} \hat{f}_1(t-1) \\ \vdots \\ \hat{b}_n(t-1) \end{pmatrix} \end{aligned}$$

Model reference algorithm

$$\begin{aligned} \hat{\theta}(t) &= \hat{\theta}(t-1) + P(t)\varphi(t)\left[y(t) - \hat{y}_M(t)\right] \\ P(t) &= \left[\sum_{s=1}^{t} \varphi(s)\varphi^T(s)\right]^{-1} \end{aligned}$$

This can be interpreted as a recursive output error method.

The algorithm has better convergence than RLS (often no bias).

Tracking time-varying dynamics

Basic algorithm

$$\hat{\theta}(t) = \hat{\theta}(t-1) + K(t)\varepsilon(t)$$

When the true parameter vector is time-varying, say, $\theta_o(t)$, attempt $\hat{\theta}(t) \approx \theta_o(t)$.

Prevent $K(t)$ from decreasing to 0: For example,

- Modify criterion
- Modify gain sequence $\gamma(t)$
- Modify model

Tracking, Bayes' approach

Modified model

$$\begin{aligned} \theta(t+1) &= \theta(t) + v(t), \quad \mathrm{cov}(v(t)) = R_1 \\ y(t) &= \varphi^T(t)\theta(t) + e(t), \quad \mathrm{cov}(e(t)) = R_2 \end{aligned}$$

Algorithm

$$\begin{aligned} \hat{\theta}(t) &= \hat{\theta}(t-1) + K(t)\varepsilon(t) \\ \varepsilon(t) &= y(t) - \varphi^T(t)\hat{\theta}(t-1) \\ K(t) &= \frac{P(t-1)\varphi(t)}{r_2 + \varphi^T(t)P(t-1)\varphi(t)} \\ P(t) &= P(t-1) + R_1 - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{r_2 + \varphi^T(t)P(t-1)\varphi(t)} \end{aligned}$$

Tracking, modified off-line approach

Modified criterion

$$V_t(\theta) = \frac{1}{2}\sum_{s=1}^{t} \lambda^{t-s}\varepsilon^2(s,\theta)$$

with forgetting factor $\lambda < 1$. Algorithm

$$\begin{aligned}
\hat{\theta}(t) &= \hat{\theta}(t-1) + K(t)\varepsilon(t)\\
\varepsilon(t) &= y(t) - \varphi^T(t)\hat{\theta}(t-1)\\
K(t) &= \frac{P(t-1)\varphi(t)}{\lambda + \varphi^T(t)P(t-1)\varphi(t)}\\
P(t) &= \left[P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{\lambda + \varphi^T(t)P(t-1)\varphi(t)}\right]\frac{1}{\lambda}
\end{aligned}$$

Using a forgetting factor, cont'd

Modified criterion

$$V_t(\theta) = \frac{1}{2}\sum_{s=1}^{t} \lambda^{t-s}\varepsilon^2(s,\theta)$$

Assume $\lambda \approx 1$.
Memory, time constant T?

$$\lambda^T = e^{T\log(\lambda)} \approx e^{T(\lambda-1)} = e^{-1}$$

gives

$$T = \frac{1}{1-\lambda}$$

Tracking, SA algorithm

Choose gain sequence such that $\gamma(t) \not\to 0$.

$$\begin{aligned}
\hat{\theta}(t) &= \hat{\theta}(t-1) + \gamma(t)R^{-1}(t)\varphi(t)\varepsilon(t)\\
R(t) &= R(t-1) + \gamma(t)[\varphi(t)\varphi^T(t) - R(t-1)]
\end{aligned}$$

Compare with

$$\begin{aligned}
\hat{\theta}(t) &= \hat{\theta}(t-1) + P(t)\varphi(t)\varepsilon(t)\\
P(t) &= \left[P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{\lambda(t) + \varphi^T(t)P(t-1)\varphi(t)}\right]\frac{1}{\lambda(t)}
\end{aligned}$$

Are the two algorithms identical?

Tracking, SA algorithms

The two algorithms (SA-based, and RPEM based, resp.) are identical, if

$$\lambda(t) = \frac{1-\gamma(t)}{\gamma(t)}\gamma(t-1)$$

Examples:

1. $\gamma(t) = \gamma_o \Rightarrow \lambda(t) = \lambda = 1 - \gamma_o$
2. $\gamma(t) = \frac{1}{t} \Rightarrow \lambda(t) = 1$

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Summary

There is only one recursive identification algorithm,

$$\hat{\theta}(t) = \hat{\theta}(t-1) + P(t)\varphi(t)\varepsilon(t)$$

$$P(t) = \left[P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{\lambda(t) + \varphi^T(t)P(t-1)\varphi(t)}\right]\frac{1}{\lambda(t)}$$

(+ a part due to the model structure)

A large number of varieties, due to various user choices.

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User choices in recursive algorithms

- Model structure/model parameterization
- Input signal [experimental condition]
- Criterion function
- Gain sequence $\gamma(t)$ [or forgetting factor $\lambda(t)$]
- Search direction [gradient vs. Newton]
- Initial conditions [$\theta(0),\ P(0)$]
- $[$Optional: instrumental variable variant$]$
- $[$Optional: PLR variant$]$