Satisfiability Modulo Theories

Tjark Weber

UPPSALA
UNIVERSITET

Mobility Seminar

January 20, 2012

Introduction

Satisfiability Modulo Theories $=$

Propositional satisfiability $+$ background theories

Example: Job-Shop Scheduling

Given: n jobs, each composed of m tasks of varying duration, that must be performed consecutively on m machines; a total maximum time max.

$d_{i,j}$	Machine 1	Machine 2
Job 1	2	1
Job 2	3	1
Job 3	2	3

$max = 8$

Is there a schedule such that the end-time of every task is $\leq$ max?

Job-Shop Scheduling: SMT Encoding

The job-shop scheduling problem has a straightforward encoding in propositional logic + linear integer arithmetic.

A schedule is specified by the start time $t_{i,j}$ for the j-th task of every job i.

Precedence constraints:

$$t_{i,1} \geq 0 \ \wedge \ t_{i,2} \geq t_{i,1} + d_{i,1} \ \wedge \ t_{i,2} + d_{i,2} \leq max \qquad (\text{for } i = 1, 2, 3)$$

Resource constraints:

$$\begin{aligned}
&(t_{1,j} \geq t_{2,j} + d_{2,j} \ \vee \ t_{2,j} \geq t_{1,j} + d_{1,j}) \ \wedge \\
&(t_{1,j} \geq t_{3,j} + d_{3,j} \ \vee \ t_{3,j} \geq t_{1,j} + d_{1,j}) \ \wedge \\
&(t_{2,j} \geq t_{3,j} + d_{3,j} \ \vee \ t_{3,j} \geq t_{2,j} + d_{2,j}) \qquad (\text{for } j = 1, 2)
\end{aligned}$$

Job-Shop Scheduling: Solution

SMT formula encoding

$d_{i,j}$	Machine 1	Machine 2
Job 1	2	1
Job 2	3	1
Job 3	2	3

$max = 8$

Solution:

$$t_{1,1} = 5,\ t_{1,2} = 7$$
$$t_{2,1} = 2,\ t_{2,2} = 6$$
$$t_{3,1} = 0,\ t_{3,2} = 3$$

Background Theories

- EUF $\quad x = y \implies f(x) = f(y)$
- Arithmetic $\quad y < 0 \implies x + y < x$
- Arrays $\quad \mathrm{select}(\mathrm{store}(a, i, x), i) = x$
- Bit-vectors $\quad 2 \cdot x \;=\; x \ll 1$

- Quantifiers
- Algebraic data types
- ...

Applications

SMT solvers are the core engine of many tools for program analysis, testing and verification.

Dynamic Symbolic Execution

Task: To find input that can steer program execution into specific branches.

Program Model Checking

Task: To prove/refute conjectures about the values of program variables in order to characterize a finite-state abstraction.

Static Program Analysis

Task: To check feasibility of certain program paths.

Program Verification

Task: To prove verification conditions that arise from claims of functional correctness.

SMT Solver Use

We've seen what SMT solvers are good for. How do you actually interact with them?

The SMT-LIB Language

SMT solvers provide a textual interface. Most solvers support a standard language, SMT-LIB.

SMT-LIB defines

- concrete syntax for input formulas, and
- a command-based scripting language.

Solver-specific syntax is often available to extend SMT-LIB, e.g., for data types.

SMT-LIB: Example

```
; This example illustrates basic arithmetic and
; uninterpreted functions

(declare-fun x () Int)
(declare-fun y () Int)
(declare-fun z () Int)
(assert (>= (* 2 x) (+ y z)))
(declare-fun f (Int) Int)
(declare-fun g (Int Int) Int)
(assert (< (f x) (g x x)))
(assert (> (f y) (g x x)))
(check-sat)
(get-model)
(push)
(assert (= x y))
(check-sat)
(pop)
(exit)
```

ask z3 *Is this formula satisfiable?* ***Click 'ask z3'!***

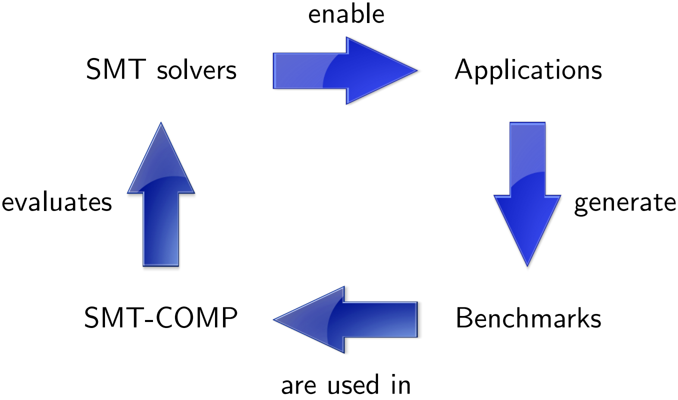
tutorial home video

Inter-Process Communication

- File-based (the basic solution)
- Stream-based (when you need online functionality)

- Web interface (mostly for quick experiments)
- In-memory API (the tightly integrated approach)
- ...

A Virtuous Circle



Choosing an SMT Solver

There are many good SMT solvers: Barcelogic, CVC, MathSAT, OpenSMT, Yices, Z3, . . .

Differences:

- Supported background theories
- Platform, license, API, incrementality, quantifiers, . . .
- Performance (cf. SMT-COMP)

Choosing an SMT Solver

Speaker's pick: Z3

- Good all-round solver, expressive input language
- Excellent performance
- Many other features: MaxSMT, fixed-point constraints, proofs
- Users can define custom theory solvers
- However, closed source (but free for academic use)

Your mileage may vary.

Algorithms

So far, we have considered SMT solvers as a black box.

SMT formula

sat (model)
unsat (proof)

This view is sufficient for many applications!

We'll now talk about what happens inside SMT solvers.

SAT: DPLL

$\vartheta := \emptyset$; *// partial Boolean valuation*
while(true) {
  $\vartheta := \vartheta \cup$ propagate(φ, ϑ); *// deduce consequences*
  if($[[\varphi]]_\vartheta ==$ true) {
    return SATISFIABLE;
  } else if($[[\varphi]]_\vartheta ==$ false) {
    $\vartheta :=$ backtrack(φ, ϑ); *// try a different branch*
    if($\vartheta == \emptyset$) { return UNSATISFIABLE; }
  } else {
    $\vartheta := \vartheta \cup$ decide(φ, ϑ); *// branch on unassigned variable*
  }
}

Interfacing Theory Solvers with SAT

Γ := abstraction function that maps atomic formulas to Boolean variables;
$\varphi := \Gamma(\varphi)$;
while(true) {
  $\vartheta :=$ dpll(φ);
  if($\vartheta ==$ UNSATISFIABLE) { return UNSATISFIABLE; }
  $\Theta := \Gamma^{-1}(\vartheta)$;
  if($\mathcal{T}(\Theta) ==$ SATISFIABLE) { return SATISFIABLE; }
  $\varphi := \varphi \wedge \neg\vartheta$; *// theory lemma*
}

Combining Theory Solvers

Nelson-Oppen combination method: for disjoint, stably infinite theories it is sufficient to propagate equalities between variables.

Example: $x \leq y \wedge y \leq x \wedge P(f(x) - f(y)) \wedge \neg P(0)$

Combining Theory Solvers

Nelson-Oppen combination method: for disjoint, stably infinite theories it is sufficient to propagate equalities between variables.

Example: $x \leq y \wedge y \leq x \wedge P(\underbrace{\underbrace{f(x)}_{v_1} - \underbrace{f(y)}_{v_2}}_{v_3}) \wedge \neg P(\underbrace{0}_{v_4})$

Arithmetic	EUF
$x \leq y$	$v_1 = f(x)$
$y \leq x$	$v_2 = f(y)$
$v_3 = v_1 - v_2$	$P(v_3)$
$v_4 = 0$	$\neg P(v_4)$

Combining Theory Solvers

Nelson-Oppen combination method: for disjoint, stably infinite theories it is sufficient to propagate equalities between variables.

Example: $x \leq y \wedge y \leq x \wedge P(\underbrace{\underbrace{f(x)}_{v_1} - \underbrace{f(y)}_{v_2}}_{v_3}) \wedge \neg P(\underbrace{0}_{v_4})$

Arithmetic	EUF
$x \leq y$	$v_1 = f(x)$
$y \leq x$	$v_2 = f(y)$
$v_3 = v_1 - v_2$	$P(v_3)$
$v_4 = 0$	$\neg P(v_4)$
$x = y$	

Combining Theory Solvers

Nelson-Oppen combination method: for disjoint, stably infinite theories it is sufficient to propagate equalities between variables.

Example: $x \leq y \wedge y \leq x \wedge P(\underbrace{\underbrace{f(x)}_{v_1} - \underbrace{f(y)}_{v_2}}_{v_3}) \wedge \neg P(\underbrace{0}_{v_4})$

Arithmetic	EUF
$x \leq y$	$v_1 = f(x)$
$y \leq x$	$v_2 = f(y)$
$v_3 = v_1 - v_2$	$P(v_3)$
$v_4 = 0$	$\neg P(v_4)$
$x = y$	$v_1 = v_2$

Combining Theory Solvers

Nelson-Oppen combination method: for disjoint, stably infinite theories it is sufficient to propagate equalities between variables.

Example: $x \leq y \wedge y \leq x \wedge P(\underbrace{\underbrace{f(x)}_{v_1} - \underbrace{f(y)}_{v_2}}_{v_3}) \wedge \neg P(\underbrace{0}_{v_4})$

Arithmetic	EUF
$x \leq y$	$v_1 = f(x)$
$y \leq x$	$v_2 = f(y)$
$v_3 = v_1 - v_2$	$P(v_3)$
$v_4 = 0$	$\neg P(v_4)$
$x = y$	$v_1 = v_2$
$v_3 = v_4$	

Combining Theory Solvers

Nelson-Oppen combination method: for disjoint, stably infinite theories it is sufficient to propagate equalities between variables.

Example: $x \leq y \wedge y \leq x \wedge P(\underbrace{\underbrace{f(x)}_{v_1} - \underbrace{f(y)}_{v_2}}_{v_3}) \wedge \neg P(\underbrace{0}_{v_4})$

Arithmetic	EUF
$x \leq y$	$v_1 = f(x)$
$y \leq x$	$v_2 = f(y)$
$v_3 = v_1 - v_2$	$P(v_3)$
$v_4 = 0$	$\neg P(v_4)$
$x = y$	$v_1 = v_2$
$v_3 = v_4$	$\perp$

Conclusion

SMT solvers are expressive and easy to use. They scale orders of magnitude beyond custom ad hoc solvers.

Use them!

Do not write your own constraint solver.