Efficiently Checking Propositional Resolution Proofs in Isabelle/HOL

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- Isabelle/HOL is used for hardware and software verification.

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Can an LCF-style theorem prover benefit from these advances?

Can we increase the degree of automation in Isabelle/HOL while keeping the trusted code base small?

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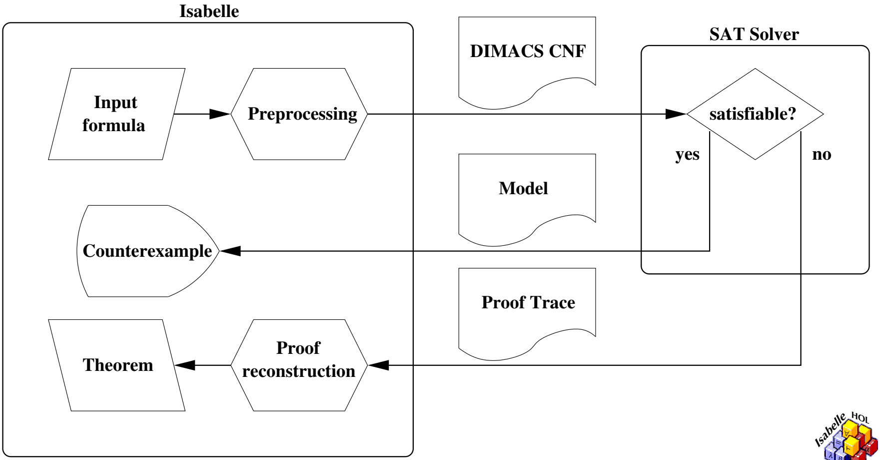
The LCF-style Approach

The external tool provides a *certificate* of its answer that is translated into an Isabelle proof.

System Overview



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The CNF transformation must be proof-producing. The result is not just a CNF formula ϕ^*, but a theorem $\vdash \phi = \phi^*$.

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- return a satisfying assignment, or . . .
- . . . a proof of unsatisfiability (since 2003 (zChaff)/2006 (MiniSat))

Proof Formats

The proofs generated by zChaff and MiniSat differ in detail, but both are based on the propositional resolution rule.

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Propositional Resolution

$$\frac{P \vee x \quad Q \vee \neg x}{P \vee Q}$$

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Propositional Resolution

$$\frac{P \vee x \qquad Q \vee \neg x}{P \vee Q}$$

Proofs: Internal Representation

```
type proof = int list Inttab.table * int
```

- "Clause n is the result of resolving clauses $n_1, \ldots, n_k$."
- "Clause m is the empty clause."

Isabelle's Previous Automation (on TPTP)

Problem	Status	auto	blast	fast
MSC007-1.008	unsat.	x	x	x
NUM285-1	sat.	x	x	x
PUZ013-1	unsat.	0.5	x	5.0
PUZ014-1	unsat.	1.4	x	6.1
PUZ015-2.006	unsat.	x	x	x
PUZ016-2.004	sat.	x	x	x
PUZ016-2.005	unsat.	x	x	x
PUZ030-2	unsat.	x	x	x
PUZ033-1	unsat.	0.2	6.4	0.1
SYN001-1.005	unsat.	x	x	x
SYN003-1.006	unsat.	0.9	x	1.6
SYN004-1.007	unsat.	0.3	822.2	2.8
SYN010-1.005.005	unsat.	x	x	x
SYN086-1.003	sat.	x	x	x
SYN087-1.003	sat.	x	x	x
SYN090-1.008	unsat.	13.8	x	x
SYN091-1.003	sat.	x	x	x
SYN092-1.003	sat.	x	x	x
SYN093-1.002	unsat.	1290.8	16.2	1126.6
SYN094-1.005	unsat.	x	x	x
SYN097-1.002	unsat.	x	19.2	x
SYN098-1.002	unsat.	x	x	x
SYN302-1.003	sat.	x	x	x

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PUZ014-1	unsat.	1.4	x	6.1	0.1
PUZ015-2.006	unsat.	x	x	x	10.5
PUZ016-2.004	sat.	x	x	x	0.3
PUZ016-2.005	unsat.	x	x	x	1.6
PUZ030-2	unsat.	x	x	x	0.7
PUZ033-1	unsat.	0.2	6.4	0.1	0.1
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SYN010-1.005.005	unsat.	x	x	x	0.4
SYN086-1.003	sat.	x	x	x	0.1
SYN087-1.003	sat.	x	x	x	0.1
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- Explicit treatment of associativity and commutativity for $\vee$, $\wedge$ required.

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$$\frac{}{\{\phi\} \vdash \phi}\text{Assume} \qquad \frac{\Gamma \vdash \psi}{\Gamma \setminus \phi \vdash \phi \Longrightarrow \psi}\text{impI} \qquad \frac{\Gamma \vdash \phi \Longrightarrow \psi \quad \Gamma' \vdash \phi}{\Gamma \cup \Gamma' \vdash \psi}\text{impE}$$

Separate Clauses (Alwen Tiu et al., 2006)

Each clause $p_1 \vee \ldots \vee p_n$ is encoded as a single theorem

$$\{p_1 \vee \ldots \vee p_n\} \vdash \overline{p_1} \Rightarrow \ldots \Rightarrow \overline{p_n} \Rightarrow \mathrm{False}$$

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Resolution is based on a derived Isabelle tactic which performs cuts.

- Proof reconstruction for MSC007-1.008: 7.8 s
- The problem is a set of clauses.
- Clauses are not viewed as sets of literals.

Sequent Representation

Each clause $p_1 \vee \ldots \vee p_n$ is encoded as a single theorem

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Resolution:

1. impI: $\Gamma_1 := \{p_1 \vee \ldots \vee p_n, \overline{p_1}, \ldots, \overline{p_n}\} \setminus \{p\} \vdash p \Rightarrow \mathrm{False}$
2. impI: $\Gamma_2 := \{q_1 \vee \ldots \vee q_m, \overline{q_1}, \ldots, \overline{q_m}\} \setminus \{\neg p\} \vdash \neg p \Rightarrow \mathrm{False}$
3. instantiate: $\vdash (p \Rightarrow \mathrm{False}) \Rightarrow (\neg p \Rightarrow \mathrm{False}) \Rightarrow \mathrm{False}$
4. impE: $\Gamma_1 \vdash (\neg p \Rightarrow \mathrm{False}) \Rightarrow \mathrm{False}$
5. impE: $\Gamma_1 \cup \Gamma_2 \vdash \mathrm{False}$

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$$\{p_1 \vee \ldots \vee p_n, \overline{p_1}, \ldots, \overline{p_n}\} \vdash \text{False}$$

- Proof reconstruction for MSC007-1.008: 1.2 s
- The problem is a set of clauses.
- Clauses are sets of literals.
- Clause hypotheses accumulate during resolution, until the set of hypotheses eventually contains every clause used in the proof.

CNF Sequent Representation

1. The whole CNF problem is assumed: $\{\bigwedge_{i=1}^{k} C_i\} \vdash \bigwedge_{i=1}^{k} C_i$.
2. Each clause is derived: $\{\bigwedge_{i=1}^{k} C_i\} \vdash C_1, \ldots, \{\bigwedge_{i=1}^{k} C_i\} \vdash C_k$.
3. Then the (modified) sequent representation is used: $\{\bigwedge_{i=1}^{k} C_i, \overline{p_1}, \ldots, \overline{p_n}\} \vdash \mathit{False}$.

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- Proof reconstruction for MSC007-1.008: 0.5 s
- The right way to do things.

Further Optimizations

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- Lemmas: Instead of proving the same intermediate clause multiple times, we store proven clauses in an array and simply retrieve them from there if they are needed again.

Evaluation on SATLIB Problems

Individual SATLIB problems typically contain several ten thousand variables and several hundred thousand clauses.

SATLIB: Pushing Isabelle to its Limits

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- Solution: small fixes to the kernel.

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Problem	Variables	Clauses
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Problem	zChaff (s)	Proof (s)	Resolutions	Total (s)
c7552mul.miter	73	70	252200	145
6pipe	167	321	268808	512
6pipe_6_ooo	308	2575	870345	3179
7pipe	495	1132	357136	1768

Conclusions

- Isabelle's automation for propositional logic has been greatly enhanced.
- Efficient proof checking for propositional logic is possible in a general LCF-style system.
- Our implementation scales well to proofs with hundreds of thousands of resolution steps.
- Our techniques are applicable to other interactive provers, e.g. to HOL 4 and HOL-Light.

Future Work

- Analysis and optimization of resolution proofs
- SAT-based decision procedures beyond propositional logic (e.g. SMT)
- Standard proof formats (for propositional logic and beyond)