Second-order fractional Fourier transform with incoherent radiation

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Based on the coherent optical theory, we extend the fractional Fourier transform of first-order correlation to a fractional Fourier transform of second-order correlation. An optical system for implementing a second-order fractional Fourier transform was designed. As a numerical example, we investigate the second-order fractional Fourier transform for a single slit.

The concept of a fractional Fourier transform (FRT) was proposed by Namias in 1980 as the generalization of a conventional Fourier transform. McBride and Kerr developed Namias’s theory to a more rigorous form. In 1993 Ozaktas and co-workers introduced the FRT in optics and designed optical systems to achieve FRTs. Since then the FRT has been used widely in signal processing, optical image encryption, beam analysis, and other applications. The discrete FRT, the scaled FRT, the fractional Hilbert transform, the fractional Hankel transform, and multifractional correlation have been investigated. Recently the first-order FRT of a partially coherent beam was studied; this FRT is related to the first-order correlation function.

One can use the second-order Fourier transform to obtain the Fourier transform of an object by measuring the second-order correlation $\langle E(x_1)E^*(x_2)E^*(x_2)E^*(x_1) \rangle$ of two arms (a coincidence-counting rate of two detectors) with the object located in one arm. Interference and imaging in the second-order correlation were proposed and experimentally achieved with entangled photon pairs and were widely used in quantum metrology, lithography, and holography. More recently, Gatti et al. and Chen and Han found that imaging and interference in a second-order correlation can be obtained with incoherent light. In this Letter we extend these studies to a second-order FRT with incoherent light, and an optical system for implementing it is proposed and investigated. The second-order FRT for a single slit is discussed.

The optical system for the second-order FRT is shown in Fig. 1. The light beam is incoherent. First it is split by a beam splitter; then the two emerging beams propagate through either path one or path two to detectors one (D1, a pointlike detector) or two (D2, an array of pixel detectors). In path one (from the light source to D1), between the beam splitter and detector 1 there is an object with transmission function $H(\nu)$, and between the object and D1 there is a lens with focal length $f_1$, and the distances from the object to the lens and from the object to D1 are both $f_1$. In path two (from the light source to D2) there is a lens with focal length $f$ between the split beam and D2, and the distances from the light source to the lens and from the lens to D2 are $l_1$ and $l_2$, respectively.

The coincidence-counting rate is proportional to the second-order correlation function, $G^{(2)}(u_1, u_2)$.

According to optical coherence theory, the second-order correlation function (coincidence-counting rate) between the two detectors obeys the following integral formula for incoherent radiation:

$$G^{(2)}(u_1, u_2) = \langle E(u_1)E^*(u_2)E^*(u_2)E^*(u_1) \rangle$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(x_1, u_1)h_1^*(x_4, u_1)h_2(x_2, u_2)$$

$$\times h_2^*(x_3, u_2)(E(x_1)E^*(x_2)E^*(x_3)E^*(x_4))dx_1dx_2dx_3dx_4$$

$$= \langle I(u_1) \rangle \langle I(u_2) \rangle + |\Gamma(u_1, u_2)|^2,$$ (1)

where $h_1(x_1, u_1)$ and $h_2(x_2, u_2)$ are the response functions of the two paths through which the incoherent light passes and

$$\langle I(u_1) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(x_1, u_1)h_1^*(x_2, u_i)$$

$$\times \langle E(x_1)E^*(x_2) \rangle dx_1dx_2, \quad i = 1, 2,$$ (2)

$$\Gamma(u_1, u_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle E(x_1)E^*(x_2) \rangle h_1(x_1, u_1)$$

$$\times h_2^*(x_3, u_2)dx_1dx_2.$$ (3)
The first-order correlation function for a completely incoherent light radiation source can be expressed as

$$
\langle E(x_1)E^*(x_2) \rangle = I(x_1)\delta(x_1 - x_2),
$$

where $I(x_1)$ is the intensity distribution of the source. In what follows, we assume that the source’s transverse size is infinitely large and the intensity distribution is uniform; then $I(x_1)$ can be expressed approximately as a constant, $I_0$. We study the influence of the source’s surface size below.

Substituting Eqs. (4)–(7) and the detailed information on the two paths, $h_1(x_1, u_1)$ and $h_2(x_2, u_2)$, into Eqs. (2) and (3), and after integration, we obtain

$$
\Gamma(u_1 = 0, u_2) = I_0 \left( \frac{i}{\lambda^2 f_1} \right)^{1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(v_1) \exp \left[ -\frac{i\pi}{\lambda^2} (x_1^2 - 2x_1v_1 + v_1^2) \right] \times \exp \left[ -\frac{i\pi}{\lambda^2} (a_2x_1^2 - 2x_1u_2 + d_2u_2^2) \right] dx_1 dv_1.
$$

We set

$$
l_1 = z + f_0 \tan(\phi/2), \quad l_2 = f_0 \tan(\phi/2),
$$

and substitute Eqs. (10) into Eq. (9). After tedious integration, Eq. (9) reduces to

$$
\Gamma(u_1 = 0, u_2) = \frac{I_0}{(\lambda^2 f_0^2 \sin \phi)^{1/2}} \int_{-\infty}^{\infty} H(v_1) \times \exp \left[ -\frac{i\pi}{\lambda^2} (u_1^2 + u_2^2) \right] dv_1.
$$

Let us recall the definition of a FRT, which is expressed as

$$
E_p(x_2) = \int_{-\infty}^{\infty} E_0(x_1) \exp \left[ -\frac{i\pi}{\lambda f_e} \tan \phi (x_1^2 + x_2^2) \right] \times \frac{2i\pi}{\lambda f_e \sin \phi} x_1 x_2 ] dx_1,
$$

where $f_e$ is the “standard focal length” (Ref. 4) and $\phi$ is defined by $\phi = p\pi/2$, where $p$ is the fractional order of the Fourier transform. The value of $p$ can be any arbitrary real number. When $p$ takes the value $4n + 1$ ($n$ is any integer), $E_p(x_2)$ reduces to the traditional Fourier field in the plane of the focal plane of the lens that has focal length $f_e$.

By comparing Eqs. (11) and (12) we easily find that the optical system shown in Fig. 1 can be used to produce a second-order FRT. In fact, as described in Ref. 4, another optical system as shown in Fig. 2 can also be used to produce a second-order FRT with the same result as that in Fig. 1. In the above calculation we assumed that the source’s transverse size was infinitely large; thus $\Gamma(u_1 = 0, u_2) = \infty$. Because both $|\Gamma(u_1 = 0, u_2)|^2$ and $\langle I(u_1 = 0)I(u_2) \rangle$ contribute to the coincidence counting, the visibility $V = |\Gamma(u_1 = 0, u_2)|^2/G^{(2)}(u_1 = 0, u_2)$ of the FRT of the object is zero. Thus to observe the second-order FRT of an object we have to consider the influence of the source’s transverse size. We assume that the intensity distribution of the light source is of the Gaussian type. Then the first-order correlation function for a completely incoherent light radiation source can be expressed as

$$
\langle E(x_1)E^*(x_2) \rangle = \exp \left[ -\frac{x_1^2 + x_2^2}{4\sigma^2} \right] \delta(x_1 - x_2),
$$

where $\sigma$ is the transverse size of the source. Substituting Eqs. (13) and (4)–(6) into Eqs. (2) and (3), we obtain
Incoherent light radiation source Beam splitter

Fig. 2. Optical system for another second-order FRT.

\[ G^{(2)}(u_1, u_2) \]

Fig. 3. Second-order FRT for a single slit with slit width \( h = 0.02 \) mm for three source’s transverse sizes, \( \sigma_1 \).

\[
\langle I(u_1 = 0) \rangle = \frac{(2\sigma_1^2 \pi)^{1/2}}{\lambda^2 f_1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |H(v_1)|^2 \\
\times \exp \left[ -\frac{2\sigma_1^2 \pi^2}{\lambda^2 z^2} (v_1 - v_2)^2 - i\pi \frac{v_1^2}{\lambda z} + i\pi \frac{v_2^2}{\lambda z} \right] dv_1 dv_2 \\
= \text{const}, \quad (14)
\]

\[
\Gamma(u_1 = 0, u_2) = \left( \frac{i\pi}{\lambda^2 f_1 b_2 A_1} \right)^{1/2} \int_{-\infty}^{\infty} H(v_1) \\
\times \exp \left[ -\frac{\pi^2}{\lambda^2 A_1} \left( \frac{v_1}{z} - \frac{u_2}{b_2} \right)^2 \right] \\
\times \exp \left[ -\frac{i\pi \lambda z}{\lambda^2} \left( v_1^2 + \frac{2\pi}{\lambda} u_2 \right) \right] dv_1, \quad (15)
\]

and \( \langle I(u_2) \rangle = (2\sigma_1^2 \pi)/\lambda |b_2| \), where \( A_1 = 1/2\sigma_1^2 + i\pi/\lambda z - ia_0 \pi/\lambda b_2 \). Then, applying Eqs. (10), we can study numerically the influence of the source’s transverse size on the quality and the visibility of the second-order FRT for an object. In Fig. 3, we show the results of our numerical calculation of the second-order FRT for a single slit with slit width \( h = 0.02 \) mm for three source transverse sizes, \( \sigma_1 \). The transmission function of the slit is \( H(v) = 1 \) for \(-h/2 < v < h/2 \) and \( H(v) = 0 \) otherwise. The other parameters used in the calculation are \( \lambda = 632.8 \) nm, \( f_e = 50 \) mm, \( z = 20 \) mm, and \( p = 1 \). It is clear from Fig. 3 that the fringes gradually disappear, whereas the visibility of the fringes increases with a decrease of the source’s \( \sigma_1 \). To observe the second-order FRT for an object in an experiment, it is necessary to select a suitable \( \sigma_1 \).

In conclusion, we have investigated the second-order fractional Fourier transform as an extension of the FRT of a first-order correlation. An optical system with which to implement the second-order FRT with incoherent sources was designed. The quality and visibility of fringes of the FRT are influenced by the source’s transverse size. The fringes gradually disappear, whereas the visibility of the fringes increases with a decrease of the source’s transverse size. When the size is reduced to a point, the source becomes coherent. Therefore, for a coherent source, the second-order FRT does not exist. It is necessary to select a suitable transverse size for the light source to obtain an observable FRT in the experiment.

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