Appendix D

Answers to Selected Exercises

1.3(a): $\mathcal{Z}\{h_{-k}\} = H(1/z); \quad \mathcal{Z}\{g_k\} = H(z)H^*(1/z^*)$

1.4(a):

$$\phi(\omega) = \frac{\sigma^2}{(1+a_1 e^{-i\omega})(1+a_1^* e^{i\omega})}\left[1 + |b_1|^2 + b_1 e^{-i\omega} + b_1^* e^{i\omega}\right]$$

$$r(0) = \frac{\sigma^2}{1-|a_1|^2}\left\{|1 - b_1 a_1^*|^2 + |b_1|^2(1-|a_1|^2)\right\}$$

$$r(k) = \frac{\sigma^2}{1-|a_1|^2}\left\{\left(1 - \frac{b_1}{a_1}\right)\left(1 - b_1^* a_1\right)\right\}(-a_1)^k, \qquad k \geq 1$$

1.9(a): $\phi_y(\omega) = \sigma_1^2 |H_1(\omega)|^2 + \rho\sigma_1\sigma_2\,[H_1(\omega)H_2^*(\omega) + H_2(\omega)H_1^*(\omega)] + \sigma_2^2|H_2(\omega)|^2$

2.3: An example is $y(t) = \{1, 1.1, 1\}$, whose unbiased ACS estimate is $\hat{r}(k) = \{1.07, 1.1, 1\}$, giving $\hat{\phi}(\omega) = 1.07 + 2.2\cos(\omega) + 2\cos(2\omega)$.

2.4(b): $\text{var}\{\hat{r}(k)\} = \sigma^4\alpha^2(k)(N-k)\left[1 + \delta_{k,0}\right]$

2.9:

(a) $E\{Y(\omega_k)Y^*(\omega_r)\} = \dfrac{\sigma^2}{N}\displaystyle\sum_{t=0}^{N-1} e^{i(\omega_r-\omega_k)t} = \begin{cases} \sigma^2 & k = r \\ 0 & k \neq r \end{cases}$

(c) $E\left\{\hat{\phi}(\omega)\right\} = \sigma^2 = \phi(\omega)$, so $\hat{\phi}(\omega)$ is an unbiased estimate.

3.2: Decompose the ARMA system as $x(t) = \frac{1}{A(z)}e(t)$ and $y(t) = B(z)x(t)$. Then $\{x(t)\}$ is an AR(n) process. To find $\{r_x(k)\}$ from $\{\sigma^2, a_1 \dots a_n\}$, write the Yule–Walker equations as

$$\begin{bmatrix} 1 & & & 0 \\ a_1 & \ddots & & \\ \vdots & & \ddots & \\ a_n & \dots & a_1 & 1 \end{bmatrix} \begin{bmatrix} r_x(0) \\ r_x(1) \\ \vdots \\ r_x(n) \end{bmatrix} + \begin{bmatrix} 0 & a_1 & \dots & a_n \\ \vdots & \vdots & & 0 \\ \vdots & a_n & 0 & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} r_x^*(0) \\ r_x^*(1) \\ \vdots \\ r_x^*(n) \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or

$$A_1 r_x + A_2 r_x^c = \begin{bmatrix} \sigma^2 \\ 0 \end{bmatrix}$$

which can be solved for $\{r_x(m)\}_{m=0}^n$. Then find $r_x(k)$ for $k > n$ from equation (3.3.4) and $r_x(k)$ for $k < 0$ from $r_x^*(-k)$. Finally,

$$r_y(k) = \sum_{j=0}^{m} \sum_{p=0}^{m} r_x(k + p - j)\, b_j b_p^*$$

3.4: $\sigma_b^2 = E\left\{|e_b(t)|^2\right\} = [1\ \theta_b^T] R_{n+1} \begin{bmatrix} 1 \\ \theta_b^c \end{bmatrix} = [1\ \theta_b^*] R_{n+1}^c \begin{bmatrix} 1 \\ \theta_b \end{bmatrix}$ giving $\theta_b = \theta_f$ and $\sigma_b^2 = \sigma_f^2$.

3.5(a):

$$R_{2m+1}^T \begin{bmatrix} c_m \\ \vdots \\ c_1 \\ 1 \\ d_1 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sigma_s^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

3.14: $c_\ell = \sum_{i=0}^{n} a_i \tilde{r}(\ell - i)$ for $0 \le \ell \le p$, where $\tilde{r}(k) = r(k)$ for $k \ge 1$, $\tilde{r}(0) = r(0)/2$, and $\tilde{r}(k) = 0$ for $k < 0$.

3.15(b): First solve for $b_1, \dots, b_m$ from

$$\begin{bmatrix} c_n & c_{n-1} & \cdots & c_{n-m+1} \\ c_{n+1} & c_n & \cdots & c_{n-m+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n+m-1} & c_{n+m-2} & \dots & c_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = - \begin{bmatrix} c_{n+1} \\ c_{n+2} \\ \vdots \\ c_{n+m} \end{bmatrix}$$

Then $a_1, \dots, a_n$ can be obtained from $a_k = c_k + \sum_{i=1}^{m} b_i c_{k-i}$.

4.2:
(a) $E\{x(t)\} = 0$; $r_x(k) = (\overline{\alpha}^2 + \sigma_\alpha^2)e^{i\omega_0 k}$
(b) Let $p(\varphi) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\varphi}$ be the Fourier series of $p(\varphi)$ for $\varphi \in [-\pi, \pi]$. Then $E\{x(t)\} = \frac{\overline{\alpha}e^{i\omega_0 t}}{2\pi}c_1$. Thus, $E\{x(t)\} = 0$ if and only if either $\overline{\alpha} = 0$ or $c_1 = 0$. In this case, $r_x(k)$ is the same as in part (a).

5.8: The height of the peak of the (unnormalized) Capon spectrum is

$$1/a^*(\omega)R^{-1}a(\omega)|_{\omega=\omega_0} = \frac{m\alpha^2 + \sigma^2}{m}$$