Uniprocessor Feasibility of Sporadic Tasks Remains coNP-complete Under Bounded Utilization

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The General Setting

Instances

Task set $T$ of sporadic (or synchronous periodic) tasks with constrained deadlines.
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Instances

Task set $T$ of sporadic (or synchronous periodic) tasks with constrained deadlines.

Question

Is $T$ feasible on a preemptive uniprocessor?
An Algorithm for Feasibility [Baruah et al., 1990]

\[ \ell(P(T)) = \text{lcm}(f_{p_j}(e; d; p)^2 T_g) \]

Feasibility

- Exp. time algorithm exists
- In coNP
- Weakly coNP-hard [Eisenbrand & Rothvoß, SODA'10]
- Conjectured pseudo-poly. time
- Strongly coNP-hard [ECRTS'15]

Feasibility

- Pseudo-poly. time algorithm if \( c < 1 \)
- In coNP
- Conjectured poly. time for all \( c < 1 \)
- Weakly coNP-hard for all \( c > 0 \)

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Feasibility is coNP-complete Under Bounded Utilization
An Algorithm for Feasibility [Baruah et al., 1990]

Feasibility is coNP-complete Under Bounded Utilization
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Feasibility

- Exp. time algorithm exists

\[ \ell \]
\[ \mathcal{P}(T) = \text{lcm}\{p \mid (e, d, p) \in T\} \]
An Algorithm for Feasibility [Baruah et al., 1990]

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An Algorithm for Feasibility [Baruah et al., 1990]

\[
\ell \leq \text{lcm} \{ p \mid (e, d, p) \in T \}
\]

Feasibility

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Feasibility is coNP-complete Under Bounded Utilization
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Expanding on the feasibility problem:

\[
\ell \overset{\text{p.p.}}{=} \text{lcm} \{ p \mid (e, d, p) \in T \}
\]

p.p. if \( U(T) \leq c < 1 \)
An Algorithm for Feasibility [Baruah et al., 1990]

Feasibility
- Exp. time algorithm exists
- In coNP

Feasibility (U(T) ≤ c)
- Pseudo-poly. time algorithm if c < 1
- In coNP

c-Feasibility

\[ \ell = \text{lcm}\{p \mid (e, d, p) \in T\} \]
An Algorithm for Feasibility [Baruah et al., 1990]

Feasibility
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C-Feasibility \((U(T) \leq c)\)
- Pseudo-poly. time algorithm if \(c < 1\)
- In coNP

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\[
\ell(P(T)) = \text{lcm}(f_{p_j}(e, d, p))^{2T_g}
\]

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\[c\text{-Feasibility } (U(T) \leq c)\]

- Pseudo-poly. time algorithm if \(c < 1\)
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An Algorithm for Feasibility [Baruah et al., 1990]

\[ P(T) = \text{lcm} \left( \prod_{j=1}^{f} \left( \sum_{k=1}^{p} \left( e_k; d_k; p_k \right) \right) \right) \]

0 ≤ \( U(T) \) ≤ 1

Feasibility

- Exp. time algorithm exists
- \( \text{coNP} \)
- Weakly \( \text{coNP} \)-hard
  - [Eisenbrand & Rothvoß, SODA'10]
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- In \( \text{coNP} \)
- Conjectured poly. time for all \( c < 1 \)
An Algorithm for Feasibility [Baruah et al., 1990]

\[ \ell(T) = \text{gcd}(p_i(e; d); p_j(e; d))^2 \]

\[ U(T) \leq c \]

\[ c \text{-Feasibility} \]

- Pseudo-poly. time algorithm if \( c < 1 \)
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An Algorithm for Feasibility [Baruah et al., 1990]

\[ P(T) = \text{lcm} \prod p_j(e, d, p)^2T_g \]

\( U(T) \) if \( U(T) \leq c < 1 \)

\( c \)-Feasibility \( (U(T) \leq c) \)

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\[ P(T) = \text{lcm}(p_j(e; d; p)) \]

\[ U(T) \leq c < 1 \]

**c-Feasibility \ (U(T) \leq c)**

- Pseudo-poly. time algorithm if \( c < 1 \)
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\[ \ell(P(T)) = \text{lcm}(f(p_j(e; d; p)))^2 T_g \]

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**Feasibility**

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An Algorithm for Feasibility [Baruah et al., 1990]

\[ P(T) = \text{lcm}(\text{f}_p(j(e,d,p))^{2T}) \]

\[ U(T) \leq c < 1 \]

\[ \ell \]

**c-Feasibility** \((U(T) \leq c)\)

- Pseudo-poly. time algorithm if \( c < 1 \)
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- Weakly \( \text{coNP} \)-hard for all \( c > 0 \)
Proving Hardness for $c$-Feasibility
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Feasibility is coNP-complete Under Bounded Utilization
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Proving Hardness for \( c \)-Feasibility

Feasibility is coNP-complete Under Bounded Utilization
Proving Hardness for \( c \)-Feasibility

1. \( T \) is feasible \( \iff \) \( T_c \) is feasible
2. \( U(T_c) \leq c \)
3. \( T_c \) is computed in poly. time
Feasibility $\propto c$-Feasibility, Step 1

Feasibility preserved
Utilization $\leq c$
Computed in poly. time

Feasibility preserved
Utilization $\leq c$
Compute in poly. time

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Feasibility $\propto c$-Feasibility, Step 1
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Feasibility preserved

Utilization $\leq c$

Computed in poly. time

Feasibility preserved

3

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Feasibility is coNP-complete Under Bounded Utilization
Feasibility $\propto c$-Feasibility, Step 1
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Feasibility preserved

Utilization $\leq c$

Computed in poly. time

Feasibility preserved

$\text{Pontus Ekberg}$

Feasibility is coNP-complete Under Bounded Utilization
Feasibility $\propto c$-Feasibility, Step 1

$\mathcal{P}(T) \leq c \mathcal{P}(T) + \mathcal{P}(T)$

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

Feasibility preserved

Pontus Ekberg

Feasibility is coNP-complete Under Bounded Utilization
Feasibility ∝ c-Feasibility, Step 1

\[ \mathcal{P}(T) + \mathcal{P}(T) + \mathcal{P}(T) = \text{Feasibility preserved} \]

Utilization \( \leq c \) Computed in poly. time

Feasibility is coNP-complete Under Bounded Utilization

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Feasibility $\propto c$-Feasibility, Step 1

\[
\ell \begin{array}{c}
\mathcal{P}(T) \\
2\mathcal{P}(T) \\
3\mathcal{P}(T)
\end{array}
\]

\[
P(T) + P(T) + P(T) = \text{Feasibility preserved}
\]

Utilization $\leq c$ Computed in poly. time

Feasibility is coNP-complete Under Bounded Utilization
Feasibility $\propto c$-Feasibility, Step 1

Feasibility preserved

Utilization $\leq c$

Computed in poly. time
Feasibility $\propto c$-Feasibility, Step 1

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

Feasibility is coNP-complete Under Bounded Utilization
Feasibility \( \propto c\text{-}Feasibility, \text{Step 1} \)

\[ \begin{align*}
\text{Feasibility preserved} & \quad \checkmark \\
\text{Utilization} \leq c & \quad \times \\
\text{Computed in poly. time} &
\end{align*} \]
Feasibility $\propto c$-Feasibility, Step 1

- Feasibility preserved
- Utilization $\leq c$
- Computed in poly. time

Feasibility is coNP-complete Under Bounded Utilization
Feasibility $\propto c$-Feasibility, Step 2

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

$\mathcal{P}(T)$ $2\mathcal{P}(T)$ $3\mathcal{P}(T)$

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Feasibility $\propto c$-Feasibility, Step 2

Feasibility preserved
Utilization $\leq c$
Computed in poly. time
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Feasibility preserved

Utilization $\leq c$

Computed in poly. time
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Feasibility $\propto c$-Feasibility, Step 2

Feasibility preserved \(\times\)

Utilization $\leq c$ \(\checkmark\)

Computed in poly. time \(\checkmark\)
Feasibility $\propto c$-Feasibility, Step 3

\[
P(T) \quad 2P(T) \quad 3P(T) \quad \ell
\]

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

Pontus Ekberg
Feasibility $\propto$ $c$-Feasibility, Step 3

Feasibility preserved

Utilization $\leq c$

Computed in poly. time
Feasibility $\propto c$-Feasibility, Step 3

Feasibility preserved

Utilization $\leq c$

Computed in poly. time
Feasibility \( \propto c\)-Feasibility, Step 3

Feasibility preserved

Utilization \( \leq c \)

Computed in poly. time

\[
P(T) \quad 2P(T) \quad 3P(T)
\]

\[
\ell
\]
Feasibility $\propto c$-Feasibility, Step 3

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

Feasibility is coNP-complete Under Bounded Utilization
**Feasibility $\propto c$-Feasibility, Step 3**

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

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Pontus Ekberg  Feasibility is coNP-complete Under Bounded Utilization
Feasibility $\propto c$-Feasibility, Step 3

Feasibility preserved

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Pontus Ekberg
Feasibility is coNP-complete Under Bounded Utilization
Feasibility $\propto$ $c$-Feasibility, Step 3

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

$P(T) \leq 2P(T)\leq 3P(T)$
**Feasibility \( \propto c\text{-Feasibility}, \text{Step 3}\)**

Feasibility preserved

Utilization \( \leq c\)

Computed in poly. time

\[ P(T) \quad 2P(T) \quad 3P(T) \quad \ell \]
Feasibility $\propto c$-Feasibility, Step 3

Feasibility preserved

Utilization $\leq c$

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Feasibility is coNP-complete Under Bounded Utilization
Feasibility $\propto c$-Feasibility, Step 3

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

$p(T) \quad 2p(T) \quad 3p(T)$

$l$
Feasibility \& c-Feasibility, Step 3

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

$\mathcal{P}(T)$ $\mathcal{2P}(T)$ $\mathcal{3P}(T)$ $\ell$

Feasibility is coNP-complete Under Bounded Utilization
Feasibility \(\propto c\)-Feasibility, Step 3

Feasibility preserved

Utilization \(\leq c\)

Computed in poly. time
Feasibility $\propto c$-Feasibility, Step 3

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

$\mathcal{P}(T)$ $2\mathcal{P}(T)$ $3\mathcal{P}(T)$ $\ell$
Feasibility $\propto$ c-Feasibility, Step 3

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

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Feasibility \( \propto \) c-Feasibility, Step 3

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Feasibility $\propto \text{c-Feasibility, Step 3}$

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Feasibility $\propto c$-Feasibility, Step 3

- Feasibility preserved
- Utilization $\leq c$
- Computed in poly. time

Feasibility is coNP-complete Under Bounded Utilization
Feasibility ∝ c-Feasibility, Step 3

Feasibility preserved

Utilization $\leq c$

Computed in poly. time

Pontus Ekberg  Feasibility is coNP-complete Under Bounded Utilization 7
Feasibility $\propto c$-Feasibility, Step 3

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Feasibility $\propto c$-Feasibility, Step 3

Feasibility preserved
Utilization $\leq c$
Computed in poly. time

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Feasibility $\propto c$-Feasibility, With log-scale Glasses

Feasibility is coNP-complete Under Bounded Utilization
Feasibility ∝ c-Feasibility, With log-scale Glasses
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Feasibility \propto c\text{-Feasibility, With log-scale Glasses}
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Feasibility is coNP-complete Under Bounded Utilization
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$P(T)$

$\log(\ell)$
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Feasibility is coNP-complete Under Bounded Utilization
When is Feasibility Decidable in Poly. Time?

1. If deadlines are implicit. [Liu & Layland, 1973]

2. If deadlines are constrained and periods are harmonic. [Bonifaci et al., 2013]

3. If $U(T) \leq c < 1$ and $\frac{\text{max period}}{\text{min period}} \leq q(n)$. 
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   \[
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   \]
## Conclusion

<table>
<thead>
<tr>
<th>Asynchronous periodic</th>
<th>General case</th>
<th>Utilization bounded by a constant $c$, $0 &lt; c &lt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Strongly coNP-complete</strong></td>
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</tr>
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<td></td>
</tr>
<tr>
<td>Synchronous periodic (or sporadic)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
∀Thank you!

∃Questions?