Complexity of partitioned scheduling for periodic tasks

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MAPSP'22

Periodic tasks

A periodic task τ_i is given by $(O_i, C_i, D_i, T_i) \in \mathbb{N}^4$, where

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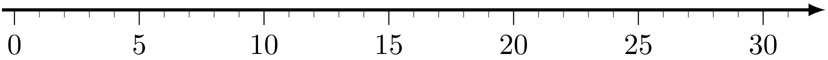

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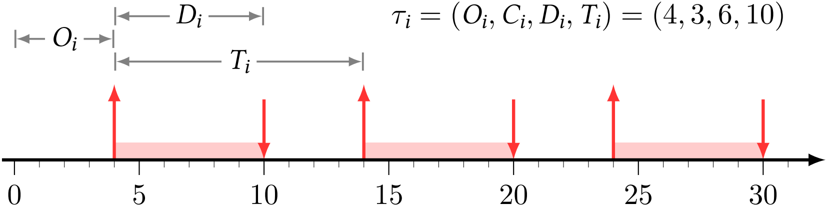

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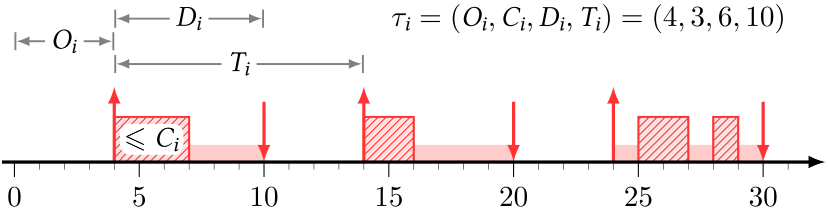

Common restrictions

Let $\mathcal{T} = \{\tau_1, \ldots, \tau_n\}$ be a set of tasks.

We say that $\mathcal{T}$ has

- *implicit deadlines*, if $D_i = T_i$ for all $\tau_i \in \mathcal{T}$,
- *constrained deadlines*, if $D_i \leqslant T_i$ for all $\tau_i \in \mathcal{T}$,
- *arbitrary deadlines*, otherwise.

Also, $\mathcal{T}$ is

- *synchronous*, if $O_i = O_j$ for all $\tau_i, \tau_j \in \mathcal{T}$,
- *asynchronous*, otherwise.

Partitioned scheduling

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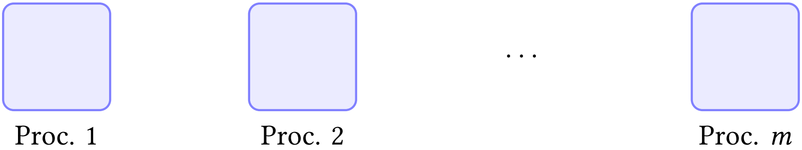
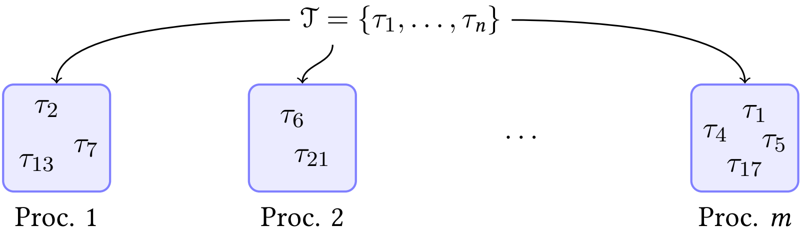


Partitioned scheduling



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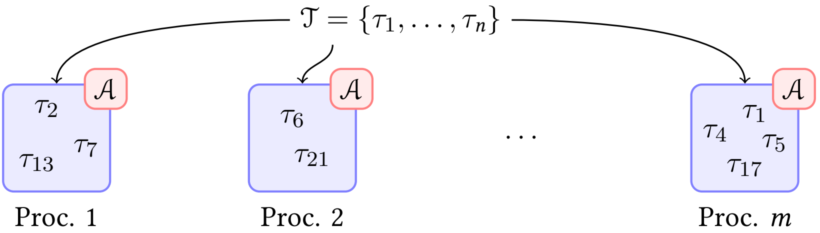

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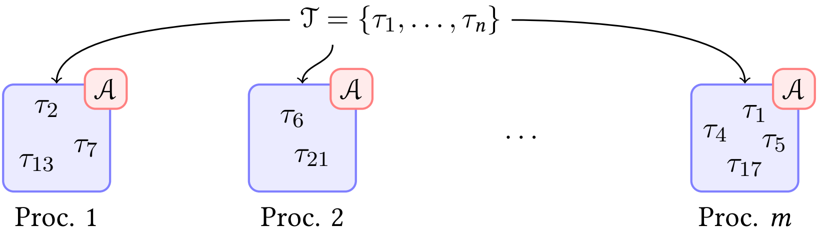

PARTITIONED $\mathcal{A}$-SCHEDULABILITY

Instance: $\langle \mathcal{T}, m \rangle$, where $\mathcal{T}$ is a set of tasks and m is the number of processors.

Question: Is there a partitioning $(\mathcal{T}_1, \ldots, \mathcal{T}_m)$ of $\mathcal{T}$ such that each $\mathcal{T}_i$ is schedulable by $\mathcal{A}$ on a single processor?

The polynomial hierarchy

$$
\begin{tikzcd}
\vdots & & \vdots \\
\Sigma_3^{\mathsf{P}} & & \Pi_3^{\mathsf{P}} \\
\Sigma_2^{\mathsf{P}} \arrow[u] \arrow[urr] & & \Pi_2^{\mathsf{P}} \arrow[u] \arrow[ull] \\
\mathsf{NP} \arrow[u] \arrow[urr] & & \mathsf{coNP} \arrow[u] \arrow[ull] \\
& \mathsf{P} \arrow[ul] \arrow[ur] &
\end{tikzcd}
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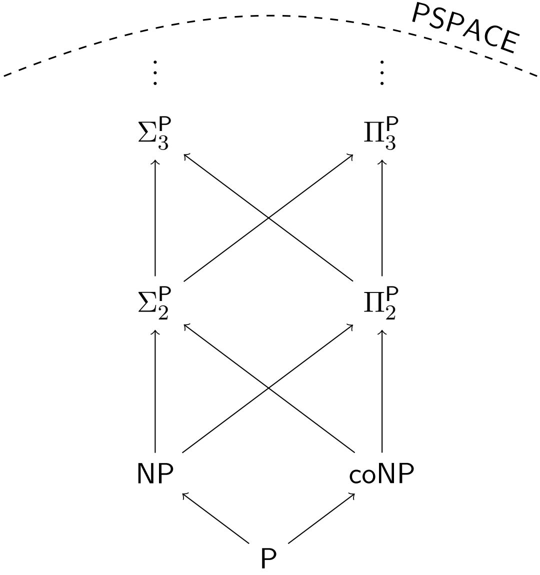

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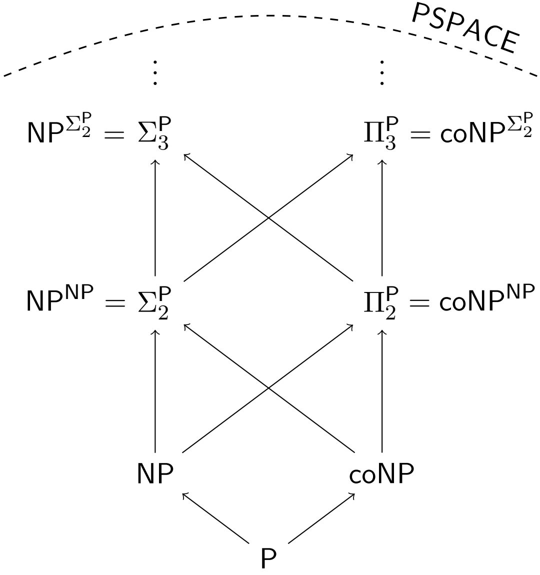

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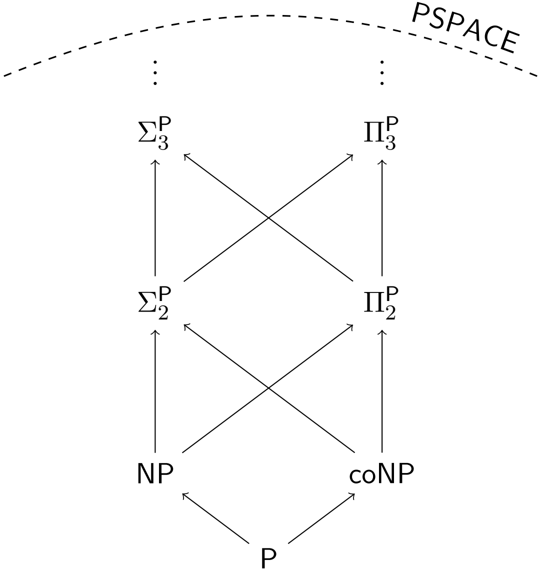

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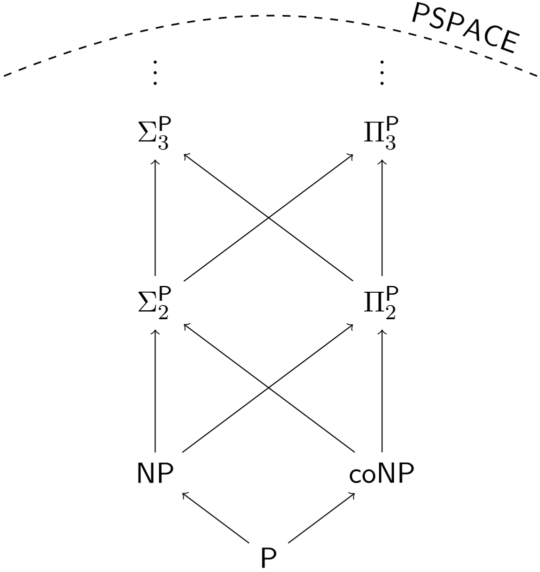
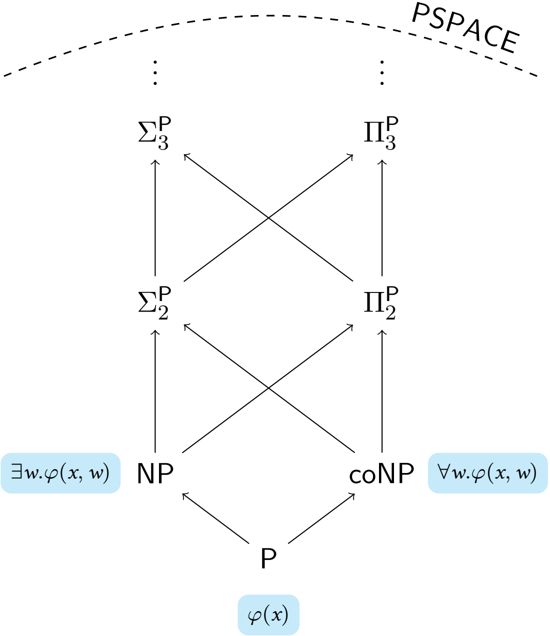


$\varphi(x)$

The polynomial hierarchy



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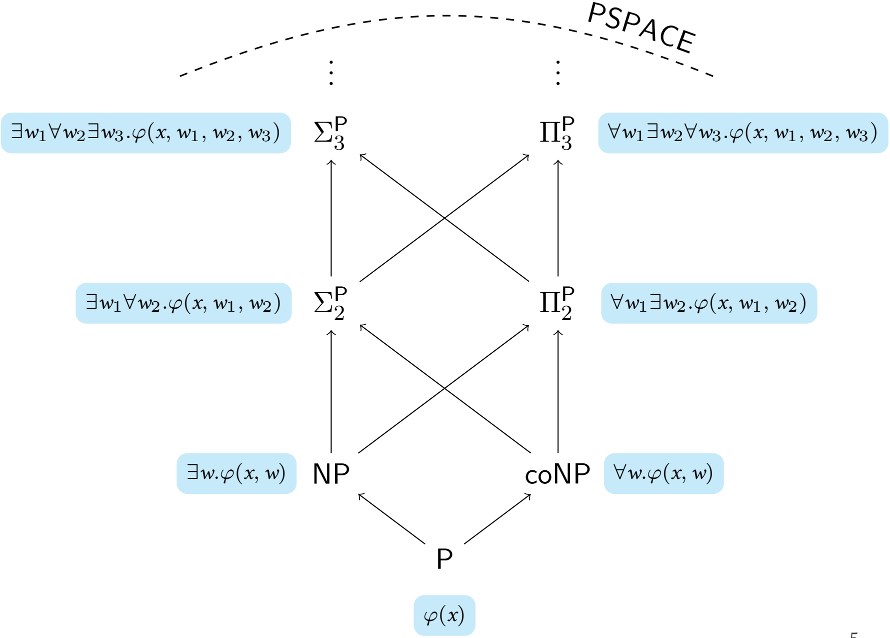

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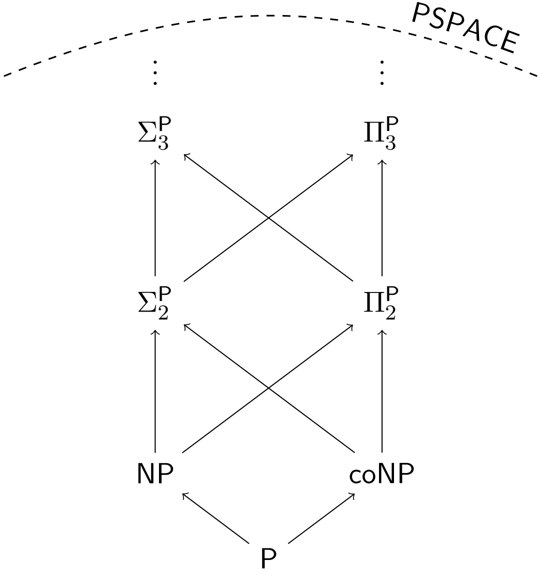

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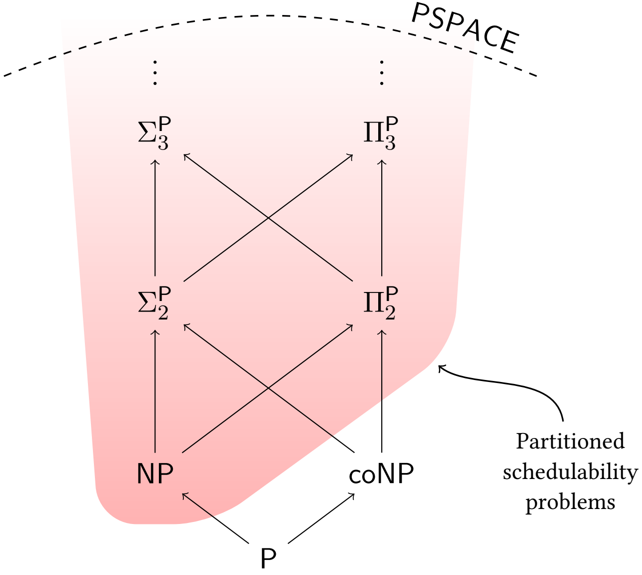

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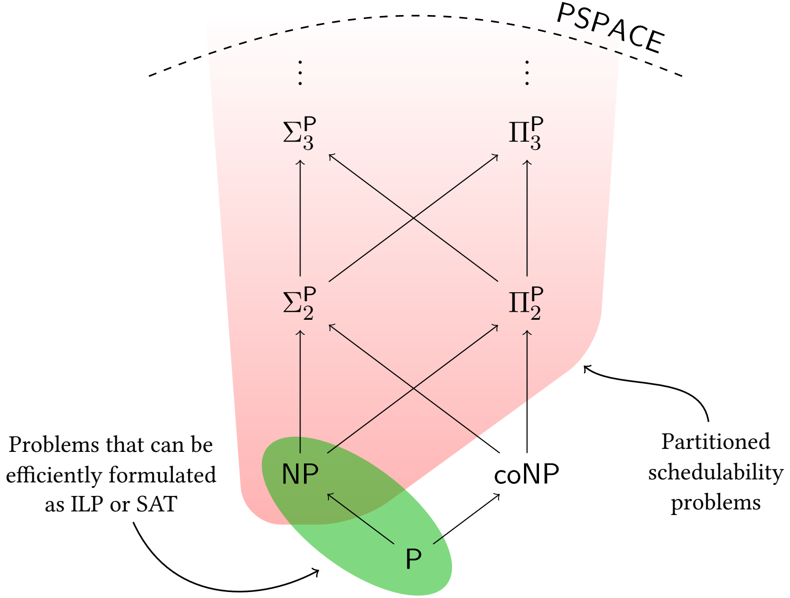

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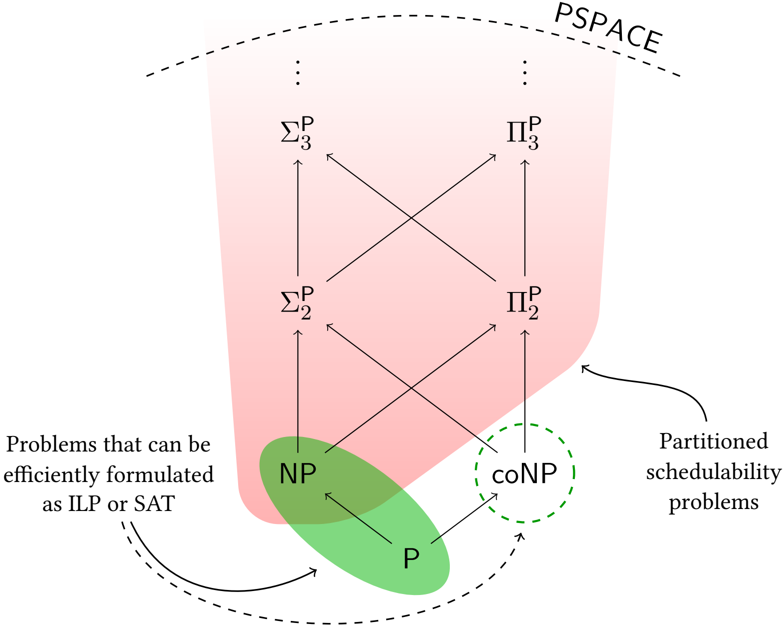

Simultaneous Congruences

The Simultaneous Congruences Problem (SCP):

Example: $A = \{(2, 4), (4, 6), (3, 8), (0, 3)\}, \qquad k = 2$

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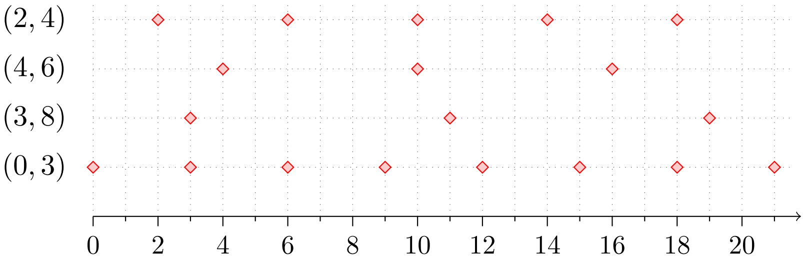

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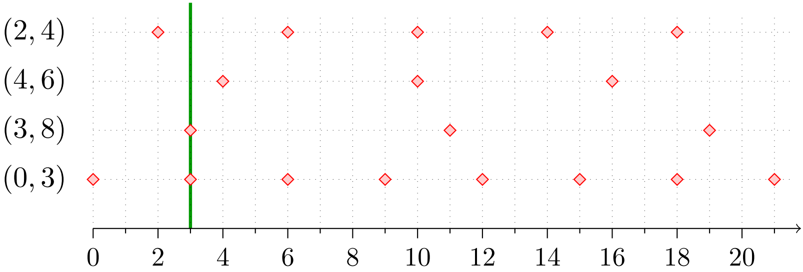

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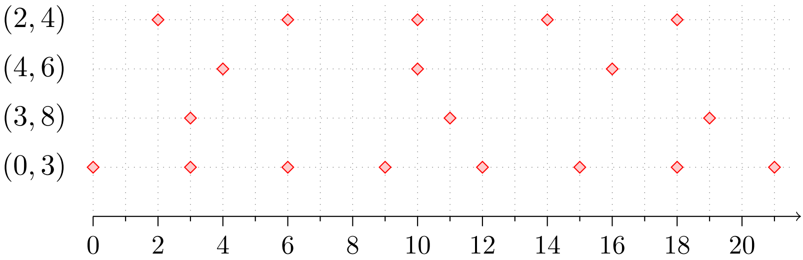

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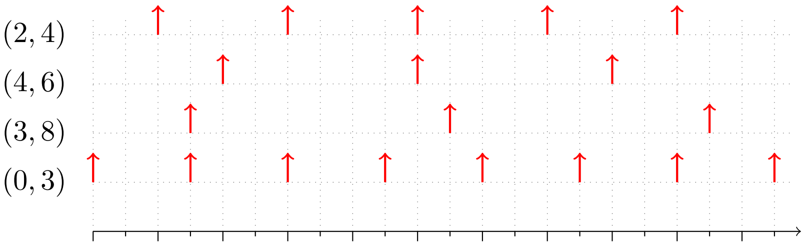

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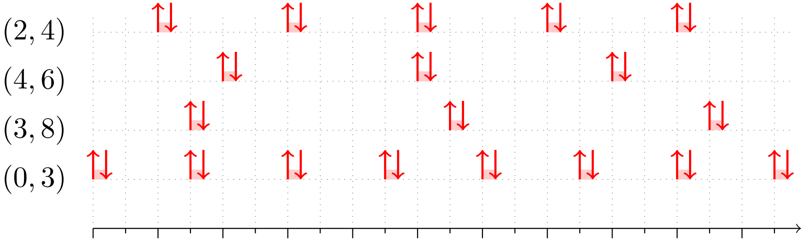

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PARTITIONED SCP

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GENERALIZED GRAPH COLORING $\longrightarrow$ PARTITIONED SCP

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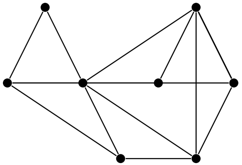
GENERALIZED GRAPH COLORING $\longrightarrow$ PARTITIONED SCP $\longrightarrow$ Some partitioned schedulability problems

Hardness of partitioned scp

GENERALIZED GRAPH COLORING

G



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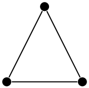

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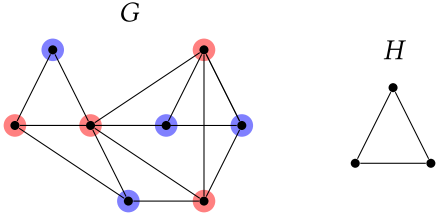

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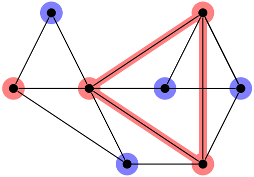

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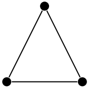

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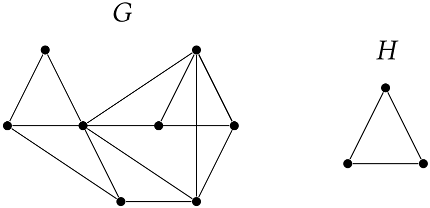

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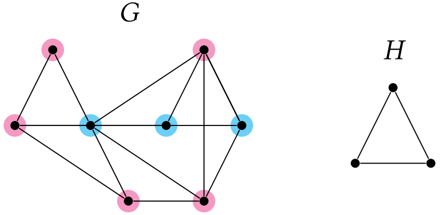
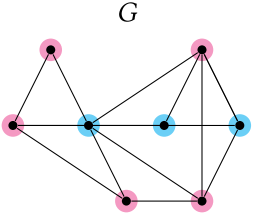
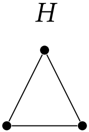


Hardness of partitioned SCP

GENERALIZED GRAPH COLORING

Σ_2^{P}-complete with two colors and H is a complete graph (Rutenburg, 1986)





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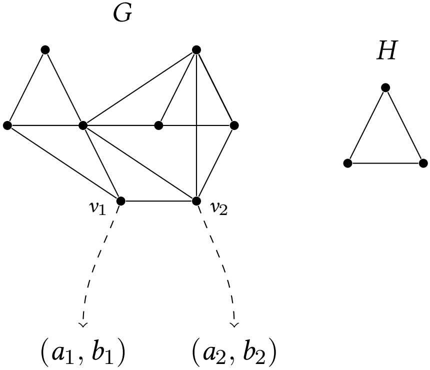

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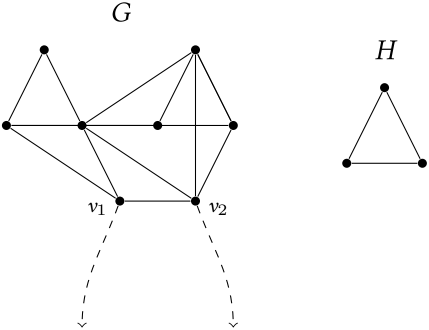

PARTITIONED SCP

(a_1, b_1) (a_2, b_2)

(a_7, b_7) (a_4, b_4)

(a_5, b_5)

(a_6, b_6)

(a_8, b_8) (a_3, b_3)

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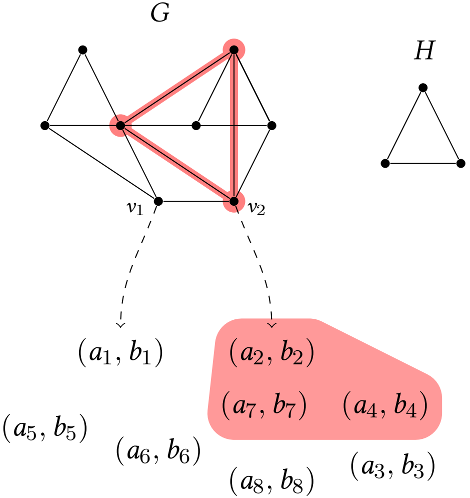

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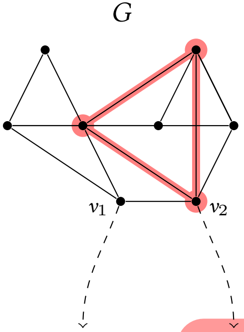

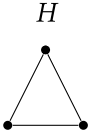

This method from Leung and Whitehead, 1982

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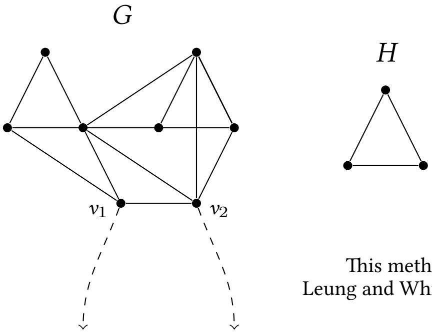
Hardness of partitioned SCP

Generalized Graph Coloring

Σ_2^{P}-complete with two colors and H is a complete graph (Rutenburg, 1986)



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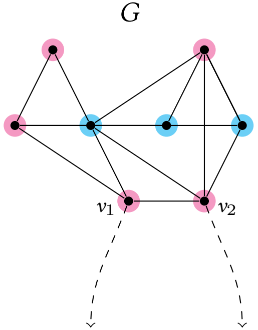

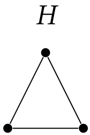

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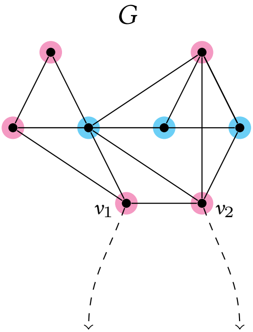

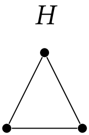

This method from Leung and Whitehead, 1982

Partitioned SCP

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(a_7, b_7) (a_4, b_4)

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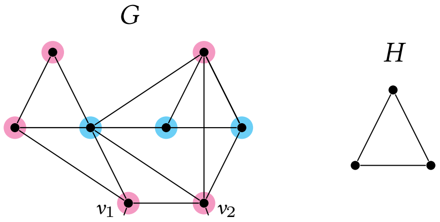
#colliding congruences $(k) = |H|$

#partitions $(m) =$ #colors

Hardness of partitioned SCP



Generalized Graph Coloring

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This method from Leung and Whitehead, 1982

Partitioned SCP

Σ_2^{P}-complete

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Example: $A = \{(2,4), (4,6), (3,8), (0,3), \ldots\}, \quad m, \quad k$

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GENERALIZED GRAPH COLORING → PARTITIONED SCP → Some partitioned schedulability problems

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GENERALIZED GRAPH COLORING $\longrightarrow$ PARTITIONED SCP $\longrightarrow$ Some partitioned schedulability problems

Σ_2^{P}-complete, Rutenburg, 1986

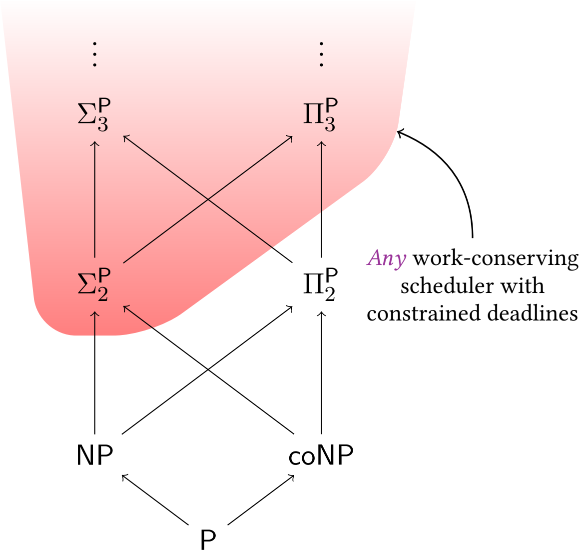
Σ_2^{P}-complete

Σ_2^{P}-hard

Complexity for asynchronous periodic tasks

$$
\begin{tikzcd}
\vdots & & \vdots \\
\Sigma_3^{\mathsf{P}} & & \Pi_3^{\mathsf{P}} \\
\Sigma_2^{\mathsf{P}} \arrow[u] \arrow[urr] & & \Pi_2^{\mathsf{P}} \arrow[u] \arrow[ull] \\
\mathsf{NP} \arrow[u] \arrow[urr] & & \mathsf{coNP} \arrow[u] \arrow[ull] \\
& \mathsf{P} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Complexity for asynchronous periodic tasks



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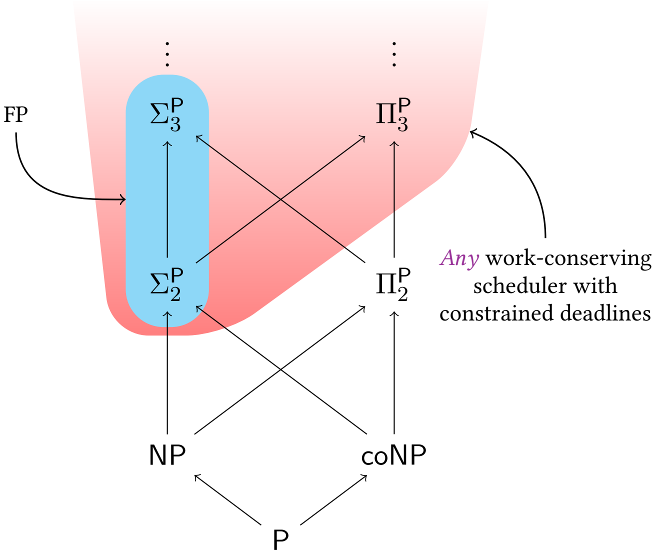

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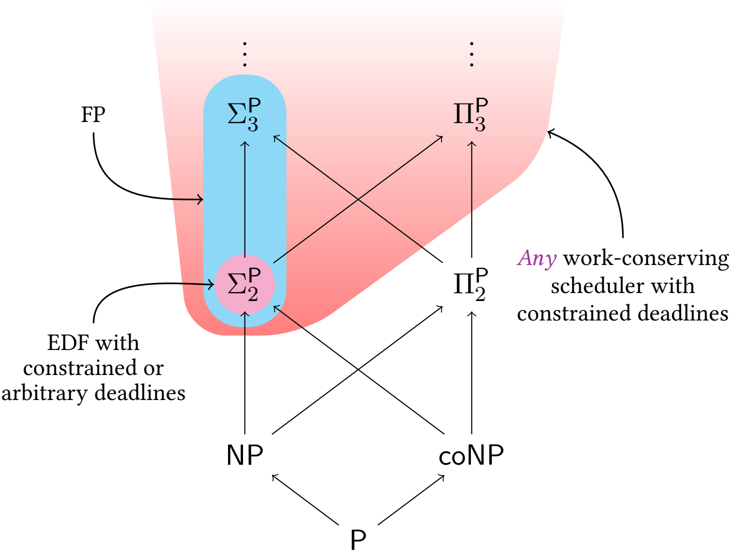

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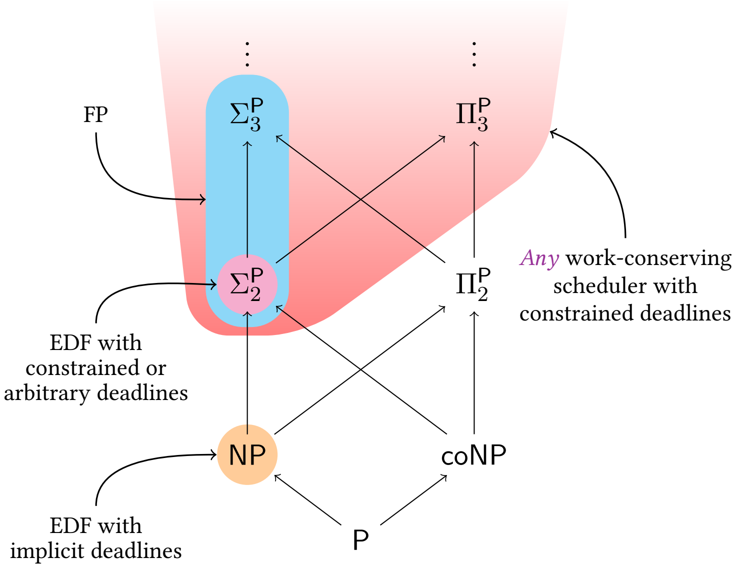

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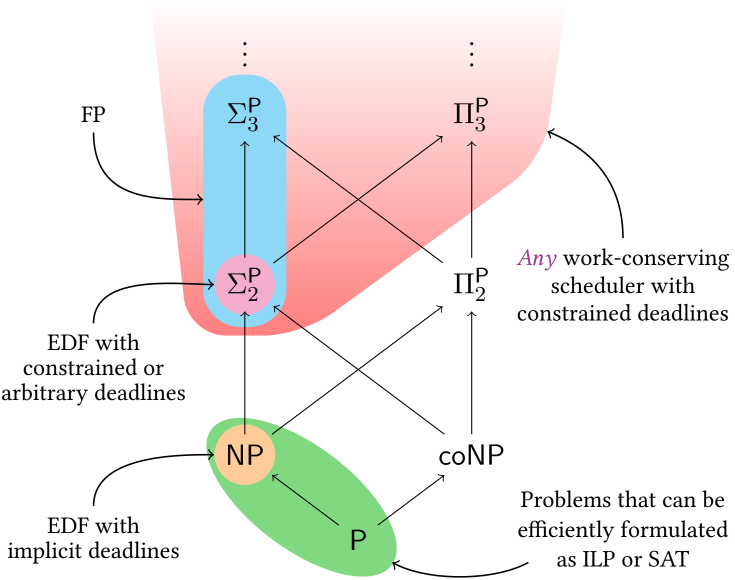

Complexity for sporadic / synchronous periodic tasks

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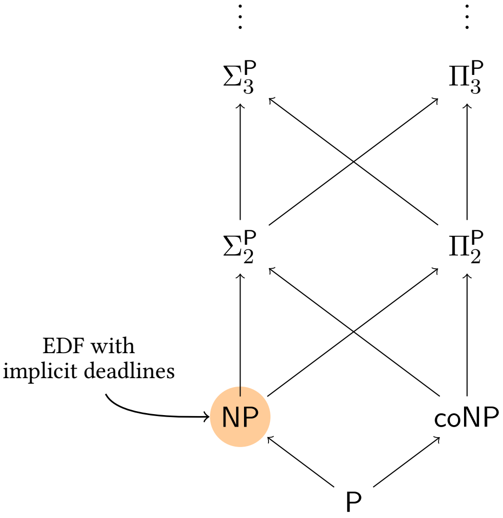

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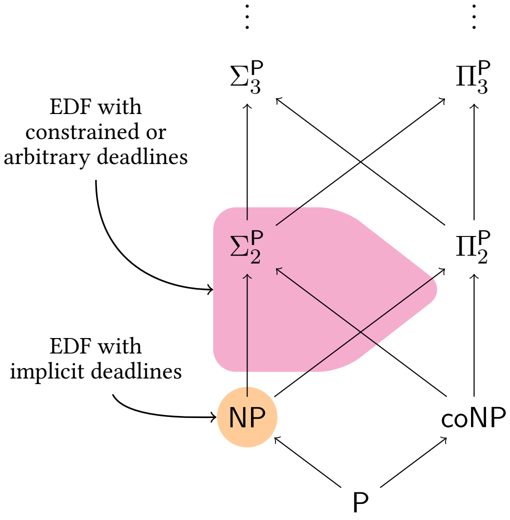

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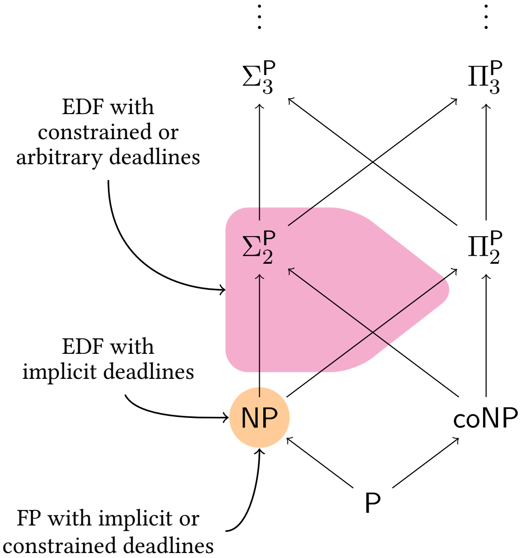

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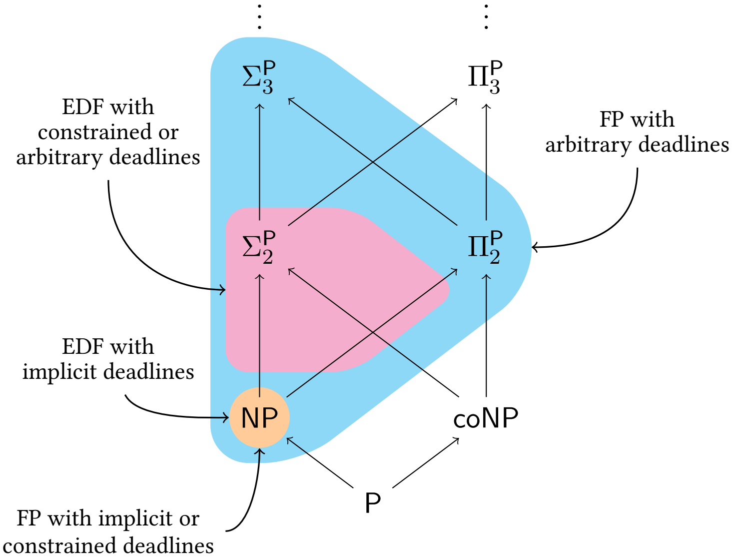

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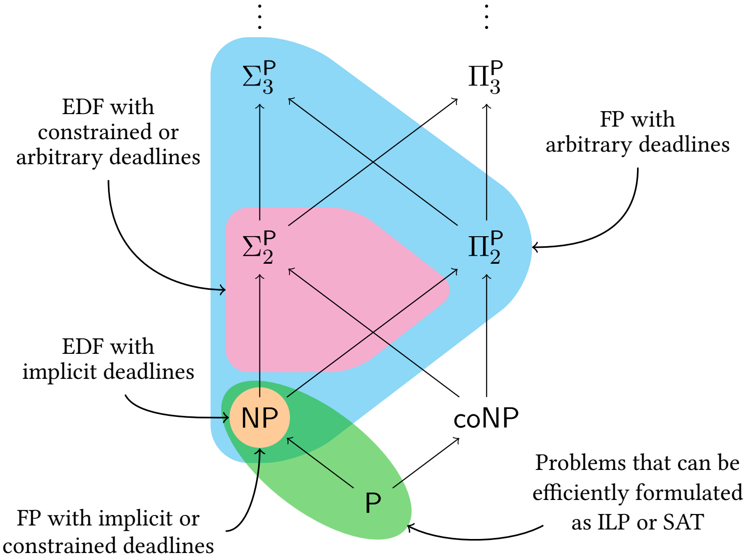

Conclusions

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- *Interesting open problems!* ☺

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$\forall$Thank you!

$\Diamond$

$\exists$Questions?