Topic 12: CP and Gecode
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Course 1DL441:
Combinatorial Optimisation and Constraint Programming,
whose part 1 is Course 1DL451:
Modelling for Combinatorial Optimisation
Outline

1. Constraint Programming (CP)
2. MiniZinc to Gecode
3. linear
4. element
5. MiniModel
6. distinct, nvalues, count
7. binpacking
8. cumulative, unary
9. circuit, path
10. extensional
11. channel
12. precede
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Reminder from Topic 1: Introduction

A solving technology offers methods and tools for:

**what:** Modelling constraint problems in *declarative* language.

and / or

**how:** Solving constraint problems *intelligently*:

- **Search:** Explore the space of candidate solutions.
- **Inference:** Reduce the space of candidate solutions.
- **Relaxation:** Exploit solutions to easier problems.

A solver is a software that takes a model & data as input and tries to solve the modelled problem instance.
Constraint Programming Technology

Constraint programming (CP) offers methods and tools for:
what: Modelling constraint problems in a high-level language.

and

how: Solving constraint problems intelligently by:
- either default systematic search upon pushing a button
- or systematic search guided by a user-given strategy
- or local search guided by a user-given strategy
with lots of inference, called propagation in the case of systematic search, but yet little relaxation.

Slogan of CP:

Constraint Program = Model [ + Search ]
CP Solving = Inference + Search

A CP solver conducts search interleaved with inference:

Each constraint has an inference algorithm.
Inference for One Constraint: Propagator

Example

Consider the constraint `CONNECTED([C_1, \ldots, C_n])`, which enforces max one stretch per colour among the \(n\) variables.

From the following current partial valuation for \(n = 6\):

\[
\begin{array}{cccccc}
\text{black} & \text{red} & C_3 & \text{red} & \text{yellow} & C_6 \\
\end{array}
\]

a propagator of the `CONNECTED` predicate can infer that \(C_3 = \text{red}\) and \(C_6 \not\in \{\text{red}, \text{black}\}\):

\[
\begin{array}{ccccccc}
\text{black} & \text{red} & \text{red} & \text{red} & \text{yellow} & C_6 \\
\end{array}
\]

Propagation is the removal of the impossible values from the current domains of the variables, and thereby accelerates otherwise blind search.
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With Gecode, which is a C++ library, one writes an imperative program that states (or: posts) — via any combination of sequential, conditional, iterative, and recursive composition — the declarative constraints, which are then given to the solver via propagators achieving chosen consistencies.
Reification

A MiniZinc reified constraint, such as $r \leftrightarrow x < y$, where $r$ is a variable of type `bool`, is modelled in Gecode by appending the reifying variable $r$, of type `Reify`, as an additional argument to the used constraint predicate:

$$rel(x, \text{IRT\_LE}, y, r)$$

Careful: Not all constraints are reifiable, as in all CP solvers!

We will use the following definition and notation:

**Definition**

The reification of a constraint $\gamma(\ldots)$ is the constraint $r \iff \gamma(\ldots)$, where $r$ is a “Boolean” variable, with the truth of $\gamma(\ldots)$ represented by 1 and its falsity by 0.

Propagation may be poor: see Topic 16: Propagators.
Inference: Propagator and Consistency

A MiniZinc inference annotation (recall Topic 8: Inference & Search in CP & LCG) to a constraint, \textit{bounds} or \textit{domain}, is modelled in Gecode by appending the consistency as an additional argument to the used constraint predicate.

The options for integers are value consistency (\textit{IPL\_VAL}), bounds consistency (\textit{IPL\_BND}), and domain consistency (\textit{IPL\_DOM}), consistency being called \textit{integer propagation level} (\textit{IPL}) in Gecode, one of them being the default (\textit{IPL\_DEF}) in case no consistency is given.

For example:

\begin{equation}
\text{distinct}(X, \text{IPL\_DOM})
\end{equation}

\textbf{For details, see Topic 13: Consistency.}
Search: Variable/Split Selection Strategies

A MiniZinc search annotation (recall Topic 8: Inference & Search in CP & LCG) to an objective, such as `int_search(X, first_fail, indomain_min)`, is modelled in Gecode by specifying or writing a brancher.

For example:

```
branch(X, INT_VAR_SIZE_MIN(), INT_VAL_MIN())
```

For details, see Topic 15: Search.
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The linear Predicate

A MiniZinc linear constraint, such as the linear equality constraint \( \text{sum}(i \ in \ 1..n)(A[i] \times X[i]) = d \), can be modelled in Gecode by using its reifiable linear predicate:

**Definition**

A \( \text{linear}([a_1, \ldots, a_n], [x_1, \ldots, x_n], R, d) \) constraint, with

- \([a_1, \ldots, a_n]\) a sequence of non-zero integer constants,
- \([x_1, \ldots, x_n]\) a sequence of integer variables,
- \(R\) in \(\{<, \leq, =, \neq, \geq, >\}\), and
- \(d\) an integer constant,

holds iff the linear relation \( \left( \sum_{i=1}^{n} a_i \cdot x_i \right) \ R \ d \) holds.

Also, \( \text{linear}([x_1, \ldots, x_n], R, d) \) holds iff \( \left( \sum_{i=1}^{n} x_i \right) \ R \ d \).
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The element Predicate

A MiniZinc constraint on an array element at an unknown index \( i \), such as `element(i, X, e)` or \( X[i] = e \) or a constraint involving the expression \( X[i] \), must be modelled in Gecode by explicitly using its non-reifiable `element` predicate:

**Definition (Van Hentenryck and Carillon, 1988)**

An `element([x_1, \ldots, x_n], i, e)` constraint, where the \( x_j \) are variables, \( i \) is an integer variable, and \( e \) is a variable, holds if and only if \( x_i = e \).

Several variants exist: see the Gecode documentation.
Example (Warehouse Location Problem)

Recall the one-way channelling constraint of Model 1 (in Topic 6: Case Studies) from the Supplier variables to the redundant Open variables:

\[
\text{constraint } \forall (s \in \text{Shops}) \left( \text{Open}[\text{Supplier}[s]] = 1 \right);
\]

This must be modelled in Gecode as in the following MiniZinc reformulation:

\[
\text{constraint } \forall (s \in \text{Shops}) \left( \text{element} (\text{Supplier}[s], \text{Open}, 1) \right);
\]
Example (Warehouse Location Problem, a last time)

Recall the objective of Model 1 in Topic 6: Case Studies:

```
solve minimize maintCost * sum(Open) + sum(s in Shops)(SupplyCost[s,Supplier[s]]);
```

This must be modelled in Gecode as in the following MiniZinc reformulation, by explicitly creating a `Cost[s]` variable and an `element` constraint for each implicit one:

```
array[Shops] of var int: Cost; % Cost[s] to supply s
constraint forall(s in Shops)
    (element(Supplier[s], SupplyCost[s,..], Cost[s]));
solve minimize maintCost * sum(Open) + sum(Cost);
```

Recall that we actually introduced these `Cost[s]` variables (in Topic 8: Inference & Search in CP & LCG) in order to state a maximal-regret search strategy on those variables.
Example (Job allocation at minimal salary cost)

Remember the model in Topic 3: Constraint Predicates:

1. array[Apps] of int: Salary;
2. array[Jobs] of var Apps: Worker; % job j by Worker[j]
3. solve minimize sum(j in Jobs)(Salary[Worker[j]]);
4. constraint ...; % qualifications, workload, etc

Line 3 must be modelled in Gecode as in the following MiniZinc reformulation, by explicitly creating a Cost[j] variable and an element constraint for each implicit one:

array[Jobs] of var int: Cost; % job j costs Cost[j]
constraint forall(j in Jobs)
  (element(Worker[j], Salary, Cost[j]));
solve minimize sum(j in Jobs)(Cost[j]); % sum(Cost)
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Using MiniModel, linear constraints can be formulated in Gecode like in MiniZinc: the appropriate linear constraints are generated by the Gecode toolchain. Another useful feature will be discussed at page 36.

Gecode has no constrained functions: everything is modelled relationally, using only constraint predicates.

Gecode does not eliminate common sub-expressions: a Gecode model automatically generated by the MiniZinc toolchain can outperform a handwritten Gecode model corresponding to the MiniZinc one.
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The **distinct Predicate**

A MiniZinc constraint of pairwise difference, such as `alldifferent(X)`, can be modelled in Gecode by using its non-reifiable `distinct` predicate:

**Definition (Laurière, 1978)**

A `distinct([x_1, \ldots, x_n])` constraint holds if and only if all the variables `x_i` take different values.

This is equivalent to \( \frac{n \cdot (n-1)}{2} \) disequality constraints:

\[
\forall i, j \in 1..n \text{ where } i < j : x_i \neq x_j
\]

Several variants exist: see the Gecode documentation.
The \texttt{nvalues} Predicate

A MiniZinc constraint on the number of distinct values within an array, such as \texttt{nvalue}(m, X), can be modelled in Gecode by using its non-reifiable \texttt{nvalues} predicate:

\begin{definition}{Pachet and Roy, 1999}

An \texttt{nvalues}([x_1, \ldots, x_n], R, m) constraint holds if and only if the number of distinct values taken by the elements of the sequence \([x_1, \ldots, x_n]\) of variables is in relation \(R\) with the variable \(m\), where \(R\) is in \(\{<, \leq, =, \neq, \geq, >\}\):

\[\lvert\{x_1, \ldots, x_n\}\rvert \ R m\]

Note that \(R\) is ‘=’ for the \texttt{nvalue} predicate of MiniZinc.

Several variants exist: see the Gecode documentation.
The count Predicate

A MiniZinc constraint on value counts within an array, such as `global_cardinality(X, V, C)`, can be modelled in Gecode by using its non-reifiable `count` predicate:

**Definition (Régis, 1996)**

A `count([x_1, \ldots, x_n], [c_1, \ldots, c_m], [v_1, \ldots, v_m])` constraint holds if and only if each variable `c_j` has the number of variables `x_i` that take the given value `v_j`.

Several variants exist: see the Gecode documentation.
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The binpacking Predicate

A MiniZinc bin-packing constraint, such as `bin_packing_load(L, B, V)`, can be modelled in Gecode by using its non-reifiable `binpacking` predicate:

**Definition**

Let item $i$ have the given weight or volume $v_i$. Let variable $b_i$ denote the bin into which item $i$ is put. Let variable $\ell_j$ denote the load of bin $j$. A `binpacking([\ell_1, \ldots, \ell_m], [b_1, \ldots, b_n], [v_1, \ldots, v_n])` constraint holds iff each $\ell_j$ is the sum of the $v_i$ where $b_i = j$. 
A MiniZinc constraint on knapsack packing, such as \texttt{knapsack}(V, P, X, v, p), can be modelled in Gecode by using two \texttt{linear} constraints:

\begin{align*}
\text{linear}(V, X, =, v) \\
\text{linear}(P, X, =, p)
\end{align*}

Recall that \texttt{linear} is reifiable.
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The cumulative Predicate

A MiniZinc constraint on the bounded cumulative resource requirement of tasks, such as `cumulative(S,D,R,u)`, can be modelled in Gecode by using its non-reifiable `cumulative` predicate:

**Definition (Aggoun and Beldiceanu, 1993)**

A `cumulative(u,[s₁,...,sₙ],[d₁,...,dₙ],[r₁,...,rₙ])` constraint, where each task $T_i$ has a starting time $s_i$, a duration $d_i$, and a resource requirement $r_i$, holds if and only if the resource upper limit $u$ is never exceeded when performing the tasks $T_i$.

Several variants exist: see the Gecode documentation.
The unary Predicate

A MiniZinc temporal non-overlap constraint on tasks, such as `disjunctive(S, D)`, can be modelled in Gecode by using its non-reifiable `unary` predicate, so called because it applies to tasks requiring a `unary` resource:

**Definition (Carlier, 1982)**

A `unary([s_1, ..., s_n], [d_1, ..., d_n])` constraint, where each task $T_i$ has a starting time $s_i$ and a duration $d_i$, holds if and only if no two tasks $T_i$ and $T_j$ overlap in time.

Several variants exist: see the Gecode documentation.
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The circuit Predicate

A MiniZinc constraint on a Hamiltonian circuit, such as \texttt{circuit}(S), can be modelled in Gecode by using its non-reifiable \texttt{circuit} predicate:

\begin{quote}
Definition (Laurièrè, 1978)

A \texttt{circuit}([s_1, \ldots, s_n]) constraint holds iff the arcs \( i \rightarrow s_i \) form a Hamiltonian circuit in the graph defined by the domains of the variables \( s_i \): each vertex is visited exactly once.
\end{quote}

Several variants exist: see the Gecode documentation.
No Subcircuit, but a Path Predicate

A MiniZinc constraint \texttt{subcircuit}(S) can be modelled in Gecode as in the MiniZinc default definition at https://www.minizinc.org/doc-lib/doc-globals.html, which is actually used by the Gecode backend to MiniZinc.

A MiniZinc constraint on a Hamiltonian path, such as \texttt{circuit}(S) \land S[t] = f, can be modelled in Gecode by using its non-reifiable \texttt{path} predicate:

\begin{definition}
A \texttt{path}([s_1, \ldots, s_n], f, t) constraint holds iff the arcs $i \rightarrow s_i$ form a Hamiltonian path from vertex $f$ to vertex $t$ in the graph defined by the domains of the variables $s_i$: each vertex is visited exactly once.
\end{definition}

Several variants exist: see the Gecode documentation.
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The extensional Predicate

A MiniZinc constraint on membership in a table \( T \) or regular language, such as \texttt{table}(X, T) or \texttt{regular}(X, R), where \( R \) is a regular expression or a deterministic finite automaton (DFA) defining a regular language, is modelled in Gecode by using its reifiable \texttt{extensional} predicate:

**Definition**

An \texttt{extensional}([\( x_1, \ldots, x_n \), \( R \)]) constraint holds if and only if the values taken by the sequence \([x_1, \ldots, x_n]\) of variables form a row of the 2d table \( R \) of constants or form a string that belongs to the regular language accepted by the regular expression (when using MiniModel) or DFA \( R \).

Several variants exist: see the Gecode documentation.
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The channel Predicate

A MiniZinc constraint on two arrays representing a function and its inverse, such as \texttt{inverse}(X, Y), can be modelled in Gecode by using its non-reifiable \texttt{channel} predicate:

\begin{definition}
A \texttt{channel}([x_1, \ldots, x_n], [y_1, \ldots, y_n]) constraint holds iff:

$$\forall i, j \in 1..n : x_i = j \iff y_j = i$$

Several variants exist: see the Gecode documentation.
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The **precede** Predicate

MiniZinc value symmetry-breaking constraints, such as

\[ \text{value_precede}(v, w, X) \]

and its generalisation

\[ \text{value_precede_chain}(V, X) \]

, can be modelled in Gecode by using its non-reifiable **precede** predicate:

**Definition**

A \[ \text{precede}([x_1, \ldots, x_n], v, w) \] constraint holds iff the first occurrence, if any, of value \( v \) precedes the first occurrence, if any, of value \( w \) among the variables \( x_i \).

**Definition**

A \[ \text{precede}([x_1, \ldots, x_n], [v_1, \ldots, v_m]) \] constraint holds iff the first occurrence, if any, of every value \( v_i \) precedes the first occurrence, if any, of value \( v_{i+1} \) among the variables \( x_i \).