Topic 10: Modelling for SAT and SMT
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Course 1DL441: Combinatorial Optimisation and Constraint Programming,
whose part 1 is Course 1DL451: Modelling for Combinatorial Optimisation
Outline

1. SAT and SMT
2. Encoding into SAT
3. Modelling for SAT and SMT in MiniZinc
4. Case Study
   Graph Colouring
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Revisit the slides on Boolean satisfiability (SAT) and SAT modulo theories (SMT) of Topic 7: Solving Technologies.

**SAT using MiniZinc:**
The currently only SAT backend, `mzn-g12sat`, allows only Boolean decision variables:

- One must manually transform a model with integer variables (see the next slides), just like when directly using a SAT solver. This will probably be automated in a future release of the MiniZinc toolchain.

- One can already flatten a model with set variables using the option `-Gnosets`.

**SMT using MiniZinc:**

- The backend `fzn2smt` transforms a FlatZinc model into the SMTlib language, which is understood by most SMT solvers, such as CVC4, Yices, and Z3.

- All predicates are decomposed by the flattening.
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Encoding into SAT

Challenges:

- How to encode an integer variable into a collection of Boolean variables?
- How to encode a constraint on integer variables into a collection of constraints on Boolean variables?
- How to transform a constraint on Boolean variables into clausal form? Most solvers do this for you.

Considerations:

- We want few variables.
- We want few clauses.

As usual, there are many possibilities and it is not always clear what is the best choice.
Encoding an Integer Variable

Well-known encodings, described on the next three slides:

- **Direct (or sparse) encoding:**
  a Boolean variable for each equality with a value.

- **Order encoding:**
  a Boolean variable for each inequality with a value.

- **Direct + order encoding:**
  channel between the direct and order encodings.

- **Bit (or binary) encoding:**
  a Boolean variable for each bit in the base-2 representation of the domain values [not covered here].
Direct Encoding of an Integer Variable

Consider an integer variable $x$ with domain $1..n$:

- Create a Boolean variable $b_{[x=k]}$ for all $k$ in $1..n$.
- The variable $b_{[x=k]}$ is true if and only if $x = k$ holds.
- Consistency constraints:
  - At least one value: $\bigvee_{k \in 1..n} b_{[x=k]}$
  - At most one value: $\bigwedge_{j,k \in 1..n, j<k} \neg (b_{[x=j]} \land b_{[x=k]})$

There are $n$ variables, $n \cdot (n - 1)/2$ binary clauses, and one $n$-ary clause.

- The constraint $x \neq k$ is encoded as $\neg b_{[x=k]}$.
- The constraint $x < k$ is encoded as $\bigwedge_{j \in k..n} \neg b_{[x=j]}$. 
Order Encoding of an Integer Variable

Consider an integer variable $x$ with domain $1..n$:

- Create a Boolean variable $b_{[x \geq k]}$ for all $k$ in $1..(n + 1)$.
- The variable $b_{[x \geq k]}$ is true if and only if $x \geq k$ holds.
- Consistency constraints:
  - Order: $\bigwedge_{k \in 1..n} (b_{[x \geq k]} \lor \neg b_{[x \geq k+1]})$
  - Bounds: $b_{[x \geq 1]} \land \neg b_{[x \geq n+1]}$

- There are $n + 1$ variables and $n$ binary clauses.
- The constraint $x = k$ is encoded as $b_{[x \geq k]} \land \neg b_{[x \geq k+1]}$.
- The constraint $x \neq k$ is encoded as $\neg b_{[x \geq k]} \lor b_{[x \geq k+1]}$.
- The constraint $x < k$ is encoded as $\neg b_{[x \geq k]}$. 
Direct + Order Encoding of Integer Variable

- Channelling constraints:

\[
\bigwedge_{k \in 1..n} \left( b_{x=k} \iff (b_{x \geq k} \land \neg b_{x \geq k+1}) \right)
\]

- Reminder: \( \alpha \iff \beta \) is equivalent to \((\neg \alpha \lor \beta) \land (\alpha \lor \neg \beta)\).

- Channelling constraints in clausal form:

\[
\bigwedge_{k \in 1..n} \left( \left( \neg b_{x=k} \lor b_{x \geq k} \right) \land \left( \neg b_{x=k} \lor \neg b_{x \geq k+1} \right) \right) \\
\land \left( b_{x=k} \lor \neg b_{x \geq k} \lor b_{x \geq k+1} \right)
\]

- There are \(2 \cdot n\) new binary and \(n\) new ternary clauses.
A constraint encoding is a constraint decomposition in clausal form.

Many possibilities exist for each constraint predicate.

There is a lot of research on how to encode constraints.

There are two general approaches:

- encode solutions;
- encode non-solutions.

Constraint encodings depend on the variable encoding.
Encoding Simple Constraints of Arity $> 1$

Constraint $x = y$, both variables with domain 1..$n$:

- **Direct encoding**: $\bigwedge_{k \in 1..n} (b_{x=k} \iff b_{y=k})$

- **Order encoding**: $\bigwedge_{k \in 1..n} (b_{x\geq k} \iff b_{y\geq k})$

Constraint $x \neq y$, both variables with domain 1..$n$:

- **Direct encoding**: $\bigwedge_{k \in 1..n} (\neg b_{x=k} \lor \neg b_{y=k})$

- **Order encoding**:

  $\bigwedge_{k \in 1..n} (\neg b_{x \geq k} \lor b_{x \geq k+1} \lor \neg b_{y \geq k} \lor b_{y \geq k+1})$
Constraint $x \leq y$, both variables with domain $1..n$:

- **Direct encoding:**
  \[
  \bigwedge_{k \in 1..n} \left( \neg b[x=k] \lor \bigvee_{j \in k..n} b[y=j] \right)
  \]

- **Order encoding:**
  \[
  \bigwedge_{k \in 1..n} \left( \neg b[x \geq k] \lor b[y \geq k] \right)
  \]

Constraint $x + c = y$, with $x \in 1..n$ and $y \in (1 + c)\ldots(n + c)$:

- **Direct encoding:**
  \[
  \bigwedge_{k \in 1..n} \left( b[x=k] \iff b[y=k+c] \right)
  \]

- **Order encoding:**
  \[
  \bigwedge_{k \in 1..n} \left( b[x \geq k] \iff b[y \geq k+c] \right)
  \]
Encoding \textit{alldifferent}

How to encode \textit{alldifferent}([X_1, \ldots, X_m]), where all the variables have domain 1..n, with usually $m \leq n$?

- **Approach 1:** Decompose \textit{alldifferent} into binary disequalities, and encode each disequality separately.

- **Approach 2:** Each value is taken by at most one var:
  - Use a decomposition into binary clauses, like in the direct encoding: this actually boils down to Approach 1.
  - Use a ladder encoding: see the next slide.
Ladder Encoding for \textit{alldifferent}

- Add a Boolean variable $A_{ik}$ for each $i \in 0..m$ & $k \in 1..n$.

- Variable $A_{ik}$ is \textbf{true} iff one of $X_1, \ldots, X_i$ has value $k$.

- Constraints:
  - Consistency:
    \[
    \bigwedge_{k \in 1..n} \bigwedge_{i \in 1..m} \left( \neg A_{(i-1)k} \lor A_{ik} \right)
    \]
  - Channelling:
    \[
    \bigwedge_{k \in 1..n} \bigwedge_{i \in 1..m} \left( B[X_i=k] \iff (\neg A_{(i-1)k} \land A_{ik}) \right)
    \]
Comparison of all-different Encodings

Decomposition encoding:

- The decomposition has $\frac{m \cdot (m-1)}{2}$ binary disequalities.
- Each disequality is encoded by $n$ binary clauses.
- Total: $\frac{n \cdot m \cdot (m-1)}{2}$ binary clauses.

Ladder encoding:

- First part: $n \cdot m$ binary clauses.
- Second part: $2 \cdot n \cdot m$ binary and $n \cdot m$ ternary clauses.
- Total: $4 \cdot n \cdot m$ clauses with 2 or 3 literals.
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Some Guidelines

Use higher-level variable types (sets, ...), even if not supported by the SAT or SMT backend.

- They enable the use of carefully designed encodings.
- It is easier for the modeller to reason about them.

Use global constraint predicates, even if not supported by the SAT or SMT backend.

- They enable the use of carefully designed encodings.
- It is easier for the modeller to reason about them.
Add implied constraints:

- They may reduce the search space a lot
- This might be redundant with the clause-learning mechanisms of modern SAT and SMT solvers.

Add symmetry-breaking constraints:

- They may reduce the search space a lot.

Limit the number of SMT theories used:

- Using several theories may decrease the inference power of the solver.
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Given a graph \((V, E)\), colour each vertex so that adjacent vertices have different colours, using at most \(n\) colours.

**Model 2, MiniZinc/CP/LCG-style**

Variable \(C[v] \text{ in } 1..n\) is \(k\) iff vertex \(v\) has colour \(k\).

1. One colour per vertex: enforced by choice of variables.
2. Different colours on edge ends:

\[
\forall ((u, v) \in E)(C[u] \neq C[v])
\]

Give SAT and SMT flattenings of this model!