Outline

1. Black-Hole Patience
2. Antenna Placement
3. Warehouse Location
4. Sport Scheduling
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1. Black-Hole Patience
2. Antenna Placement
3. Warehouse Location
4. Sport Scheduling
Move all the cards into the black hole. A fan top card can be moved if it is one rank apart from the black-hole top card, independently of suit (♣, ♦, ♥); aces (A,1) and kings (K,13) are a rank apart.


The cards $c_1$ and $c_2$ are one rank apart if and only if

$$(c_1 \mod 13) - (c_2 \mod 13) \in \{-12, -1, 1, 12\}$$

Avoiding *mod* on variables and defining a help predicate:

```plaintext
predicate rankApart(var 1..52: c1, var 1..52: c2) =
  let { array[1..52] of int: R = [i mod 13 | i in 1..52] }
  in  R[c1] - R[c2] in {-12,-1,1,12};
```

Avoiding implicit *element* constraints for better inference:

```plaintext
predicate rankApart(var 1..52: c1, var 1..52: c2) =
  table([c1,c2], [ |1,2|1,13|...|1,52|2,1|...|52,40|52,51|]);
```
Model

Move all the cards into the black hole. A fan top card can be moved if it is one rank apart from the black-hole top card, independently of suit (♣, ♦, ♥); aces (A,1) and kings (K,13) are a rank apart.

Let Card[p] denote the card at position p in the black hole:

- constraint Card[1] = A♠
- constraint forall(p in 1..51) (rankApart(Card[p], Card[p+1]));

The card order in the fans is respected in the black hole:

- constraint forall ("fan with card" c1 "on top of" c2 "on top of" c3) (let { var 2..52: p1; var 2..52: p2; var 2..52: p3 } in Card[p1] = c1 \ Card[p2] = c2 \ Card[p3] = c3 \ p1 < p2 \ p2 < p3);%

or, equivalently:

- constraint forall ("fan with card" c1 "on top of" c2 "on top of" c3) (value_precede_chain([c1, c2, c3], Card));%

% constraint alldifferent(Card); % implied by correct data!
Redundant Variables and Channelling

Let $\text{Pos}[c]$ denote the position of card $c$ in the black hole. The card order in the fans is respected in the black hole:

\[
\text{constraint } \text{Pos}[1] = 1; \quad \% \text{ the position of card } \text{A}\spadesuit \text{ is } 1
\]
\[
\text{constraint } \forall (\text{"fan with card" } c1 \text{ "on top of" } c2 \text{ "on top of" } c3) (\text{Pos}[c1] < \text{Pos}[c2] \wedge \text{Pos}[c2] < \text{Pos}[c3]);
\]
\[
\% \text{ constraint alldifferent}(\text{Pos}); \quad \% \text{ implied by correct data!}
\]
\[
\% \text{ How to model "adjacent black-hole cards are a rank apart"?}
\]

Let us use both the Card variables and the Pos variables, and channel between them.

Observe that $\forall c, p \in 1..52 : \text{Card}[p] = c \leftrightarrow \text{Pos}[c] = p$.

Seen as functions, Card and Pos are each other's inverse. This is the semantics of inverse$(\text{Card, Pos})$ and implies both alldifferent$(\text{Card})$ and alldifferent$(\text{Pos})$.

The model with redundant variables and their channelling constraint is much more efficient (at least on a CP or LCG solver).
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1. Black-Hole Patience

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Antenna Placement Problem

Given:
- a region divided into zones, each with an expected gain if covered,
- the maintenance costs and covering areas of antenna types,
- a targeted number of antennae,

find non-overlapping antenna placements and types such that the total expected gain of actual coverage minus the total maintenance cost is maximal.

We now show that we can pre-compute a 3d array with the net gain for each possible antenna placement and type, making it much easier to express the objective function and much faster to solve problem instances.
Model without Pre-Computation

1 ...  
2 set of int: Zones = 1..z; % the region has z by z zones  
3 array[Zones,Zones] of int: ExpGain =  
   [ |79,18,3,6,8,6,2,9,17,24| ... | ]; % expected gain per zone  
4 set of int: Types = 1..t; % type i covers i by i zones  
5 array[Types] of int: Cost; % maintenance costs  
6 set of int: Antennae = 1..antennae;  
7 % Variables (for lower-left coordinates) and constraints:  
8 array[Antennae] of var Zones: X; % X[a] = x-coordinate of a  
9 array[Antennae] of var Zones: Y; % Y[a] = y-coordinate of a  
10 array[Antennae] of var Types: Type; % Type[a] = type of a  
11 constraint diffn(X,Y,Type,Type); % no coverage overlaps  
12 constraint forall(a in Antennae)  
     (X[a]+Type[a] <= z+1 \ Y[a]+Type[a] <= z+1);  
13 % Objective:  
14 array[Antennae] of var int: Gain; % Gain[a] = gain of a  
15 constraint forall(a in Antennae) (Gain[a]=sum(x,y in Zones)  
   (ExpGain[x,y] * bool2int(X[a] <= x \ x < X[a]+Type[a]  
   \ Y[a] <= y \ y < Y[a]+Type[a])));  
16 var int: cost=sum(a in Antennae)(Cost[Type[a]]); % total cost  
17 solve maximize sum(Gain) - cost;
This model yields better **inference** and faster solving.

**Solving to optimality with Gecode (CP) for \( z=10 \):**

<table>
<thead>
<tr>
<th>pre-computation</th>
<th>antennae</th>
<th>types</th>
<th># nodes</th>
<th>seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>without</td>
<td>1</td>
<td>4</td>
<td>927</td>
<td>0.007</td>
</tr>
<tr>
<td>with</td>
<td>1</td>
<td>4</td>
<td>65</td>
<td>0.001</td>
</tr>
<tr>
<td>without</td>
<td>2</td>
<td>4</td>
<td>11,445,833</td>
<td>106.936</td>
</tr>
<tr>
<td>with</td>
<td>2</td>
<td>4</td>
<td>361</td>
<td>0.005</td>
</tr>
<tr>
<td>without</td>
<td>3</td>
<td>4</td>
<td>timeout</td>
<td>timeout</td>
</tr>
<tr>
<td>with</td>
<td>3</td>
<td>4</td>
<td>961</td>
<td>0.015</td>
</tr>
<tr>
<td>with</td>
<td>5</td>
<td>4</td>
<td>188,844</td>
<td>2.642</td>
</tr>
</tbody>
</table>
Outline

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The Warehouse Location Problem (WLP)

A company considers opening warehouses at some candidate locations in order to supply its existing shops:

- Each candidate warehouse has the same maintenance cost.
- Each candidate warehouse has a supply capacity, which is the maximum number of shops it can supply.
- The supply cost to a shop depends on the warehouse.

Determine which warehouses to open, and which of them should supply the various shops, so that:

1. Each shop is supplied by exactly one actually opened warehouse.
2. Each actually opened warehouse supplies a number of shops at most equal to its capacity.
3. The sum of the actually incurred maintenance costs and supply costs is minimised.
WLP: Sample Instance Data

\[ \text{Shops} = \{\text{Shop}_1, \text{Shop}_2, \ldots, \text{Shop}_{10}\} \]

\[ \text{Whs} = \{\text{Berlin, London, Ankara, Paris, Rome}\} \]

\[ \text{maintCost} = 30 \]

<table>
<thead>
<tr>
<th>Capacity</th>
<th>Berlin</th>
<th>London</th>
<th>Ankara</th>
<th>Paris</th>
<th>Rome</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Shop</th>
<th>Berlin</th>
<th>London</th>
<th>Ankara</th>
<th>Paris</th>
<th>Rome</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>24</td>
<td>11</td>
<td>25</td>
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<tr>
<td>2</td>
<td>28</td>
<td>27</td>
<td>82</td>
<td>83</td>
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</tr>
<tr>
<td>3</td>
<td>74</td>
<td>97</td>
<td>71</td>
<td>96</td>
<td>70</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>55</td>
<td>73</td>
<td>69</td>
<td>61</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>10</td>
<td>47</td>
<td>65</td>
<td>55</td>
<td>71</td>
<td>95</td>
</tr>
</tbody>
</table>
WLP Model 1: Decision Variables

Automatic enforcement of the total-function constraint (1):

\[ \text{Supplier} = \begin{array}{cccc}
\text{Shop}_1 & \text{Shop}_2 & \cdots & \text{Shop}_{10} \\
\in \text{Whs} & \in \text{Whs} & \cdots & \in \text{Whs}
\end{array} \]

\text{Supplier}[s] \text{ denotes the supplier warehouse for shop } s.\]

Redundant decision variables:

\[ \text{Open} = \begin{array}{ccccc}
\text{Berlin} & \text{London} & \text{Ankara} & \text{Paris} & \text{Rome} \\
\in 0..1 & \in 0..1 & \in 0..1 & \in 0..1 & \in 0..1
\end{array} \]

\[ \text{Open}[w] = 1 \text{ if and only if (iff) warehouse } w \text{ is opened.} \]

\[ \text{Our chosen array names always reflect functions.} \]
WLP Model 1: Objective

\[
\text{solve minimize } \text{maintCost} \star \sum(\text{Open}) + \sum(s \text{ in Shops})(\text{SupplyCost}[s, \text{Supplier}[s]])
\]

The first term is the total maintenance cost, expressed as the product of the warehouse maintenance cost by the number of actually opened warehouses.

The second term is the total supply cost, expressed as the sum over all shops of their actually incurred supply costs.

Notice the implicit use of the \element predicate, as the index \text{Supplier}[s] is a decision variable.

If warehouse \text{w} has maintenance cost \text{MaintCost}[w], then the first term becomes
\[
\sum(w \text{ in Whs})(\text{MaintCost}[w] \star \text{Open}[w]).
\]
WLP Model 1: Channelling Constraint

One-way channelling constraint from the Supplier variables to the redundant Open variables:

\[ \text{forall}(s \text{ in } \text{Shops})(\text{Open}[\text{Supplier}[s]] = 1) \]

The supplier warehouse of each shop is actually opened.

Notice the implicit use of the element predicate, as the index Supplier[s] is a decision variable.

How do the remaining Open[w] variables become 0? Upon minimisation.
WLP Model 1: Channelling Constraint

**Alternative:** Two-way channelling constraint between the Supplier variables and the redundant Open variables:

```
forall(w in Whs)
  (Open[w] = bool2int(exists(s in Shops)(Supplier[s]=w)))
```

A warehouse is opened iff there exists a shop it supplies.

Make experiments to find out which channelling is better. We will revisit this issue in Topic 8: Inference & Search in CP & LCG, and in Topic 9: Modelling for CBLS.

Also consider making Open an array of Boolean variables and using bool2int for its summation in the objective.
WLP Model 1: Capacity Constraint

Capacity constraint (2):

\[
\text{global_cardinality_low_up_closed} \\
\text{(Supplier, Whs, [0 \mid i \text{ in } Whs], Capacity)}
\]

Each warehouse is a supplier of a number of shops at most equal to its capacity.
WLP Model 2

Drop the array Open of redundant decision variables as well as its channelling constraint, and reformulate the first term of the objective function as follows:

\[
\text{maintCost} \times \sum(w \in \text{Whs})(\text{bool2int} (\exists(s \in \text{Shops})(\text{Supplier}[s]=w)))
\]

We can also use the \text{nvalue} constrained function:

\[
\text{maintCost} \times \text{nvalue}(\text{Supplier})
\]

This model cannot be generalised for warehouse-specific maintenance costs. For a speed comparison, see Topic 8: Inference & Search in CP & LCG. Redundancy elimination may pay off, but it may just as well be the converse. But this is hard to guess, as human intuition may be weak.
WLP Model 3: Decision Variables

No automatic enforcement of total-function constraint (1):

\[
\text{Supply} = \begin{bmatrix}
\text{Berlin} & \text{London} & \text{Ankara} & \text{Paris} & \text{Rome} \\
\in [0..1] & \in [0..1] & \in [0..1] & \in [0..1] & \in [0..1]
\end{bmatrix}
\]

\[\text{Supply}[s, w] = 1 \text{ iff shop } s \text{ is supplied by warehouse } w.\]

Redundant decision variables (as in Model 1):

\[
\text{Open} = \begin{bmatrix}
\text{Berlin} & \text{London} & \text{Ankara} & \text{Paris} & \text{Rome} \\
\in [0..1] & \in [0..1] & \in [0..1] & \in [0..1] & \in [0..1]
\end{bmatrix}
\]

\[\text{Open}[w] = 1 \text{ if and only if warehouse } w \text{ is opened.}\]
WLP Model 3: Objective

The objective can now be expressed in linear fashion:

\[
\text{solve minimize} \quad \text{maintCost} \times \sum(\text{Open}) \\
\quad + \quad \sum(s \text{ in Shops, } w \text{ in Whs}) \quad (\text{Supply}[s,w] \times \text{SupplyCost}[s,w])
\]

The first term is the total maintenance cost, expressed (as in Model 1) as the product of the warehouse maintenance cost by the number of actually opened warehouses.

The second term is the total supply cost, expressed as the sum over all shops and warehouses of their actually incurred supply costs: each parameter \(\text{SupplyCost}[s,w]\) is weighted by the decision variable \(\text{Supply}[s,w]\).
WLP Model 3: Constraints

The total-function constraint (1) now needs to be modelled, and can be expressed in linear fashion without \texttt{count}:

\[
\text{forall}(s \text{ in } \text{Shops})(\text{sum}(\text{Supply}[s,..]) = 1)
\]

Each shop is supplied by exactly one actually opened warehouse.
WLP Model 3: Constraints (end)

Capacity constraint (2), in isolation:

\[
\text{forall } (w \in \text{Whs}) \\
(\text{sum}(\text{Supply}[..,w]) \leq \text{Capacity}[w])
\]

Two-way channelling constraint, in isolation:

\[
\text{forall } (w \in \text{Whs}) \\
(\text{sum}(\text{Supply}[..,w]) > 0 \iff \text{Open}[w] = 1)
\]

or, one-way without reification, upon exploiting minimisation:

\[
\text{forall } (w \in \text{Whs}) \\
(\text{forall } (s \in \text{Shops})(\text{Supply}[s,w]\leq\text{Open}[w]))
\]

Capacity (2) & one-way channelling constraints combined:

\[
\text{forall } (w \in \text{Whs}) \\
(\text{sum}(\text{Supply}[..,w]) \leq \text{Capacity}[w] \times \text{Open}[w])
\]

All constraints are linear (in)equalities: this is an IP model!
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The Sport Scheduling Problem (SSP)

Find schedule in $\text{Periods} \times \text{Weeks} \rightarrow \text{Teams} \times \text{Teams}$ for

- $|\text{Teams}| = n$ and $n$ is even
- $|\text{Weeks}| = n-1$
- $|\text{Periods}| = n \div 2$

subject to the following constraints:

1. Each possible game is played exactly once.
2. Each team plays exactly once per week.
3. Each team plays at most twice per period.

Idea for a model, and a solution for $n=8$

<table>
<thead>
<tr>
<th></th>
<th>Wk 1</th>
<th>Wk 2</th>
<th>Wk 3</th>
<th>Wk 4</th>
<th>Wk 5</th>
<th>Wk 6</th>
<th>Wk 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>P 1</td>
<td>1 vs 2</td>
<td>1 vs 3</td>
<td>2 vs 6</td>
<td>3 vs 5</td>
<td>4 vs 7</td>
<td>4 vs 8</td>
<td>5 vs 8</td>
</tr>
<tr>
<td>P 2</td>
<td>3 vs 4</td>
<td>2 vs 8</td>
<td>1 vs 7</td>
<td>6 vs 7</td>
<td>6 vs 8</td>
<td>2 vs 5</td>
<td>1 vs 4</td>
</tr>
<tr>
<td>P 3</td>
<td>5 vs 6</td>
<td>4 vs 6</td>
<td>3 vs 8</td>
<td>1 vs 8</td>
<td>1 vs 5</td>
<td>3 vs 7</td>
<td>2 vs 7</td>
</tr>
<tr>
<td>P 4</td>
<td>7 vs 8</td>
<td>5 vs 7</td>
<td>4 vs 5</td>
<td>2 vs 4</td>
<td>2 vs 3</td>
<td>1 vs 6</td>
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</tr>
</tbody>
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- $|\text{Weeks}| = n-1$
- $|\text{Periods}| = n \div 2$

subject to the following constraints:

1. Each possible game is played exactly once.
2. Each team plays exactly once per week.
3. Each team plays at most twice per period.

Idea for a model, and a solution for $n=8$, with a dummy week $n$ of duplicate games:

<table>
<thead>
<tr>
<th>Wk 1</th>
<th>Wk 2</th>
<th>Wk 3</th>
<th>Wk 4</th>
<th>Wk 5</th>
<th>Wk 6</th>
<th>Wk 7</th>
<th>Wk 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>P 1</td>
<td>1 vs 2</td>
<td>1 vs 3</td>
<td>2 vs 6</td>
<td>3 vs 5</td>
<td>4 vs 7</td>
<td>4 vs 8</td>
<td>5 vs 8</td>
</tr>
<tr>
<td>P 2</td>
<td>3 vs 4</td>
<td>2 vs 8</td>
<td>1 vs 7</td>
<td>6 vs 7</td>
<td>6 vs 8</td>
<td>2 vs 5</td>
<td>1 vs 4</td>
</tr>
<tr>
<td>P 3</td>
<td>5 vs 6</td>
<td>4 vs 6</td>
<td>3 vs 8</td>
<td>1 vs 8</td>
<td>1 vs 5</td>
<td>3 vs 7</td>
<td>2 vs 7</td>
</tr>
<tr>
<td>P 4</td>
<td>7 vs 8</td>
<td>5 vs 7</td>
<td>4 vs 5</td>
<td>2 vs 4</td>
<td>2 vs 3</td>
<td>1 vs 6</td>
<td>3 vs 6</td>
</tr>
</tbody>
</table>
SSP Model 1: Data

Parameter:

- **int**: `n; assert(n>1 \ n mod 2 =0, "Odd n")`

Useful Ranges and Sets:

- **Teams** = 1..n
- **Weeks** = 1..(n-1)
- **ExtendedWeeks** = 1..n
- **Periods** = 1..(n div 2)
- **Slots** = 1..2
- **Games** = `{f*n+s | f,s in Teams where f<s}`, thereby breaking some symmetries, such that the game between teams `f` and `s` is uniquely identified by the natural number `f * n + s`.

**Example**: For `n=4`, we get `Games={6,7,8,11,12,16}`.
SSP Model 1: Decision Variables

A 3d matrix Team[Periods,ExtendedWeeks,Slots] of variables in Teams, denoted $T$ below, over a schedule extended by a dummy week where teams play fictitious duplicate games in the period where they would otherwise play only once, thereby transforming constraint (3) into:

(3’) Each team plays exactly twice per period.

Predicate $\text{global_cardinality_low_up_closed}$ need not be used and can be replaced by a stronger predicate.

Team =

\[
\begin{array}{cccccccc}
\text{Wk 1} & \cdots & \cdots & \text{Wk } n-1 & \text{Wk } n \\
1 & 2 & \cdots & \cdots & 1 & 2 & 1 & 2 \\
\text{P 1} & \in T & \in T & \cdots & \cdots & \in T & \in T & \in T & \in T \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\text{P } n/2 & \in T & \in T & \cdots & \cdots & \in T & \in T & \in T & \in T \\
\end{array}
\]

$\text{Team}[p,w,s]$ is the numeric name of the team that plays in period $p$ of week $w$ in game slot $s$. 
SSP Model 1: Constraints

Twice-per-period constraint (3'):

```
forall(p in Periods)
  (global_cardinality_closed
   (array1d(Team[p,..,..]), Teams, [2 | i in 1..n]))
```

In each period, each team name occurs exactly twice within the slots of the weeks in $\text{Team}$.

Once-per-week constraint (2):

```
forall(w in ExtendedWeeks)
  (alldifferent(Team[..,w,..])) % works on 2d slices
```

In each week, incl. the dummy week, there are no duplicate team names within the slots of the periods in $\text{Team}$. 
Try to state the each-game-once constraint (1) using $\text{Team}!$

Declare a 2d matrix $\text{Game}[\text{Periods}, \text{Weeks}]$ of redundant decision variables in $\text{Games}$ over the non-extended weeks:

$$
\begin{array}{c}
\text{Period 1} \\
\vdots \\
\text{Period } n/2 \\
\end{array}
\begin{array}{ccc}
\in \text{Games} & \cdots & \in \text{Games} \\
\vdots & \ddots & \vdots \\
\in \text{Games} & \cdots & \in \text{Games} \\
\end{array}
$$

$\text{Game}[p, w]$ is the game played in period $p$ of week $w$. 

SSP Model 1: Decision Variables (revisited)
SSP Model 1: Constraints (end)

Each-game-once constraint (1):

\texttt{alldifferent}(\texttt{Game})

There are no duplicate game numbers in \texttt{Game}.

Channelling constraint (but rather use \texttt{table}:

\texttt{forall}(p \ \texttt{in} \ Periods, \ w \ \texttt{in} \ Weeks)

\texttt{(Team}[p,w,1]*n+Team[p,w,2] = Game[p,w])

The game number in \texttt{Game} of each period and week corresponds to the teams scheduled at that time in \texttt{Team}.

Constraints (2) and (3’) are hard to formulate using \texttt{Game}.
A **round-robin schedule** suffices to break many of the remaining symmetries:

- Fix the games of the first week to the set $$\{(1, 2)\} \cup \{(t + 1, n + 2 - t) \mid 1 < t \leq n/2\}$$

- For the remaining weeks, transform each game \((f, s)\) of the previous week into a game \((f', s')\), where

$$f' = \begin{cases} 
1 & \text{if } f = 1 \\
2 & \text{if } f = n \\
f + 1 & \text{otherwise}
\end{cases}$$

and

$$s' = \begin{cases} 
2 & \text{if } s = n \\
s + 1 & \text{otherwise}
\end{cases}$$

The constraints (1) and (2) are now automatically enforced: we must determine the period of each game, **not** its week!
Interested in More Details?

For more details on WLP & SSP and their modelling, see:

Van Hentenryck, Pascal. 

Van Hentenryck, Pascal. 
*Constraint and integer programming in OPL.*

Van Hentenryck, Pascal; Michel, Laurent; Perron, Laurent; and Régis, Jean-Charles. 
*Constraint programming in OPL.*