Topic 4: Modelling (for CP & LCG)
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Course 1DL441:
Combinatorial Optimisation and Constraint Programming,
whose part 1 is Course 1DL451:
Modelling for Combinatorial Optimisation
Outline

1. Viewpoints
2. Implied Constraints
3. Redundant Variables & Channelling Constraints
4. Pre-Computation
Outline

1. Viewpoints

2. Implied Constraints

3. Redundant Variables & Channelling Constraints

4. Pre-Computation
Recap

1. **Modelling**: express problem in terms of
   - parameters,
   - decision variables,
   - constraints, and
   - objective.

2. **Solving**: solve using a state-of-the-art solver.
Example (Student Seating Problem)

Given:
- \( s \) students, and
- \( c \) chairs positioned around tables.

Find a seating arrangement such that:
- Each table has either at least half its chairs occupied, or none.
- Each table has at least as many students as any table behind it.
- A maximum number of student preferences on being seated at the same table are satisfied.

What are suitable decision variables for this problem?
A viewpoint is a choice of decision variables.

**Example (Student Seating Problem)**

**Viewpoint 1:**
For each student, which chair is the student assigned to?

% Chair[i] = the chair of student i:
array[1..s] of var 1..c: Chair;
constraint alldifferent(Chair);

**Viewpoint 2:**
For each chair, which student, if any, is seated on it?

% Student[i] = the student on chair i:
array[1..c] of var 0..s: Student; % dummy 0
constraint alldifferent_except_0(Student);

Let us now look at a generic problem in order to see how viewpoints differ when we start formulating constraints.
Example (Objects, Shapes, and Colours)

There are $n$ objects, $s$ shapes, and $c$ colours, with $s \geq n$. Assign a shape and a colour to each object such that:

1. the objects have distinct shapes;
2. the numbers of objects of the used colours are distinct;
3. other constraints, yielding NP-hardness and distinguishing objects and shapes, are satisfied.

This problem can be modelled from different viewpoints:

1. Which colour, if any, does each shape have?
2. Which shapes, if any, does each colour have?
3. Which shape and colour does each object have?
4. . .

Each viewpoint comes with benefits and drawbacks.
Example (Objects, Shapes, and Colours)

Viewpoint 1: Which colour, if any, does each shape have?

1 int: \( n \); % number of objects
2 int: \( s \); % number of shapes
3 int: \( c \); % number of colours
4 constraint assert(\( s \geq n \), "Not enough shapes");
5 % Colour\([i]\) = the colour of the object of shape \( i \):
6 array\([1..s]\) of var 0..\( c \): Colour; % 0 is a dummy colour
7 % There are \( n \) objects:
8 constraint exactly(\( s-n \),Colour,0);
9 % The numbers of objects of the used colours are distinct:
10 constraint
   alldifferent_except_0(global_cardinality(Colour,1..\( c \)));
11 % The objects have distinct shapes:
12 % implied by lines 6 and 8!
13 % ... add here the other constraints ...
14 solve satisfy;

Colour 0 is used when there is no object of the given shape.
So what are the shape and colour of a particular object?!
Map the objects onto the shapes with a non-0 colour!
Example (Objects, Shapes, and Colours)

Viewpoint 2: Which shapes, if any, does each colour have?

1. `int: n; % number of objects`
2. `int: s; % number of shapes`
3. `int: c; % number of colours`
4. `constraint assert(s >= n, "Not enough shapes");`
5. `% NbrObj[i,j] = the number of objects of colour i & shape j:`
6. `array[1..c,1..s] of var 0..1: NbrObj;`
7. `% There are n objects:`
8. `constraint n = sum(NbrObj);`
9. `% The numbers of objects of the used colours are distinct:`
10. `constraint alldifferent_except_0(colour in 1..c) (sum(NbrObj[colour,..]));`
11. `% The objects have distinct shapes:`
12. `constraint forall(shape in 1..s)(sum(NbrObj[..,shape])<=1);`
13. `% ... add here the other constraints ...`
14. `solve satisfy;`

Post-process: map the objects onto actually used shapes.
Example (Objects, Shapes, and Colours)

Viewpoint 3: Which shape & colour does each object have?

1 \texttt{int: n; \% number of objects}
2 \texttt{int: s; \% number of shapes}
3 \texttt{int: c; \% number of colours}
4 \texttt{constraint assert(s >= n, "Not enough shapes");}
5 \texttt{array[1..n] of var 1..s: Shape; \% Shape[i] = shape of obj. i}
6 \texttt{array[1..n] of var 1..c: Colour; \% Colour[i] = colour of i}
7 \% There are n objects:
8 \% implied by lines 5 and 6!
9 \% The numbers of objects of the used colours are distinct:
10 \texttt{constraint alldifferent_except_0}
11 \quad (global_cardinality_closed(Colour,1..c));
12 \% The objects have distinct shapes:
13 \texttt{constraint alldifferent(Shape)};
14 \% \ldots add here the other constraints \ldots
15 \texttt{solve satisfy;}

We have used two parallel arrays with the same index set but different domains in order to represent pair variables.
Which viewpoint is better in terms of:

- Size of the search space:
  - Viewpoint 1: $O((c + 1)^s)$, which is independent of $n$
  - Viewpoint 2: $O(2^{s\cdot c})$, which is independent of $n$
  - Viewpoint 3: $O(s^n \cdot c^n)$

Does this actually matter?

- Ease of formulating the constraints and the objective:
  - It depends on the unstated other constraints.
  - Ideally, we want a viewpoint that allows global-constraint predicates to be used.

- Performance:
  - Hard to tell: we have to run experiments!

- Readability:
  - Who is going to read the model?
  - What is their background?

There are no correct answers here:
we actually need to think about this and run experiments.
Outline

1. Viewpoints

2. Implied Constraints

3. Redundant Variables & Channelling Constraints

4. Pre-Computation
Example (The Magic Series Problem)

The element at index $i$ in $I = 0..(n-1)$ is the number of occurrences of $i$. Solution: $\text{Magic} = [1,2,1,0]$ for $n=4$.

**Variables:**

$\text{Magic} = \begin{array}{cccc}
0 & 1 & \cdots & n-1 \\
\in 0..n & \in 0..n & \cdots & \in 0..n
\end{array}$

**Constraint:**

forall($i$ in $I$)($\text{Magic}[i] = \text{sum}(j$ in $I$)($\text{bool2int}($$\text{Magic}[j]=$$i$)))

or, logically equivalently but better:

forall($i$ in $I$)($\text{count}($$\text{Magic},i,\text{Magic}[i]$))

or, logically equivalently and even better:

$\text{global_cardinality_closed}($$\text{Magic},I,\text{Magic}$)

**Implied Constraint:**

$\text{sum}(\text{Magic})=n$ /
\$\text{sum}(i$ in $I$)($\text{Magic}[i]$*$i$)$=n$

For $n=80$, using a CP solver: only 7 search nodes are explored instead of 302; the solving is 1,000 times faster.
Definition
An implied constraint, also called a redundant constraint, is a constraint that logically follows from other constraints.

Benefit:
Solving may be faster, without losing any solutions. However, not all implied constraints accelerate the solving.

Good practice in MiniZinc:
Flag implied constraints using the `implied_constraint` predicate. This allows backends to handle them differently, if wanted (see Topic 9: Modelling for CBLS):

```plaintext
predicate implied_constraint(var bool: c) = c; vs
predicate implied_constraint(var bool: c) = true;
```

Example

```plaintext
constraint implied_constraint(sum(Magic) = n);
```

In Topic 5: Symmetry, we see the equally recommended `symmetry_breaking_constraint` predicate.
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Redundant Decision Variables

Example (n-queens)

Use both the $n^2$ decision variables $\text{Queen}[i, j]$ in $0..1$ and the $n$ decision variables $\text{Row}[q]$ in $1..n$.

Definition

A redundant decision variable is a decision variable that represents information that is already represented by some other decision variables. It reflects a different viewpoint.

Benefit: Easier modelling of some constraints, or faster solving, or both.

Examples (see Topic 6: Case Studies)

- Model of Black-Hole Patience
- Models 1 & 3 of Warehouse Location Problem
Channelling Constraints

Example (n-queens)

Channelling between the $n$ decision variables $Row[i]$ in $1..n$ and the $n^2$ decision variables $Queen[i,j]$ in $0..1$:

$\forall (i \in 1..n) (Row[i] = \sum (j \in 1..n) (j \times Queen[i,j]))$

Definition

A channelling constraint establishes the coherence of the values of mutually redundant decision variables.

Examples (see Topic 6: Case Studies)

- Model of Black-Hole Patience
- Models 1 & 3 of Warehouse Location Problem
- Experiment with channelling between the viewpoints for the Objects, Shapes, and Colours problem (slide 7).
Example (Prize-Pool Division)

Consider a maximisation problem where the objective function is the division of an unknown prize pool by an unknown number of winners:

```plaintext
array[1..5] of int: Pools = [1000,5000,15000,20000,25000];
var 1..5: x; % index of the actual prize pool within Pools
var 1..500: nbrWinners; % the number of winners

solve maximize Pools[x] div nbrWinners; % implicit: element!
```

**Observation:** We should beware of using the `div` function on decision variables, because:

- It yields weak inference, at least in CP & LCG solvers.
- Its inference takes unnecessary time and memory.
- It is not supported by all MiniZinc backends.

**Idea:** We can pre-compute all possible objective values.
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### Example (Prize-Pool Division, revisited)

Pre-compute a 2d array, indexed by 1..5 and 1..500, for each possible value pair of \( x \) and \( \text{nbrWinners} \):

```plaintext
...  
array[1..5] of int: Pools = [1000,5000,15000,20000,25000];  
var 1..5: x; % index of the actual prize pool within Pools  
var 1..500: nbrWinners; % the number of winners  
...  
array[1..5,1..500] of int: objVal = array2d(1..5,1..500,  
    [Pools[p] div n | p in 1..5, n in 1..500]);  
solve maximize objVal[x,nbrWinners]; % implicit: 2d-element!
```