Topic 4: Modelling (for CP & LCG)\(^1\)
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Modelling for Combinatorial Optimisation

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Outline

1. Viewpoints
2. Implied Constraints
3. Redundant Variables & Channelling Constraints
4. Pre-Computation
Outline

1. Viewpoints

2. Implied Constraints

3. Redundant Variables & Channelling Constraints

4. Pre-Computation
Recap

1. **Modelling**: express problem in terms of
   - parameters,
   - decision variables,
   - constraints, and
   - objective.

2. **Solving**: solve using a state-of-the-art solver.
Example (Student Seating Problem)

Given:

- \( s \) students, and
- \( c \) chairs positioned around tables.

Find a seating arrangement such that:

- Each table has either at least half its chairs occupied, or none.
- Each table has at least as many students as any table behind it.
- A maximum number of student preferences on being seated at the same table are satisfied.

What are suitable decision variables for this problem?

\( s = 15 \) students
\( c = 20 \) chairs
A viewpoint is a choice of decision variables.

Example (Student Seating Problem)

**Viewpoint 1:**
For each student, which chair is the student assigned to?

% Chair[i] = the chair of student i:
array[1..s] of var 1..c: Chair;
constraint alldifferent(Chair);

**Viewpoint 2:**
For each chair, which student, if any, is seated on it?

% Student[i] = the student on chair i:
array[1..c] of var 0..s: Student; % dummy 0
constraint alldifferent_except_0(Student);

Let us now look at a generic problem in order to see how viewpoints differ when we start formulating constraints.
There are $n$ objects, $s$ shapes, and $c$ colours, with $s \geq n$. Assign a shape and a colour to each object such that:

1. the objects have distinct shapes;
2. the numbers of objects of the used colours are distinct;
3. other constraints, yielding NP-hardness and distinguishing objects and shapes, are satisfied.

This problem can be modelled from different viewpoints:

1. Which colour, if any, does each shape have?
2. Which shapes, if any, does each colour have?
3. Which shape and colour does each object have?
4. . . .

Each viewpoint comes with benefits and drawbacks.
Example (Objects, Shapes, and Colours)

Viewpoint 1: Which colour, if any, does each shape have?

```plaintext
1 int: n; % number of objects
2 int: s; % number of shapes
3 int: c; % number of colours
4 constraint assert(s >= n, "Not enough shapes");
5 % Colour[i] = the colour of the object of shape i:
6 array[1..s] of var 0..c: Colour; % 0 is a dummy colour
7 % There are n objects:
8 constraint count(Colour,0) = s - n;
9 % The numbers of objects of the used colours are distinct:
10 constraint
   alldifferent_except_0(global_cardinality(Colour,1..c));
11 % The objects have distinct shapes:
12 % implied by lines 6 and 8!
13 % ... add here the other constraints ...
14 solve satisfy;
```

Colour 0 is used when there is no object of the given shape.
So what are the shape and colour of a particular object?! 
臊 Map the objects onto the shapes with a non-0 colour!
Example (Objects, Shapes, and Colours)

Viewpoint 2: Which shapes, if any, does each colour have?

1 int: n; % number of objects
2 int: s; % number of shapes
3 int: c; % number of colours
4 constraint assert(s >= n, "Not enough shapes");
5 % Shapes[i] = the set of shapes of colour i:
6 array[1..c] of var set of 1..s: Shapes;
7 % There are n objects:
8 constraint n = sum(colour in 1..c)(card(Shapes[colour]));
9 % The numbers of objects of the used colours are distinct:
10 constraint alldifferent_except_0(colour in 1..c) (card(Shapes[colour]));
11 % The objects have distinct shapes:
12 constraint n = card(array_union(Shapes));
13 % ... add here the other constraints ...
14 solve satisfy;

Post-process: map the objects onto actually used shapes. Can we also model this viewpoint without set variables? 🤔 Yes, see the next slide!
Example (Objects, Shapes, and Colours)

Viewpoint 2: Which shapes, if any, does each colour have?

```plaintext
int: n; % number of objects
int: s; % number of shapes
int: c; % number of colours
constraint assert(s >= n, "Not enough shapes");
NbrObj[i,j] = the number of objects of colour i & shape j:
array[1..c,1..s] of var 0..1: NbrObj;
% There are n objects:
n = sum(NbrObj);
% The numbers of objects of the used colours are distinct:
constraint alldifferent_except_0(colour in 1..c)
    (sum(NbrObj[colour,..]));
% The objects have distinct shapes:
constraint forall(shape in 1..s)(sum(NbrObj[..,shape])<=1);
% ... add here the other constraints ...
solve satisfy;
```

Which model for viewpoint 2 is clearer or better?
★ Ask and try!
Example (Objects, Shapes, and Colours)

Viewpoint 3: Which shape & colour does each object have?

1. int: n; % number of objects
2. int: s; % number of shapes
3. int: c; % number of colours
4. constraint assert(s >= n, "Not enough shapes");
5. array[1..n] of var 1..s: Shape; % Shape[i] = shape of obj. i
6. array[1..n] of var 1..c: Colour; % Colour[i] = colour of i
7. % There are n objects:
8. % implied by lines 5 and 6!
9. % The numbers of objects of the used colours are distinct:
10. constraint alldifferent_except_0
(    global_cardinality_closed(Colour,1..c));
11. % The objects have distinct shapes:
12. constraint alldifferent(Shape);
13. % ... add here the other constraints ...
14. solve satisfy;

We have used two parallel arrays with the same index set but different domains in order to represent pair variables.
Which viewpoint is better in terms of:

- **Size of the search space:**
  - Viewpoint 1: $O((c + 1)^s)$, which is independent of $n$
  - Viewpoint 2: $O(2^{s \cdot c})$, which is independent of $n$
  - Viewpoint 3: $O(s^n \cdot c^n)$

Does this actually matter?

- **Ease of formulating the constraints and the objective:**
  - It depends on the unstated other constraints.
  - Ideally, we want a viewpoint that allows global-constraint predicates to be used.

- **Performance:**
  - Hard to tell: we have to run experiments!

- **Readability:**
  - Who is going to read the model?
  - What is their background?

There are no correct answers here: we actually need to think about this and run experiments.
Outline

1. Viewpoints

2. Implied Constraints

3. Redundant Variables & Channelling Constraints

4. Pre-Computation
Example (The Magic Series Problem)

The element at index $i$ in $I = 0..(n-1)$ is the number of occurrences of $i$. Solution: Magic = $[1, 2, 1, 0]$ for $n=4$.

Variables: $\begin{array}{cccc}
0 & 1 & \cdots & n-1 \\
\in 0..n & \in 0..n & \cdots & \in 0..n
\end{array}$

Constraint:
$\forall (i \in I)(\text{Magic}[i] = \sum(j \in I)(\text{Magic}[j]=i))$

or, logically equivalently but better:
$\forall (i \in I)(\text{count}($Magic,i, Magic[i] ))$

or, logically equivalently and even better:
$\text{global_cardinality_closed}($Magic, I, Magic)$

Implied Constraint:
$\sum($Magic$) = n \land \sum(i \in I)(\text{Magic}[i] \times i) = n$

For $n=80$, using a CP solver: only 7 search nodes are explored instead of 302; the solving is 1,000 times faster.
Definition

An implied constraint, also called a redundant constraint, is a constraint that logically follows from other constraints.

**Benefit:**
Solving may be faster, without losing any solutions. However, not all implied constraints accelerate the solving.

**Good practice in MiniZinc:**
Flag implied constraints using the `implied_constraint` predicate. This allows backends to handle them differently, if wanted (see Topic 9: Modelling for CBLS):

```mini_zinc
predicate implied_constraint(var bool: c) = c; vs
predicate implied_constraint(var bool: c) = true;
```

**Example**

```mini_zinc
constraint implied_constraint(sum(Magic) = n);
```

In Topic 5: Symmetry, we see the equally recommended `symmetry_breaking_constraint` predicate.
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Redundant Decision Variables

Example (n-queens)

Use both the $n^2$ decision variables $\text{Queen}[i,j]$ in 0..1 and the $n$ decision variables $\text{Row}[q]$ in 1..n.

Definition

A redundant decision variable is a decision variable that represents information that is already represented by some other decision variables. It reflects a different viewpoint.

Benefit: Easier modelling of some constraints, or faster solving, or both.

Examples (see Topic 6: Case Studies)

- Model of Black-Hole Patience
- Models 1 & 3 of Warehouse Location Problem
Channelling Constraints

Example (n-queens)

Channelling between the \( n \) decision variables \( \text{Row}[i] \) in \( 1 \ldots n \) and the \( n^2 \) decision variables \( \text{Queen}[i,j] \) in \( 0 \ldots 1 \):

\[
\forall (i \in 1 \ldots n) \left( \text{Row}[i] = \sum (j \in 1 \ldots n) (j \times \text{Queen}[i,j]) \right)
\]

Definition

A channelling constraint establishes the coherence of the values of mutually redundant decision variables.

Examples (see Topic 6: Case Studies)

- Model of Black-Hole Patience
- Models 1 & 3 of Warehouse Location Problem
- Experiment with channelling between the viewpoints for the Objects, Shapes, and Colours problem (slide 7).
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Example (Prize-Pool Division)

Consider a maximisation problem where the objective function is the division of an unknown prize pool by an unknown number of winners:

```plaintext
... 
array[1..5] of int: Pools = [1000,5000,15000,20000,25000];
var 1..5: x; % index of the actual prize pool within Pools
var 1..500: nbrWinners; % the number of winners
... 
solve maximize Pools[x] div nbrWinners; % implicit: element!
```

**Observation:** We should beware of using the `div` function on decision variables, because:

- It yields weak inference, at least in CP & LCG solvers.
- Its inference takes unnecessary time and memory.

**Idea:** We can pre-compute all possible objective values.
Idea: We can pre-compute all possible objective values.

Example (Prize-Pool Division, revisited)

Pre-compute a 2d array, indexed by 1..5 and 1..500, for each possible value pair of x and nbrWinners:

```plaintext
1 ... 
2 array[1..5] of int: Pools = [1000,5000,15000,20000,25000]; 
3 var 1..5: x; % index of the actual prize pool within Pools 
4 var 1..500: nbrWinners; % the number of winners 
5 ... 
6 array[1..5,1..500] of int: objVal = array2d(1..5,1..500, 
        [Pools[p] div n | p in 1..5, n in 1..500]); 
7 solve maximize objVal[x,nbrWinners]; % implicit: 2d-element!
```