Outline

1. Modelling Viewpoints

2. Implied Constraints

3. Redundant Variables & Channelling Constraints

4. Pre-Computation
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Recap

1. **Modelling**: express problem in terms of
   - Parameters,
   - Decision variables,
   - Constraints, and
   - Objective.

2. **Solving**: solve with existing state-of-the-art solvers.
Example (Student Placement Problem)

Given:

- $n$ students, and
- $m$ chairs positioned around tables.

$\begin{align*}
  n &= 15 \text{ students} \\
  m &= 20 \text{ chairs}
\end{align*}$
Example (Student Placement Problem)

Given:

- $n$ students, and
- $m$ chairs positioned around tables.

Find a seating arrangement such that:

- Each table is either at least half full or empty.
- Each table has at least as many students as any table behind it.
- The total diversity of each table is maximised.

A choice of decision variables represents a viewpoint.

$n = 15$ students
$m = 20$ chairs
Example (Student Placement Problem)

Given:

- $n$ students, and
- $m$ chairs positioned around tables.

Find a seating arrangement such that:

- Each table is either at least half full or empty.
- Each table has at least as many students as any table behind it.
- The total diversity of each table is maximised.

What are the decision variables for this problem?

A choice of decision variables represents a viewpoint.
A viewpoint is a choice of decision variables.

Example (Student Placement Problem)

Viewpoint 1:
For each student, which chair is the student assigned to?

- Define variables $S_1, S_2, \ldots, S_{15} \in \{1, \ldots, 20\}$.
- A solution where $S_5 = 7$ represents that student 5 is sitting on chair 7.

Viewpoint 2:
For each chair, which student, if any, is sitting in it?

- Define variables $C_1, C_2, \ldots, C_{20} \in \{0, \ldots, 15\}$.
- A solution where $C_7 = 5$ represents that student 5 is sitting on chair 7, and $C_7 = 0$ that chair 7 is empty.

Let’s look at a generic problem to see how viewpoints differ when we start formulating constraints.
Example (Objects, Shapes, and Colours)

There are $n$ objects, $s$ shapes, and $c$ colours, with $s \geq n$. Assign a shape and a colour to each object such that:

1. the objects have distinct shapes;
2. the numbers of objects of the used colours are distinct;
3. other constraints, yielding NP-hardness, are satisfied.

This problem can be modelled from different viewpoints:

- Which colour, if any, does each shape have?
- Which shapes, if any, does each colour have?
- Which shape and colour does each object have?
- . . .

Each viewpoint comes with benefits and drawbacks.
Example (Objects, Shapes, and Colours)

Viewpoint 1: Which colour, if any, does each shape have?

```plaintext
1 int: n; % number of objects
2 int: s; % number of shapes
3 int: c; % number of colours
4 constraint assert(s >= n, "Not enough shapes");
5
6 array[1..s] of var 0..c: Colour; % 0 is a dummy colour
7
8 % There are n objects:
9 constraint count(Colour,0) = s-n;
10 % The numbers of objects of the used colours are distinct:
11 constraint
   alldifferent_except_0(global_cardinality(Colour,1..c));
12 % The objects have distinct shapes:
13 % implied by lines 6 and 9
14 % ... add here the other constraints ...
15 solve satisfy;
```

Colour 0 is used when there is no object of the given shape. So what are the shape and colour of a particular object?!  Map the objects onto the shapes with a non-0 colour!
Example (Objects, Shapes, and Colours)

Viewpoint 2: Which shapes, if any, does each colour have?

```plaintext
int: n; % number of objects
int: s; % number of shapes
int: c; % number of colours
constraint assert(s >= n, "Not enough shapes");

array[1..c] of var set of 1..s: Shapes;

% There are n objects:
constraint n = sum(colour in 1..c)(card(Shapes[colour]));
% The numbers of objects of the used colours are distinct:
constraint alldifferent_except_0(colour in 1..c)
  (card(Shapes[colour]));
% The objects have distinct shapes:
constraint n = card(array_union(Shapes));
% ... add here the other constraints ...
solve satisfy;
```

Post-process: map the objects onto actually used shapes.
Can we also model this viewpoint without set variables? Yes, see the next slide!
Example (Objects, Shapes, and Colours)

Viewpoint 2: Which shapes, if any, does each colour have?

```
1 int: n; % number of objects
2 int: s; % number of shapes
3 int: c; % number of colours
4 constraint assert(s >= n, "Not enough shapes");

5 array[1..c,1..s] of var 0..1: NbrObjects;

% There are n objects:
6 constraint n = sum(colour in 1..c, shape in 1..s) (NbrObjects[colour,shape]);

% The numbers of objects of the used colours are distinct:
7 constraint alldifferent_except_0(colour in 1..c)(
     sum(shape in 1..s) (NbrObjects[colour,shape]));

% The objects have distinct shapes:
8 constraint forall(shape in 1..s)(1 >=
     sum(colour in 1..c) (NbrObjects[colour,shape]));

% ... add here the other constraints ...
10 solve satisfy;
```

Which model for viewpoint 2 is clearer or better? Ask & try!
Example (Objects, Shapes, and Colours)

Viewpoint 3: Which shape & colour does each object have?

1 int: n; % number of objects
2 int: s; % number of shapes
3 int: c; % number of colours
4 constraint assert(s >= n, "Not enough shapes");
5
6 array[1..n] of var 1..s: Shape;
7 array[1..n] of var 1..c: Colour;
8
9 % There are n objects:
10 % implied by lines 6 and 7
11 % The numbers of objects of the used colours are distinct:
12 constraint
13 alldifferent_except_0(global_cardinality(Colour,1..c));
14 % The objects have distinct shapes:
15 constraint alldifferent(Shape);
16 % ... add here the other constraints ...
17 solve satisfy;

We have used two parallel arrays with the same index set but different domains in order to represent pair variables.
Which viewpoint is better in terms of:

- Search space

- Expressing constraints and objective

- Performance

- Readability
Which viewpoint is better in terms of:

- **Search space**
  - Viewpoint 1: $O((c + 1)^s)$
  - Viewpoint 2: $O(2^{s \cdot c})$
  - Viewpoint 3: $O(s^n \cdot c^n)$

- **Expressing constraints and objective**

- **Performance**

- **Readability**

Does this actually matter?

Expressing constraints and objective depends on the unstated constraints. Ideally, we want a viewpoint that allows global constraint predicates to be used.

Performance is hard to tell; we have to run experiments!

Readability: Who is going to read our model? What is their background? There is no correct answer here; we need to actually think about this and do experiments.
Which viewpoint is better in terms of:

- **Search space**
  - Viewpoint 1: $\mathcal{O}((c + 1)^s)$
  - Viewpoint 2: $\mathcal{O}(2^{s\cdot c})$
  - Viewpoint 3: $\mathcal{O}(s^n \cdot c^n)$

- Does this actually matter?

- **Expressing constraints and objective**

- **Performance**

- **Readability**
Which viewpoint is better in terms of:

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  - Viewpoint 1: $O((c + 1)^s)$
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- **Performance**

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  - Depends on the unstated constraints.
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- **Performance**
  - Hard to tell, we have to run experiments!

- **Readability**
  - Who is going to read our model?
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*There is no correct answer here, we need to actually think about this and do experiments.*
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Example (The Magic Series Problem)

The element at index $i$ in $I = 0..(n-1)$ is the number of occurrences of $i$. Solution: $\text{Magic} = [1,2,1,0]$ for $n=4$.

**Variables:**

\[
\text{Magic} = \begin{bmatrix} 0 & 1 & \cdots & n-1 \\ \in 0..n & \in 0..n & \cdots & \in 0..n \end{bmatrix}
\]

**Constraint:**

\[
\forall (i \in I) \left( \text{Magic}[i] = \sum (j \in I) (\text{bool2int} (\text{Magic}[j]=i)) \right)
\]

or, logically equivalently but better:

\[
\forall (i \in I) \left( \text{count} (\text{Magic}, i, \text{Magic}[i]) \right)
\]

or, logically equivalently and even better:

\[
\text{global_cardinality_closed} (\text{Magic}, I, \text{Magic})
\]

**Implied Constraint:**

\[
\sum (\text{Magic}) = n \ \land \ \sum (i \in I) (\text{Magic}[i] \times i) = n
\]

For $n=80$, using a CP solver: only 7 search nodes are explored instead of 302; the solving is 1,000 times faster.
**Definition**

An **implied constraint**, also called a **redundant constraint**, is a constraint that logically follows from other constraints.

**Benefit:**
Solving may be faster, without losing any solutions. However, not all implied constraints accelerate the solving.

**Good practice in MiniZinc:**
Flag implied constraints using the `implied_constraint` predicate. This allows backends to handle them differently, if wanted (see Topic 9: Modelling for CBLS):

```
predicate implied_constraint(var bool: c) = c;
vs
predicate implied_constraint(var bool: c) = true;
```

**Example**

```
constraint implied_constraint(sum(Magic) = n);
```

In Topic 7: Symmetry, we will see the equally recommended `symmetry_breaking_constraint` predicate.
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Redundant Decision Variables

Example (n-queens)

Use both the $n^2$ decision variables $\text{Queen}[i,j]$ in $0..1$ and the $n$ decision variables $\text{Row}[q]$ in $1..n$.

Definition

A redundant decision variable is a decision variable that represents information that is already represented by some other decision variables. It reflects a different viewpoint.

Benefit: Easier modelling of some constraints, or faster solving, or both.

Examples (see Topic 6: Case Studies)

- Model of Black-Hole Patience
- Models 1 & 3 of Warehouse Location Problem
Channelling Constraints

Example (n-queens)

Channelling between the $n$ decision variables $Row[i]$ in $1..n$ and the $n^2$ decision variables $Queen[i,j]$ in $0..1$:

$$\forall i \in 1..n \ (Row[i] = \sum j \in 1..n (j \times Queen[i,j]))$$

Definition

A channelling constraint establishes the coherence of the values of mutually redundant decision variables.

Examples (see Topic 6: Case Studies)

- Model of Black-Hole Patience
- Models 1 & 3 of Warehouse Location Problem
- Experiment with channelling between the viewpoints for the Objects, Shapes, and Colours problem (slide 7).
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Example (Prize-Pool Division)

Consider a maximisation problem where the objective function is the division of an unknown prize pool by an unknown number of winners:

```plaintext
array[1..5] of int: Pools = [1000, 5000, 15000, 20000, 25000];
var 1..5: x;
var 1..500: numWinners;

solve maximize Pools[x] div numWinners; % implicit: element!
```

Observation: We should avoid using the `div` function on decision variables, because:

- It yields weak inference, at least in CP & LCG solvers.
- Its inference takes unnecessary time and memory.
- It is not supported by all MiniZinc backends.

Idea: We can pre-compute all possible objective values.
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Example (Prize-Pool Division, revisited)

Pre-compute a 2D array, indexed by 1..5 and 1..500, for each possible value pair of x and numWinners:

```plaintext
...  2 array[1..5] of int: Pools = [1000,5000,15000,20000,25000];  3 var 1..5: x;  4 var 1..500: numWinners;  5 ...
  6 array[1..5,1..500] of int: objFun = array2d(1..5,1..500,  
      [Pools[p] div n | p in 1..5, n in 1..500]);  7 solve maximize objFun[x,numWinners]; % implicit: 2D-element!
```