Topic 7: Symmetry
(Version of 1st November 2017)

Pierre Flener

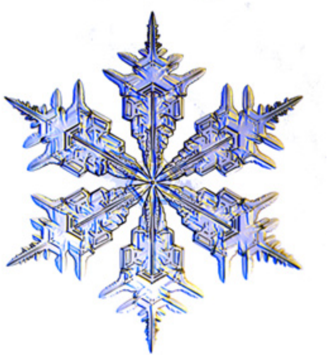
ASTRA Research Group
on Combinatorial Optimisation
Uppsala University
Sweden

Course 1DL448:
Modelling for Combinatorial Optimisation

Outline

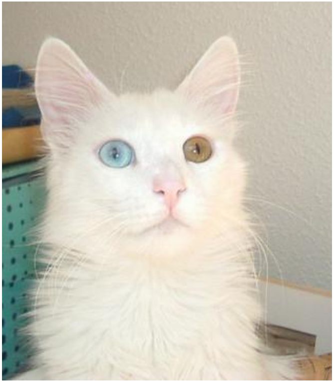
Outline

Symmetry in Nature



Johannes Kepler, *On the Six-Cornered Snowflake*, 1611: six-fold rotational symmetry of snowflakes, role of symmetry in human perception and the arts, fundamental importance of symmetry in the laws of physics.

Broken Symmetry in Nature



The Angora cat originated in the Turkish city of Ankara. It is admired for its long silky coat and quiet graceful charm. It is often bred to favour a pale milky colouring, as well as one blue and one amber eye. (*Turkish Daily News*, 14 Oct 2001)

The Nobel Prize in Physics 2008

"for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics"

"for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature"

Photo: University of Chicago

Yoichiro Nambu

Photo: U. Montan

Makoto Kobayashi

Photo: U. Montan

Toshihide Maskawa

Value Symmetry

Example (Map colouring)

Use k colours to paint the countries of a map such that no neighbour countries have the same colour. The model where the countries (as decision variables) take colours (as values) has $k!$ value symmetries because any permutation of the colours of a (non-)solution transforms that (non-)solution into another (non-)solution: the values (the colours) are not distinguished. (Continued on page 33)

Example (Partitioned map colouring)

The colours of map colouring are partitioned into subsets, such that only the colours of the same subset are not distinguished. (Continued on page 33)

UPPSALA UNIVERSITET

Introduction

Symmetry Breaking by Reformulation

Static Symmetry Breaking

Conclusion

Variable Symmetry

Example (n-Queens)

The model with a 2D array of Boolean decision variables has 4 reflection symmetries and 4 rotation symmetries, which are variable symmetries, as any reflection or rotation of an $n \times n$ board with n queens transforms that (non-) solution into another (non-)solution. (Continued on page 32)

Example (Subset)

Find an n-element subset of a given set S, such that some constraints are satisfied.
The model encoding the subset as an array of n distinct decision variables of domain S has $n!$ variable symmetries because the order of the elements does not matter in a set, but does matter in an array. (Continued on page 23)

Symmetries can be introduced!

- The symmetries in the (partitioned) map colouring and *n*-queens models are actually problem symmetries: they are detectable in *every* model.
- The symmetries in the subset model are *not* problem symmetries but model symmetries: they are *not* detectable in every model.
- There can also be instance symmetries, which are detectable in the instance data of a problem. Example: cargo boats with the same capacity.

Observation:

A solver may waste a lot of effort on gazillions of (partial) non-solutions that are symmetric to already visited ones, whereas found solutions can be transformed without search into symmetric solutions in polynomial time.

Definition (also see Cohen *et al.*, *CP 2005*)

A symmetry is a permutation of values or decision variables (or both) that preserves solutions: it transforms (partial) solutions into (partial) solutions, and it transforms (partial) non-solutions into (partial) non-solutions.

Symmetries form a group:

- The inverse of a symmetry is a symmetry.
- The identity permutation is a symmetry.
- The composition of two symmetries is a symmetry.

(Computational) group theory is the way to study symmetry.

Example (The Sport Scheduling Problem, SSP)

Find schedule in $\texttt{Periods} \times \texttt{Weeks} \to \texttt{Teams} \times \texttt{Teams}$ for

- $|\texttt{Teams}| = \texttt{n}$ and $\texttt{n}$ is even
- $|\texttt{Weeks}| = \texttt{n-1}$
- $|\texttt{Periods}| = \texttt{n div 2}$

subject to the following constraints:

1. Each game is played exactly once.
2. Each team plays exactly once per week.
3. Each team plays at most twice per period.

Idea for a model, and a solution for $\texttt{n=8}$:

	Wk 1	Wk 2	Wk 3	Wk 4	Wk 5	Wk 6	Wk 7
P 1	1 vs 2	1 vs 3	2 vs 6	3 vs 5	4 vs 7	4 vs 8	5 vs 8
P 2	3 vs 4	2 vs 8	1 vs 7	6 vs 7	6 vs 8	2 vs 5	1 vs 4
P 3	5 vs 6	4 vs 6	3 vs 8	1 vs 8	1 vs 5	3 vs 7	2 vs 7
P 4	7 vs 8	5 vs 7	4 vs 5	2 vs 4	2 vs 3	1 vs 6	3 vs 6

Example (SSP: the symmetries)

Observation: The periods, weeks, game slots, and teams of a sport schedule are not distinguished:

	Wk 1	Wk 2	Wk 3	Wk 4	Wk 5	Wk 6	Wk 7
P 1	1 vs 2	1 vs 3	2 vs 6	3 vs 5	4 vs 7	4 vs 8	5 vs 8
P 2	3 vs 4	2 vs 8	1 vs 7	6 vs 7	6 vs 8	2 vs 5	1 vs 4
P 3	5 vs 6	4 vs 6	3 vs 8	1 vs 8	1 vs 5	3 vs 7	2 vs 7
P 4	7 vs 8	5 vs 7	4 vs 5	2 vs 4	2 vs 3	1 vs 6	3 vs 6

- The periods/rows can be permuted: 4! variable syms.
- The weeks/columns can be permuted: 7! var syms.
- The game slots can be permuted: $2!^{28}$ variable syms.
- The team names can be permuted: 8! value syms.

All these permutations preserve solutions.

Example (The Social Golfer Problem, SGP)

Find schedule $\texttt{Weeks} \times \texttt{Groups} \times \texttt{Slots} \to \texttt{Players}$ for

- $|\texttt{Weeks}| = \texttt{w}$
- $|\texttt{Groups}| = \texttt{g}$
- $|\texttt{Slots}| = \texttt{s}$
- $|\texttt{Players}| = \texttt{g} \cdot \texttt{s}$

subject to the following constraint:

1. Any two players are at most once in the same group.

Idea for a model and a solution for $\langle \texttt{w}, \texttt{g}, \texttt{s} \rangle = \langle 4, 4, 3 \rangle$:

	Group 1	Group 2	Group 3	Group 4
Week 1	[1,2, 3]	[4,5, 6]	[7,8, 9]	[10,11,12]
Week 2	[1,4, 7]	[2,5,10]	[3,8,11]	[6, 9,12]
Week 3	[1,8,10]	[2,4,12]	[3,5, 9]	[6, 7,11]
Week 4	[1,9,11]	[2,6, 8]	[3,4,10]	[5, 7,12]

By the way, there is no solution when adding a fifth week.

Example (SGP: the symmetries)

Observation: The weeks, groups, group slots, and players of a social golfer schedule are not distinguished:

	Group 1	Group 2	Group 3	Group 4
Week 1	[1,2, 3]	[4,5, 6]	[7,8, 9]	[10,11,12]
Week 2	[1,4, 7]	[2,5,10]	[3,8,11]	[6, 9,12]
Week 3	[1,8,10]	[2,4,12]	[3,5, 9]	[6, 7,11]
Week 4	[1,9,11]	[2,6, 8]	[3,4,10]	[5, 7,12]

- The weeks/rows can be permuted: 4! variable symmetries.
- The groups can be permuted within a week: $4!^4$ var symmetries.
- The group slots can be permuted: $3!^{16}$ variable symmetries.
- The player names can be permuted: 12! value symmetries.

All these permutations preserve solutions.

Terminology, for Variable and Value Syms

Definition (Special cases of symmetry)

- Full symmetry: any permutation preserves solutions. The full symmetry group S_n has n! symmetries over a sequence of n elements.
- Partial symmetry: any piecewise permutation preserves solutions. This often occurs in instances. **Examples:** weekdays vs weekend; same-size boats.
- Wreath symmetry: any wreath permutation preserves solutions. **Example:** the composition of the first two variable symmetries of the social golfer problem.
- Rotational symmetry: any rotation preserves solutions. The cyclic symmetry group C_n has n symmetries over a sequence of n elements.

Definition (Special cases of symmetry, end)

- Index symmetry: any permutation of slices of an array of decision variables preserves solutions:
  full vs partial row symmetry, column symmetry, ...
- Conditional or dynamic symmetry: a symmetry that appears while solving a problem.
  Such symmetries are beyond the scope of this topic.

Careful: Symmetries multiply up!

If there is full row and column symmetry in an $m \times n$ array (that is, if there are $m!$ row syms and $n!$ column syms), then there are ~~$m! + n!$~~ $m! \cdot n!$ compositions of symmetries, and at most $m! \cdot n! - 1$ symmetric solutions per solution.
For example, none of the $2^{1 \cdot 4} = 16$ Boolean 1×4 arrays can have $1! \cdot 4! - 1 = 23$ symmetric arrays.

Challenges Raised by Symmetries

Definition

Symmetry handling has two aspects:

- Detecting ~~the~~ symmetries of the problem (in a model) as well as ~~the~~ symmetries introduced when modelling.
- Breaking (better: exploiting) ~~the~~ detected symmetries so that less effort is spent on the solving.

Automated detection is beyond the scope of this topic.

Classification of Symmetry Breaking

Definition

A symmetry class is an equivalence class of candidate solutions under all the considered symmetries, including their compositions.

Aim: While solving, keep ideally one member per symmetry class, as this may make a problem "less intractable":

- Symmetry breaking by reformulation:
  the elimination of ~~the~~ symmetries detectable in model.
- Static symmetry breaking:
  the elimination of symmetric solutions by constraints.
- Dynamic symmetry breaking:
  the elimination of symmetric solutions by search.
  This is beyond the scope of this topic: see MiniSearch.

Definition

Structural symmetry breaking exploits the combinatorial structure of a problem by using the key strengths of constraint-based modelling, namely constraint predicates if not search strategies, towards eliminating, ideally in polynomial time and space, all symmetric solutions, even if there are exponentially many symmetries.

Careful: Size does not matter!

A number of symmetries is no indicator of the difficulty of breaking them! For example, consider variable symmetry:

- The full group S_n has $n!$ easily broken symmetries: see page 23.
- The cyclic group C_n has only n symmetries, which are more difficult to break.

Outline

Symmetry Breaking by Reformulation

Example (The sport scheduling problem)

Let the **domain** of the decision variables of an $\frac{n}{2} \times n$ array be $\{f \cdot n + s \mid 1 \leq f < s \leq n\}$: the game between teams f and s is uniquely identified by $f \cdot n + s$.

A **round-robin schedule** breaks many of the other syms:

- Fix the games of the first week to the set $\{(1,2)\} \cup \{(t+1, n+2-t) \mid 1 < t \leq n/2\}$
- For the other weeks, transform each (f, s) into (f', s'):

$$f' = \begin{cases} 1 & \text{if } f = 1 \\ 2 & \text{if } f = n \\ f+1 & \text{otherwise} \end{cases}, \text{ and } s' = \begin{cases} 2 & \text{if } s = n \\ s+1 & \text{otherwise} \end{cases}$$

Determine the period of each game, but not its week!

Example (The social golfer problem)

Break the slot symmetries within each group by switching from the 3D $\mathtt{w} \times \mathtt{g} \times \mathtt{s}$ array of integer decision variables:

	Group 1	Group 2	Group 3	Group 4
Week 1	[1,2, 3]	[4,5, 6]	[7,8, 9]	[10,11,12]
Week 2	[1,4, 7]	[2,5,10]	[3,8,11]	[6, 9,12]
Week 3	[1,8,10]	[2,4,12]	[3,5, 9]	[6, 7,11]
Week 4	[1,9,11]	[2,6, 8]	[3,4,10]	[5, 7,12]

to a 2D $\mathtt{w} \times \mathtt{g}$ array of set decision variables (see Topic 2: Basic Modelling):

	Group 1	Group 2	Group 3	Group 4
Week 1	$\{1,2,3\}$	$\{4,5,6\}$	$\{7,8,9\}$	$\{10,11,12\}$
Week 2	$\{1,4,7\}$	$\{2,5,10\}$	$\{3,8,11\}$	$\{6,9,12\}$
Week 3	$\{1,8,10\}$	$\{2,4,12\}$	$\{3,5,9\}$	$\{6,7,11\}$
Week 4	$\{1,9,11\}$	$\{2,6,8\}$	$\{3,4,10\}$	$\{5,7,12\}$

and adding the constraint that all sets must be of size $\mathtt{s}$.

Outline

Useful Predicates: Lexicographic Ordering

Example

`lex_lesseq([1,2,34,5,678], [1,2,36,45,78])`,
because `34 < 36`, even though `not(678 < 78)`:
this is not the point-wise ordering.

Definition

The `lex_lesseq(A,B)` constraint, where `A` and `B` are same-length 1D arrays of variables, say both with indices in `1..n`, holds iff `A` is lexicographically at most equal to `B`:

- either `n=0`,
- or `A[1]<B[1]`,
- or `A[1]=B[1]` & `lex_lesseq(A[2..n],B[2..n])`.

Variant predicates exist.

Static Symmetry Breaking (SSB)

Definition

SSB = the elimination of symmetric solutions by constraints.

Classification:

- Lex-leader scheme (Crawford *et al.*, *KR'96*) (page 28): state a `lex_lesseq` constraint for each symmetry.

The lex-leader scheme is general and may take exponential space if there are exponentially many symmetries. Hence:

- Static structural symmetry breaking (SSSB) (page 33): exploit the combinatorial structure of a problem for stating fewer, not necessarily `lex_lesseq` constraints.

Careful: Potential interference of SSB with search: see Topic 8: Inference & Search in CP & LCG.

Lexicographic ordering along one dimension of an array breaks the index symmetry of that dimension.

Example (The sport scheduling problem)

Breaking all the variable symmetries of the $\frac{n}{2} \times n$ array:

- Each row is lexicographically less than the next, if any.
- Each col. is lexicographically less than the next, if any.
- The first team of each game has a smaller number than the second team of the game (this constraint can also be enforced by the definition on page 21 of the domain of the decision variables).

This breaks all the variable symmetries in this case, because the array values are all different: this is also why one should use `lex_less` here.

When lexicographically ordering an array along every dimension with index symmetry:

- No symmetry class is of size 0.
- However, in general, not all sym classes are of size 1, except if all the array values are different, etc.

Counterexample

Symmetric matrices with lex ordered rows and columns:

0	1
0	1
1	0

→ Swap the columns

← Swap row 1 and 3

0	1
1	0
1	0

The Lex-Leader Scheme

For any group G of variable symmetries on decision variables `x1,...,xn` of domain D:

1. Choose a variable ordering, say `x1,...,xn`.
2. Choose a total value ordering on D, say $\preceq$.
3. Choose a lexicographic-ordering predicate induced by $\preceq$, say `lex_lesseq`.
4. For every symmetry $\sigma \in G$, add the constraint

   `lex_lesseq([x1,...,xn],[xσ(1),...,xσ(n)])`

   to the problem model.
5. Simplify the resulting constraints, locally and globally.

This yields exactly one solution per symmetry class.

Example (2 $\times$ 3 array with full row and column sym)

Constraints for the variable ordering `x1,x2,x3,x4,x5,x6`:

```
lex_lesseq([x1,x2,x3,x4,x5,x6],[x2,x1,x3,x5,x4,x6]);
lex_lesseq([x1,x2,x3,x4,x5,x6],[x1,x3,x2,x4,x6,x5]);
lex_lesseq([x1,x2,x3,x4,x5,x6],[x4,x5,x6,x1,x2,x3]);
lex_lesseq([x1,x2,x3,x4,x5,x6],[x6,x4,x5,x3,x1,x2]);
lex_lesseq([x1,x2,x3,x4,x5,x6],[x5,x6,x4,x2,x3,x1]);
lex_lesseq([x1,x2,x3,x4,x5,x6],[x4,x6,x5,x1,x3,x2]);
lex_lesseq([x1,x2,x3,x4,x5,x6],[x5,x4,x6,x2,x1,x3]);
lex_lesseq([x1,x2,x3,x4,x5,x6],[x6,x5,x4,x3,x2,x1]);
lex_lesseq([x1,x2,x3,x4,x5,x6],[x3,x2,x1,x6,x5,x4]);
lex_lesseq([x1,x2,x3,x4,x5,x6],[x2,x3,x1,x5,x6,x4]);
lex_lesseq([x1,x2,x3,x4,x5,x6],[x3,x1,x2,x6,x4,x5]);
```

UPPSALA UNIVERSITET

Example (2 $\times$ 3 array with full row and column sym)

Simplified constraints for the variable ordering `x1,x2,x3,x4,x5,x6`:

```
lex_lesseq([x1      ,x4      ],[x2      ,x5      ]);
lex_lesseq([   x2      ,x5   ],[   x3      ,x6   ]);
lex_lesseq([x1,x2,x3         ],[x4,x5,x6         ]);
lex_lesseq([x1,x2,x3         ],[x6,x4,x5         ]);
lex_lesseq([x1,x2,x3,x4      ],[x5,x6,x4,x2      ]);
lex_lesseq([x1,x2,x3         ],[x4,x6,x5         ]);
lex_lesseq([x1,x2,x3         ],[x5,x4,x6         ]);
lex_lesseq([x1,x2,x3         ],[x6,x5,x4         ]);
```

Example (Full variable symmetry)

For the $n!$ symmetries of the full symmetry group S_n, the $n! - 1$ n-ary `lex_lesseq` constraints (over arrays of size n) simplify into $n - 1$ binary `<=` constraints (over integers):

```
x1 <= x2; x2 <= x3; ... <= xn;
```

If the chosen variable ordering forms an array `X`, then `increasing(X)` should be used.

In practice:

Breaking all the symmetries may increase the solving time:

- Break only some symmetries, but which ones?
- Double-lex often works well on a 2D array `X` with full row and column symmetry: the `lex2(X)` constraint breaks the row symmetries and column symmetries, but not their compositions. See the example on p. 26.

Example (Rotation and reflection symmetry)

A magic square of order n is an $n \times n$ array containing all integers 1 to n^2 exactly once, so that the sums of the rows, columns, and main diagonals are equal (namely $\frac{n^2 \cdot (n^2+1)}{2 \cdot n}$).

For instance, a magic square `X` of order 3 has row sum 15:

2	9	4
7	5	3
6	1	8

Rotation and reflection symmetry-breaking constraint:

```
X[1,1] < X[1,n] /\ X[1,n] < X[n,1] /\ X[1,1] < X[n,n]
```

Static Structural Symmetry Breaking

Example (Full / partial value sym; Law & Lee, *CP'04*)

Consider decision variables `X` in the domain `D = 1..k`:

- **Full value symmetry** over the domain `D`:
  The first occurrences of the domain values are ordered:

  ```
  forall(i in 1..k-1)(value_precede(i,i+1,X))
  ```

  or, logically equivalently but better:

  ```
  value_precede_chain(D,X)
  ```

- **Partial value symmetry** over the partitioned domain `D = D[1] ∪ D[2] ∪ ··· ∪ D[m]`:

  ```
  forall(i in 1..m)(value_precede_chain(D[i],X))
  ```

Example (Partial var sym + full value sym; *CP 2006*)

- Make study groups for two sets of five indistinguishable students each. There are six indistinguishable tables.
- The arrays `P` and `M` of decision variables, both with indices in `1..5`, correspond to the students and are to be given table values from the domain `1..6`.
- Variable-symmetry-breaking constraint:

```
increasing(P) /\ increasing(M)
```

- Constraint on the signatures, which are integer pairs:

```
global_cardinality_closed(P,1..6,CP) /\
global_cardinality_closed(M,1..6,CM)
```

- Value-symmetry-breaking constraint using signatures:

```
forall(i in 1..(6-1))
  (lex_greatereq([CP[i],CM[i]],[CP[i+1],CM[i+1]]))
```

Example (continued)

Consider the solution

$$\begin{aligned} \mathtt{P} &= [1, 1, 2, 3, 4] \\ \mathtt{M} &= [1, 2, 2, 3, 5] \end{aligned}$$

The variable-symmetry-breaking constraint is satisfied:

```
increasing(P) /\ increasing(M)
```

and the value-symmetry-breaking constraint is satisfied:

$$[2, 1] \geq_{\text{lex}} [1, 2] \geq_{\text{lex}} [1, 1] \geq_{\text{lex}} [1, 0] \geq_{\text{lex}} [0, 1] \geq_{\text{lex}} [0, 0]$$

Note that a pointwise ordering would not have sufficed.

Example (continued)

If student `M[5]` is at table 6 instead of table 5, producing a symmetrically equivalent candidate solution because the tables are fully interchangeable:

$$\begin{aligned} \texttt{P} &= [1, 1, 2, 3, 4] \\ \texttt{M} &= [1, 2, 2, 3, 6] \end{aligned}$$

then the value-symmetry-breaking constraint is violated:

$$[2, 1] \geq_{\text{lex}} [1, 2] \geq_{\text{lex}} [1, 1] \geq_{\text{lex}} [1, 0] \geq_{\text{lex}} [0, 0] \not\geq_{\text{lex}} [0, 1]$$

Example (end)

If students `M[4]` and `M[5]` permute their tables, producing a symmetrically equivalent candidate solution because those two students are indistinguishable:

$$\begin{aligned} \texttt{P} &= [1, 1, 2, 3, 4] \\ \texttt{M} &= [1, 2, 2, 5, 3] \end{aligned}$$

then the signatures do not change and hence the value-symmetry-breaking constraint remains satisfied, but the variable-symmetry-breaking constraint is violated, because

$$\texttt{M[1]} \leq \texttt{M[2]} \leq \texttt{M[3]} \leq \texttt{M[4]} \not\leq \texttt{M[5]}$$

Static Symmetry Breaking in MiniZinc

Good practice in MiniZinc:

Flag symmetry-breaking constraints using the `symmetry_breaking_constraint` predicate.
This allows backends to handle them differently, if wanted (see Topic 9: Modelling for CBLS):

```
predicate symmetry_breaking_constraint(var bool: c) = c;
```
vs
```
predicate symmetry_breaking_constraint(var bool: c) = true;
```

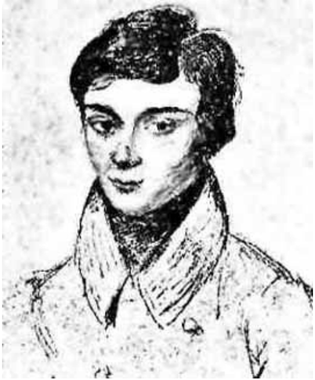
Examples

```
constraint symmetry_breaking_constraint(increasing(X));
constraint symmetry_breaking_constraint(lex_lesseq(A,B));
```

Especially for MIP backends, try commenting away some symmetry-breaking constraints, as the first definition above of `symmetry_breaking_constraint` is currently used.

Outline

Évariste Galois (1811–1832)



Évariste Galois was one of the parents of group theory.

Insight: The structure of the symmetries of an equation determines whether it has solutions or not.

Marginal note in his last paper: "*Il y a quelque chose à compléter dans cette démonstration. Je n'ai pas le temps.*" (There is something to complete in this demonstration. I do not have the time.)

UPPSALA UNIVERSITET

In Practice

- Are there few symmetries in real-life problems?
- Keep in mind the objective:
  first solution, all solutions, or best solution?
  Symmetry breaking might not pay off when searching for the first solution.
- Problem constraints can sometimes be simplified in the presence of symmetry-breaking constraints.
  Example: `z = abs(x-y)` can be simplified into `z = x-y` if symmetry breaking requires `x >= y`, thereby eliminating an undesirable disjunction.