Chapter 6
Abstract Datatypes

(Version of 17 November 2005)

6.1. Application: correctly parenthesised texts

Determine whether a text is correctly parenthesised

Specification

function parenthesised text
TYPE: char list ⟶ bool
PRE: (none)
POST: **true** if text is correctly parenthesised
false otherwise

Examples and counter-examples

((a+b) ∗ (c—d))	**true**
((a+b ∗ (c—d))	**false**
(a+b)) ∗ ((c—d)	**false**
(a[(b+c) ∗ d] + e) ∗ f	**true**
(a[(b+c) ∗ d) + e] ∗ f	**false**
(ab [+b(c ∗)])	**true**

Analysis

- Ignore everything except the parentheses
- The number of left parentheses of each kind must be equal to the number of right parentheses of the same kind
- Each right parenthesis must correspond to a left parenthesis of the same kind, which precedes it
- Each left parenthesis must correspond to a right parenthesis of the same kind, which follows it
- Between two corresponding parentheses, the text must be correctly parenthesised

Strategy: Generalise the problem

function parGen S L
TYPE: ?? → char list → bool
PRE: S is a ‘list’ of left-parenthesis characters
POST: **true** if ‘S@L’ is correctly parenthesised
false otherwise

Example

parGen [#"(", #"{"] [#"[", #"a", #"]", #"+", #"b", #"}", #"−", #"c", #")"]
shall be **true** because ({[a]+b}−c) is correctly parenthesised

Construction

Variant: the length of `L`

Base case: `L` is `[]`

Then `S@L` is correctly parenthesised iff `S` is empty

General case: `L` is of the form `(x::xs)`

If `x` is not a parenthesis,
then `S@L` is correctly parenthesised iff
`S@xs` is correctly parenthesised
Hence the recursive call `parGen S xs`

If `x` is a left parenthesis,
then `S@L` is correctly parenthesised iff
`S.x@xs` is correctly parenthesised,
where `S.x` denotes the addition of `x` to the end (top) of `S`
(so `S` seems to be a *stack*)
Hence the recursive call `parGen (S.x) xs`

If `x` is a right parenthesis,
then `S@L` is correctly parenthesised iff

- `S` has the corresponding left parenthesis at its top, and
- `S’@xs` is correctly parenthesised,
  where `S’` is `S` without its top element

Hence the recursive call `parGen S’ xs`

Hence:

- Necessity of an *abstract datatype* for stacks
- Necessity of the auxiliary functions
  leftParenthesis, rightParenthesis, and corresponds

```
function parGen  S  L
TYPE: char stack ⟶ char list ⟶ bool
PRE: S only has left-parenthesis characters
POST:  true    if  rev(showStack S)@L  is correctly parenthesised
       false   otherwise

function leftParenthesis x
TYPE: char ⟶ bool
PRE: (none)
POST:  true    if  x  is a left parenthesis
       false   otherwise

function rightParenthesis x
TYPE: char ⟶ bool
PRE: (none)
POST:  true    if  x  is a right parenthesis
       false   otherwise

function corresponds x y
TYPE: char ⟶ char ⟶ bool
PRE: (none)
POST:  true    if  x  is a left parenthesis
               and  y  is a right parenthesis corresponding to x
       false   otherwise
```

6.2. An abstract datatype for stacks

Stacks of objects of type α: α stack

Operations

value emptyStack
TYPE: α stack
VALUE: the empty stack

function isEmptyStack S
TYPE: α stack $\longrightarrow$ bool
PRE: (none)
POST: **true** if S is empty
false otherwise

function push v S
TYPE: $\alpha \longrightarrow \alpha$ stack $\longrightarrow \alpha$ stack
PRE: (none)
POST: the stack S with v added as new top element

function top S
TYPE: α stack $\longrightarrow \alpha$
PRE: S is non-empty
POST: the top element of S

function pop S
TYPE: α stack $\longrightarrow \alpha$ stack
PRE: S is non-empty
POST: the stack S without its top element

‘Formal’ semantics

```
isEmptyStack emptyStack = true
∀v,S : isEmptyStack (push v S) = false
top emptyStack = ... error ...
∀v,S : top (push v S) = v
pop emptyStack = ... error ...
∀v,S : pop (push v S) = S
```

Example: correctly parenthesised text (`parentheses.sml`)

```
local
   fun leftParenthesis v = ...
   fun rightParenthesis w = ...
   fun corresponds v w = ...
   fun parGen S [ ] = isEmptyStack S
     | parGen S (x::xs) =
         if leftParenthesis x
         then parGen (push x S) xs
         else if rightParenthesis x
         then not (isEmptyStack S) andalso
              corresponds (top S) x andalso
              parGen (pop S) xs
         else parGen S xs
in
   fun parenthesised text = parGen emptyStack text
end
```

6.3. Realisation of the stack abstract datatype

Version 1

Representation of a stack by a *list*:

type α stack = α **list**

REPRESENTATION CONVENTION: the head of the list is the top of the stack, the 2nd element of the list is the element below the top, etc

Realisation of the operations (`stack1.sml`)

```
val  emptyStack = [ ]

fun  isEmptyStack S = (S = [ ])

fun  push v S = v::S

fun  top [ ] = error "top: empty stack"
  |  top (x::xs) = x

fun  pop [ ] = error "pop: empty stack"
  |  pop (x::xs) = xs
```

- This realisation does *not* force the usage of the stack type
- The operations can *also* be used with objects of type α **list**, even if they do not represent stacks!
- It is possible to access the elements of the stack *without* using the operations specified above: no encapsulation!

Version 2

Definition of a *new* constructed type using the **list** type:

datatype α stack = Stack **of** α **list**
REPRESENTATION CONVENTION: the head of the list is the top of the stack, the 2nd element of the list is the element below the top, etc

Realisation of the operations

```
val  emptyStack = Stack [ ]

fun  isEmptyStack (Stack S) = (S = [ ])

fun  push v (Stack S) = Stack (v::S)

fun  top (Stack [ ]) = error "top: empty stack"
  |  top (Stack (x::xs)) = x

fun  pop (Stack [ ]) = error "pop: empty stack"
  |  pop (Stack (x::xs)) = Stack xs
```

- The operations are now *only* defined for stacks
- It is *still* possible to access the elements of the stack *without* using the operations specified above, namely by pattern matching

An abstract datatype (`stack2.sml`)

Objective: encapsulate the definition of the stack type and its operations in a parameterised *abstract datatype*

```
abstype 'a stack = Stack of 'a list
with
   val emptyStack = Stack [ ]
   fun isEmptyStack (Stack S) = (S = [ ])
   fun push v (Stack S) = Stack (v::S)
   fun top (Stack [ ]) = error "top: empty stack"
     | top (Stack (x::xs)) = x
   fun pop (Stack [ ]) = error "pop: empty stack"
     | pop (Stack (x::xs)) = Stack xs
end
```

- The stack type is an *abstract datatype* (ADT)
- The concrete representation of a stack is *hidden*
- An object of the stack type can *only* be manipulated via the functions defined in its ADT declaration
- The Stack constructor is *invisible* outside the ADT
- It is now *impossible* to access the representation of a stack outside the declarations of the functions of the ADT
- The parameterisation allows the usage of stacks of integers, reals, strings, integer functions, etc, from a *single* definition!

```
-   abstype 'a stack = Stack of 'a list with ... ;
    type 'a stack
    val 'a emptyStack = - : 'a stack
    val ''a isEmptyStack = fn : ''a stack -> bool
    ...
-   push 1 (Stack [ ]) ;
    Error: unbound variable or constructor: Stack
-   push 1 emptyStack ;
    val it = - : int stack
```

It is *impossible* to compare two stacks:

```
-   emptyStack = emptyStack ;
    Error: operator and operand don't agree
    [equality type required]
```

It is *impossible* to see the contents of a stack without popping its elements, so let us add a visualisation function:

```
function showStack S
TYPE: α stack → α list
PRE:  (none)
POST: the representation of S in list form, with the top of S as head, etc

abstype 'a stack = Stack of 'a list
with
   ...
   fun showStack (Stack S) = S
end
```

- The result of showStack is *not* of the stack type
- One can thus *not* apply the stack operations to it

Version 3

Definition of a *recursive* new constructed type:

```
datatype α stack = EmptyStack
                 | >> of α stack ∗ α
infix >>
```

EXAMPLE: EmptyStack >> 3 >> 5 >> 2 represents the stack with top 2
REPRESENTATION CONVENTION: the right-most value is the top of the stack, its left neighbour is the element below the top, etc

An abstract datatype `(stack3.sml)`

```
abstype 'a stack = EmptyStack | >> of 'a stack ∗ 'a
with
  infix >>
  val emptyStack = EmptyStack
  fun isEmptyStack EmptyStack = true
    | isEmptyStack (S>>v) = false
  fun push v S = S>>v
  fun top EmptyStack = error "top: empty stack"
    | top (S>>v) = v
  fun pop EmptyStack = error "pop: empty stack"
    | pop (S>>v) = S
  fun showStack EmptyStack = [ ]
    | showStack (S>>v) = v :: (showStack S)
end
```

We have thus defined a new list constructor,
but with access to the elements *from the right*!

6.4. An abstract datatype for FIFO queues

First-in first-out (FIFO) queues of objects of type α: α queue

- Addition of elements to the rear (*tail*)
- Deletion of elements from the front (*head*)

Operations

value emptyQueue
TYPE: α queue
VALUE: the empty queue

function isEmptyQueue Q
TYPE: α queue $\longrightarrow$ bool
PRE: (none)
POST: **true** if Q is empty
false otherwise

function enqueue v Q
TYPE: $\alpha \longrightarrow \alpha$ queue $\longrightarrow \alpha$ queue
PRE: (none)
POST: the queue Q with v added as new tail element

function head Q
TYPE: α queue $\longrightarrow \alpha$
PRE: Q is non-empty
POST: the head element of Q

function dequeue Q
TYPE: α queue $\rightarrow$ α queue
PRE: Q is non-empty
POST: the queue Q without its head element

function showQueue Q
TYPE: α queue $\rightarrow$ α list
PRE: (none)
POST: the representation of Q in list form, with the head of Q as head, etc

'Formal' semantics

```
isEmptyQueue emptyQueue = true
∀v,Q : isEmptyQueue (enqueue v Q) = false
head emptyQueue = ... error ...
∀v,Q : head (enqueue v Q) =  if isEmptyQueue Q then v
                             else head Q
dequeue emptyQueue = ... error ...
∀v,Q : dequeue (enqueue v Q) =  if isEmptyQueue Q then emptyQueue
                                else enqueue v (dequeue Q)
```

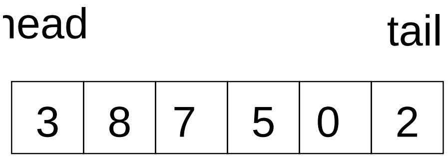
6.5. Realisation of the queue abstract datatype

Version 1

Representation of a FIFO queue by a *list*:

type α queue = α **list**
REPRESENTATION CONVENTION: the head of the list is the head of the queue, the 2nd element of the list is behind the head of the queue, and so on, and the last element of the list is the tail of the queue

Example: the queue



is represented by the list [3,8,7,5,0,2]

Exercises

- Realise the queue ADT using this representation
- What is the time complexity of enqueuing an element?
- What is the time complexity of dequeuing an element?

Version 2

Representation of a FIFO queue by a *pair of lists*:

datatype α queue = Queue **of** α **list** $* \, \alpha$ **list**
REPRESENTATION CONVENTION: the term
Queue $([x_1, x_2, \ldots, x_n], [y_1, y_2, \ldots, y_m])$
represents the queue

ıead tail

$$\boxed{x_1 \; x_2 \quad \ldots \, x_n \; y_m \qquad \ldots \quad y_2 \; y_1}$$

REPRESENTATION INVARIANT: (see next slide)

- It is *now* possible to enqueue in $\Theta(1)$ time
- It is still possible to dequeue in $\Theta(1)$ time, but *only* if $n \geq 1$
- What if $n = 0$ while $m > 0$?!
- The same queue can thus be represented in *different* ways
- How to test the equality of two queues?

Normalisation

Objective: avoid the case where $n = 0$ while $m > 0$

When this case appears, transform (or: normalise)
the representation of the queue:
transform Queue $([\,], [y_1, \ldots, y_m])$ with $m > 0$
into Queue $([y_m, \ldots, y_1], [\,])$,
which indeed represents the same queue

We thus have:

REPRESENTATION INVARIANT: a non-empty queue is never represented by Queue $([\,], [y_1, \ldots, y_m])$

```
function normalise Q
TYPE: $\alpha$ queue $\rightarrow$ $\alpha$ queue
PRE: (none)
POST: if Q is of the form Queue $([\,], [y_1, \ldots, y_m])$
        then Queue $([y_m, \ldots, y_1], [\,])$
        else Q
```

Realisation of the operations (`queue2.sml`)

Construction of an abstract datatype:
the normalise function may be *local* to the ADT,
as it is only used for realising some operations on queues

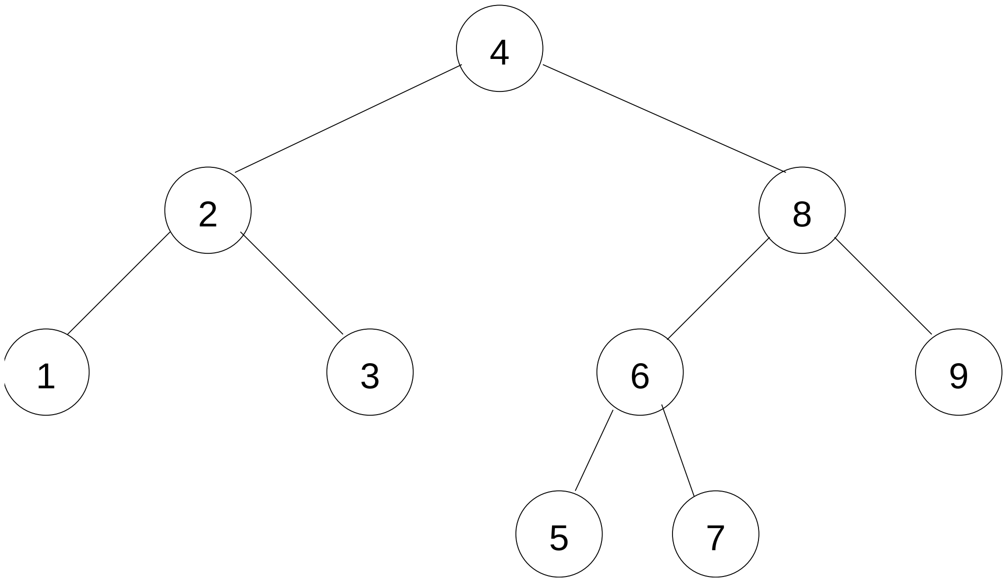
```
abstype 'a queue = Queue of 'a list * 'a list
with
  val emptyQueue = Queue ([ ],[ ])
  fun isEmptyQueue (Queue ([ ],[ ])) = true
    | isEmptyQueue (Queue (xs,ys)) = false
  fun head (Queue (x::xs,ys)) = x
    | head (Queue ([ ],[ ])) = error "head: empty queue"
    | head (Queue ([ ],y::ys)) = error "head: non-normalised queue"
  local
    fun normalise (Queue ([ ],ys)) = Queue (rev ys,[ ])
      | normalise Q = Q
  in
    fun enqueue v (Queue (xs,ys)) = normalise (Queue (xs,v::ys))
    fun dequeue (Queue (x::xs,ys)) = normalise (Queue (xs,ys))
      | dequeue (Queue ([ ],[ ])) = error "dequeue: empty queue"
      | dequeue (Queue ([ ],y::ys)) = error "dequeue: non-norm. queue"
  end
  fun showQueue (Queue (xs,ys)) = xs @ (rev ys)
  fun equalQueues Q1 Q2 = (showQueue Q1 = showQueue Q2)
end
```

- Why do the head and dequeue functions *not* normalise the queue instead of stopping the execution with an error?
- The normalisation and representation invariant are *hidden* in the realisation of the abstract datatype
- On average, the time of enqueuing and dequeuing is $\Theta(1)$
- This representation is thus *very* efficient!

6.6. An abstract datatype for binary trees

Concepts and terminology

Binary trees of objects of type α: α bTree



Operations

value emptyBtree
TYPE: α bTree
VALUE: the empty binary tree

function isEmptyBtree T
TYPE: α bTree $\longrightarrow$ bool
PRE: (none)
POST: **true** if T is empty
false otherwise

function consBtree v L R
TYPE: $\alpha \to \alpha$ bTree $\to \alpha$ bTree $\to \alpha$ bTree
PRE: (none)
POST: the binary tree with root v, left sub-tree L, and right sub-tree R

function left T
TYPE: α bTree $\to \alpha$ bTree
PRE: T is non-empty
POST: the left sub-tree of T

function right T
TYPE: α bTree $\to \alpha$ bTree
PRE: T is non-empty
POST: the right sub-tree of T

function root T
TYPE: α bTree $\to \alpha$
PRE: T is non-empty
POST: the root of T

'Formal' semantics

isEmptyBtree emptyBtree = **true**
∀v,L,R : isEmptyBtree (consBtree v L R) = **false**

root emptyBtree = ... error ...
∀v,L,R : root (consBtree v L R) = v

left emptyBtree = ... error ...
∀v,L,R : left (consBtree v L R) = L

right emptyBtree = ... error ...
∀v,L,R : right (consBtree v L R) = R

6.7. Realisation of the bTree abstract datatype

Representation

```
datatype  bTree = Void
                | Bt of int ∗ bTree ∗ bTree
```

REPRESENTATION CONVENTION: a binary tree with root x, left subtree L, and right subtree R is represented by Bt(x,L,R)

```
EXAMPLE: Bt(4, Bt(2, Bt(1,Void,Void), Bt(3,Void,Void)),
               Bt(8, Bt(6, Bt(5,Void,Void), Bt(7,Void,Void)), Bt(9,Void,Void)))
```

Realisation of the operations (`bTree.sml`)

```
abstype  'a bTree = Void
                  | Bt of 'a ∗ 'a bTree ∗ 'a bTree
with
   val emptyBtree = Void
   fun isEmptyBtree Void = true
     | isEmptyBtre (Bt(v,L,R)) = false
   fun consBtree v L R = Bt(v,L,R)
   fun left Void = error "left: empty bTree"
     | left (Bt(v,L,R)) = L
   fun right Void = error "right: empty bTree"
     | right (Bt(v,L,R)) = R
   fun root Void = error "root: empty bTree"
     | root (Bt(v,L,R)) = v
end
```

Walk operations `(inorder.sml)`

function inorder T

TYPE: α bTree $\rightarrow$ α list

PRE: (none)

POST: the nodes of T upon an inorder walk

```
fun  inorder  Void  =  [ ]
  |  inorder  (Bt(v,L,R))  =  (inorder  L)  @  (v  ::  inorder  R)
```

No tail recursion!

It takes $\Theta(n \log n)$ time for a binary tree of n nodes…

function inorderGen T acc

TYPE: α bTree $\rightarrow$ α list $\rightarrow$ α list

PRE: (none)

POST: (the nodes of T upon an inorder walk) @ acc

```
fun  inorderGen  Void  acc  =  acc
  |  inorderGen  (Bt(v,L,R))  acc  =
        let  val  rAcc  =  inorderGen  R  acc
        in  inorderGen  L  (v::rAcc)  end
```

```
fun  inorder  t  =  inorderGen  t  [ ]
```

One tail recursion! No call to @ (concatenation)!

It takes $\Theta(n)$ time for a binary tree of n nodes

Exercises

- Efficiently realise the preorder and postorder walks of a binary tree, and analyse the underlying algorithms
- How to test the equality of two binary trees?

Other operations

function exists k T

TYPE: $\alpha^= \longrightarrow \alpha^=$ bTree $\longrightarrow$ bool

PRE: (none)

POST: **true** if T contains node k
false otherwise

function insert k T

TYPE: $\alpha^= \longrightarrow \alpha^=$ bTree $\longrightarrow \alpha^=$ bTree

PRE: (none)

POST: T with node k

function delete k T

TYPE: $\alpha^= \longrightarrow \alpha^=$ bTree $\longrightarrow \alpha^=$ bTree

PRE: (none)

POST: if exists k T, then T without one occurrence of node k, otherwise T

function nbNodes T

TYPE: α bTree $\longrightarrow$ int

PRE: (none)

POST: the number of nodes of T

function nbLeaves T

TYPE: α bTree $\longrightarrow$ int

PRE: (none)

POST: the number of leaves of T

Exercises

- Efficiently realise these five functions
- Show that their algorithms at worst take $\Theta(n)$ time, if not $\Theta(1)$ time, on a binary tree with initially n nodes

Height of a binary tree `(height.sml)`

- The *height of a node* is the length of the longest path (measured in its number of nodes) from that node to a leaf
- The *height of a tree* is the height of its root

```
function height T
TYPE: α bTree → int
PRE: (none) ; POST: the height of T

fun height Void = 0
  | height (Bt(v,L,R)) = 1 + Int.max (height L, height R)
```

No tail recursion!
It takes $\Theta(n)$ time for a binary tree of n nodes.
Note that heightGen T acc $=$ acc + height T does *not* suffice to get a tail recursion: why?!

```
function heightGen T acc hMax
TYPE: α bTree → int → int → int
PRE: (none) ; POST: max (acc + height T, hMax)

fun heightGen Void acc hMax = Int.max (acc, hMax)
  | heightGen (Bt(v,L,R)) acc hMax =
      let val h1 = heightGen R (acc+1) hMax
      in heightGen L (acc+1) h1 end

fun height2 bt = heightGen bt 0 0
```

One tail recursion!
It also takes $\Theta(n)$ *time* for a binary tree of n nodes,
but it takes less *space*!

6.8. An ADT for Binary Search Trees (BSTs)

Concepts and terminology (see the tree on slide 6.19)

Binary search trees of nodes of type $(\alpha^=, \beta)$: $(\alpha^=, \beta)$ bsTree where:

- $\alpha^=$ is the type of the *keys* (need for an equality test)
- β is the type of the *satellite data* for each key

are binary trees with:

```
(REPRESENTATION) INVARIANT: for a binary search tree
with (k,s) in the root, left subtree L, and right subtree R:
 - every element of L has a key smaller than k
 - every element of R has a key  larger than k
```

Note that we (arbitrarily) ruled out duplicate keys

Benefits

- The inorder walk of a binary search tree lists its nodes by increasing order of their keys!
- The basic operations at worst take $\Theta(n)$ time on a binary search tree with (initially) n nodes, but they take $O(\lg n)$ time on *randomly* built binary search trees

Let us restrict our realisation to *integer* keys: β bsTree

Some operations

value emptyBsTree
TYPE: β bsTree
VALUE: the empty binary search tree

function isEmptyBsTree T
TYPE: β bsTree $\longrightarrow$ bool
PRE: (none)
POST: **true** if T is empty
false otherwise

function exists k T
TYPE: int $\longrightarrow$ β bsTree $\longrightarrow$ bool
PRE: (none)
POST: **true** if T contains a node with key k
false otherwise

function insert (k,s) T
TYPE: (int $*\beta$) $\longrightarrow$ β bsTree $\longrightarrow$ β bsTree
PRE: (none)
POST: if exists k T, then T with s as satellite data for key k
otherwise T with node (k,s)

function retrieve k T
TYPE: int $\longrightarrow$ β bsTree $\longrightarrow$ β
PRE: exists k T
POST: the satellite data associated to key k in T

function delete k T
TYPE: int $\longrightarrow$ β bsTree $\longrightarrow$ β bsTree
PRE: (none)
POST: if exists k T, then T without the node with key k, otherwise T

6.9. Realisation of the bsTree ADT

Representation

```
datatype 'b bsTree = Void
                   | Bst of (int * 'b) * 'b bsTree * 'b bsTree
```

REPRESENTATION CONVENTION: a BST with (k,s) in the root, left subtree L, and right subtree R is represented by Bst((k,s),L,R)

REPRESENTATION INVARIANT: (see slide 6.25)

Realisation of the operations (`bsTree.sml`)

```
val emptyBsTree = Void

fun isEmptyBsTree Void = true
  | isEmptyBsTree (Bst((key,sat),L,R)) = false

fun exists k Void = false
  | exists k (Bst((key,sat),L,R)) =
      if k = key then true
      else if k < key then exists k L
      else (* k > key *) exists k R

fun insert (k,s) Void = Bst((k,s),Void,Void)
  | insert (k,s) (Bst((key,sat),L,R)) =
      if k = key then Bst((k,s),L,R)
      else if k < key then Bst((key,sat), (insert (k,s) L), R)
      else (* k > key *) Bst((key,sat), L, (insert (k,s) R))

fun retrieve k Void = error "retrieve: non-existing node"
  | retrieve k (Bst((key,sat),L,R)) =
      if k = key then sat
      else if k < key then retrieve k L
      else (* k > key *) retrieve k R
```

When deleting a node (key,sat) whose subtrees L and R are *both* non-empty, we must not violate the repr. invariant!

1. Replace (key,sat) by the node with the *maximal* key of L, whose key is smaller than the key of *any* node of R (one could also replace by the node with the *minimal* key of R)
2. Remove this node with the maximal key from L

So we need a deleteMax function:

```
function deleteMax T
TYPE: β bsTree → (int ∗ β) ∗ β bsTree
PRE:  T is non-empty
POST: (max, NT), where max is the node of T with the maximal key,
      and NT is T without max

fun deleteMax Void = error "deleteMax: empty bsTree"
  | deleteMax (Bst(r,L,Void): 'b bsTree) = (r, L)
  | deleteMax (Bst(r,L,R)) =
      let val (max, newR) = deleteMax R
      in (max, Bst(r,L,newR)) end

fun delete k Void = Void
  | delete k (Bst((key,sat),L,R)) =
      if k < key then Bst((key,sat), (delete k L), R)
      else if k > key then Bst((key,sat), L, (delete k R))
      else (∗ k = key ∗)
        case (L,R) of
            (Void, _ ) => R
          | ( _ ,Void) => L
          | ( _ , _ ) => let val (max, newL) = deleteMax L
                         in Bst(max,newL,R) end
```