Topic 10: Propagation for a DISTINCT Constraint[1]
(Version of 30th October 2017)

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Course 1DL441:
Combinatorial Optimisation using
Constraint Programming

[1] Based also on some material by Christian Schulte

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Outline

1. Prelude
2. The DISTINCT Predicate
3. Naïve DC Propagation
4. Efficient DC Propagation
5. Efficient BC Propagation
6. Bibliography

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Domain Consistency Revisited

All solutions to the constraint need to be considered:

- A naïve propagator, computing first all solutions, takes too much time and space. Recall Topic 3: the store $\{x \mapsto \{2,\ldots,7\}, y \mapsto \{0,1,2\}, z \mapsto \{-1,\ldots,2\}\}$ has solutions $\langle 3,1,0\rangle$, $\langle 5,0,1\rangle$, $\langle 6,2,0\rangle$ to $x = 3 \cdot y + 5 \cdot z$; hence $\{x \mapsto \{3,5,6\}, y \mapsto \{0,1,2\}, z \mapsto \{0,1\}\}$ is DC.
- An efficient DC propagator does in general not exist (for LINEAR$(A, X, =, d)$ it takes time exponential in $|X|$), but there exist polynomial-time DC propagators for some predicates, such as ELEMENT (☞ see Topic 6).
- Are there efficient DC propagators for more complex predicates than ELEMENT, such as DISTINCT? ☞ Yes!

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Definition

The DISTINCT($\{x_1, \ldots, x_n\}$) constraint, also known as the ALLDIFFERENT($\{x_1, \ldots, x_n\}$) constraint, has the semantics of a conjunction of $\frac{n \cdot (n-1)}{2}$ disequality constraints:

$$\textbf{forall}(i, j \textbf{ in } 1..n \textbf{ where } i < j)(x_i \neq x_j)$$

Example (DISTINCT($\{u, v, w, x, y, z\}$))

From the store

$$\left\{ \begin{array}{c} u \mapsto \{0,1\},\ v \mapsto \{1,2\},\ w \mapsto \{0,2\}, \\ x \mapsto \{1,3\},\ y \mapsto \{2,3,4,5\},\ z \mapsto \{5,6\} \end{array} \right\}$$

the pink values are pruned upon DC.

Is DC Needed for DISTINCT?

Example (Golomb Rulers)

Design a ruler with n ticks such that:

- The distances between any 2 distinct ticks are distinct.
- The length of the ruler is minimal.

For $n = 6$, an optimal ruler is $[0, 1, 4, 10, 12, 17]$.
This very hard problem has applications in crystallography.

n	value consistency	domain consistency
7	950 nodes	474 nodes
8	7,622 nodes	3,076 nodes
9	55,930 nodes	16,608 nodes
10	413,922 nodes	97,782 nodes
11	6,330,568 nodes	1,448,666 nodes

Good search-tree reduction: worth looking for a propagator!

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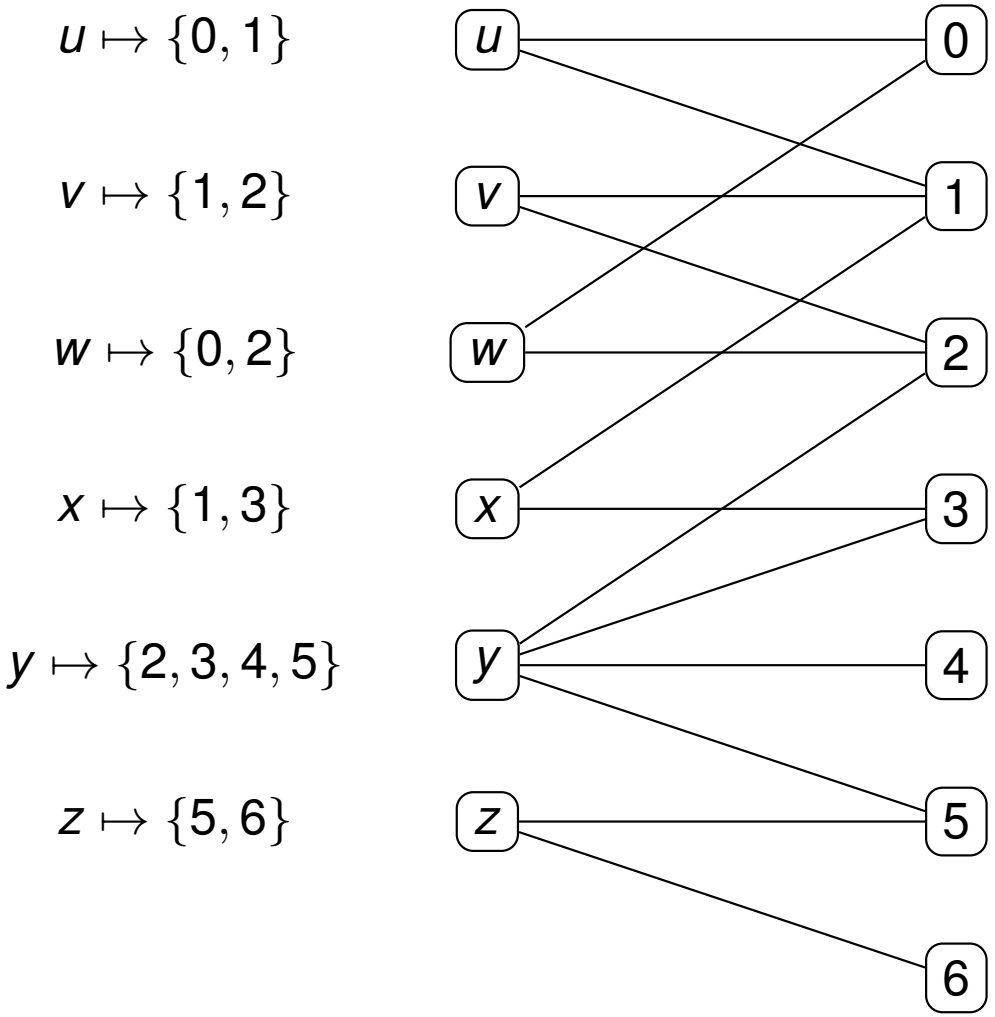
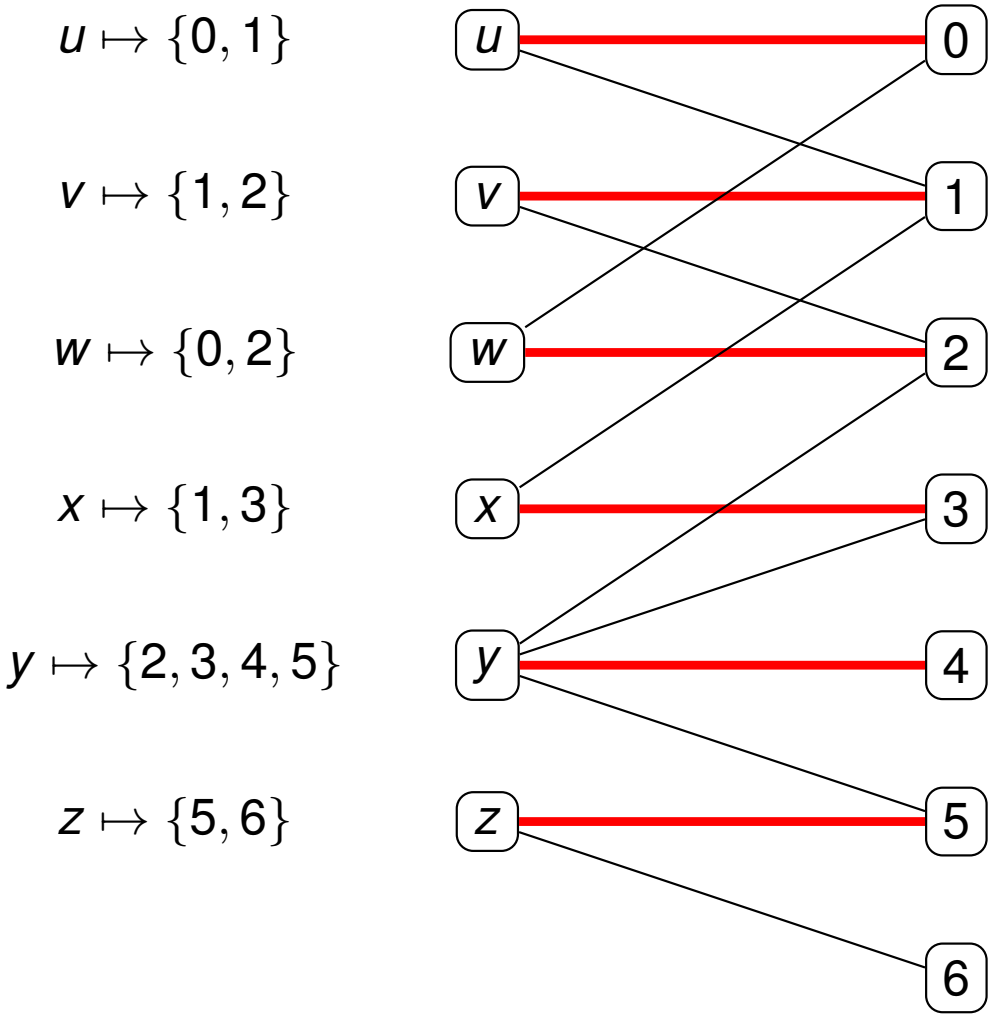
Variable-Value Graph:

Construct from the current domains a bipartite graph:



Variable-Value Graph: Matchings Are Solutions:

A (maximum) matching is a (maximum) subset of edges so that no vertex is incident to two of its edges. Example 1:



Variable-Value Graph: Matchings Are Solutions:

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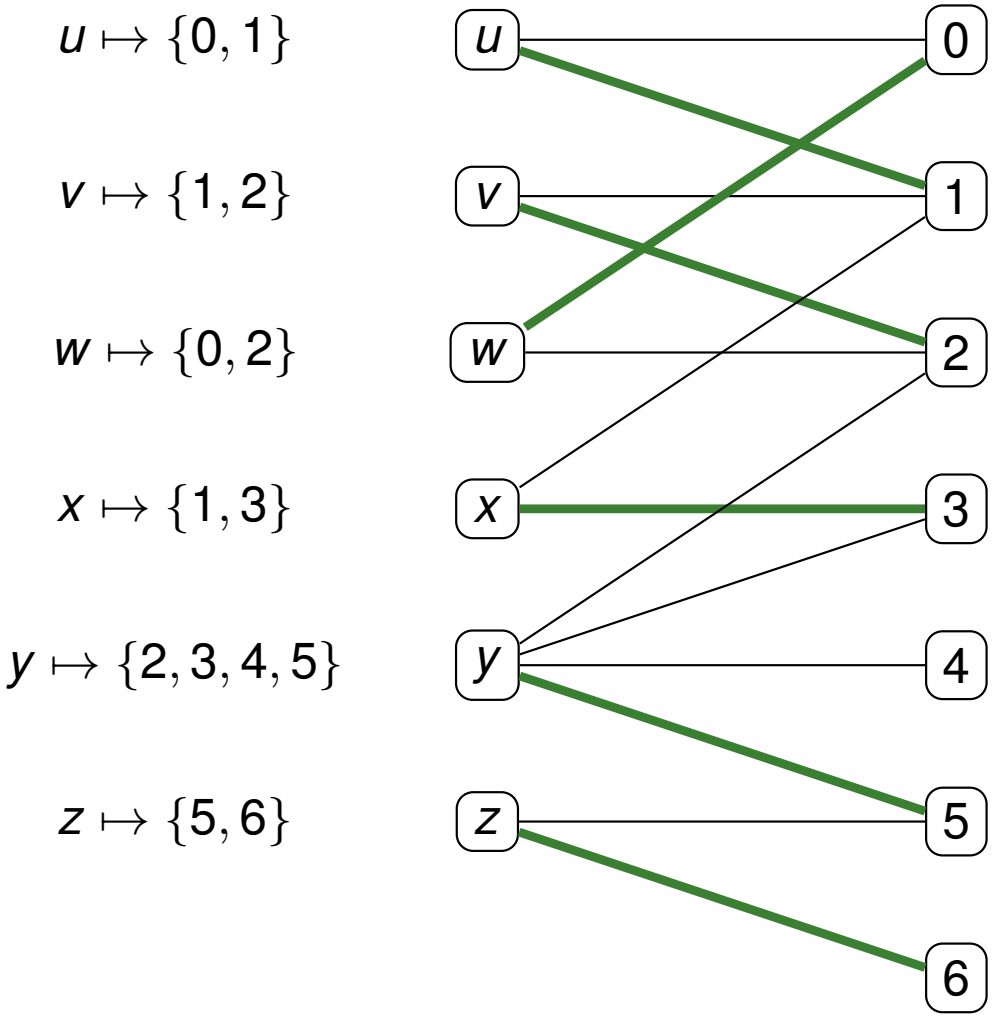

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Naïve DC propagator:

1. If no maximum matching exists, then fail.
2. Compute all maximum matchings & mark their edges.
3. For every unmarked edge between a decision variable α and a value d: prune value d from dom(α).

But there are as many maximum matchings as solutions!

☞ We have solved the space issue, but not the time issue.

Matching theory to the rescue!

There is a relationship between the edges in a maximum matching and the edges in all other maximum matchings!

☞ Hence we need only compute one maximum matching!

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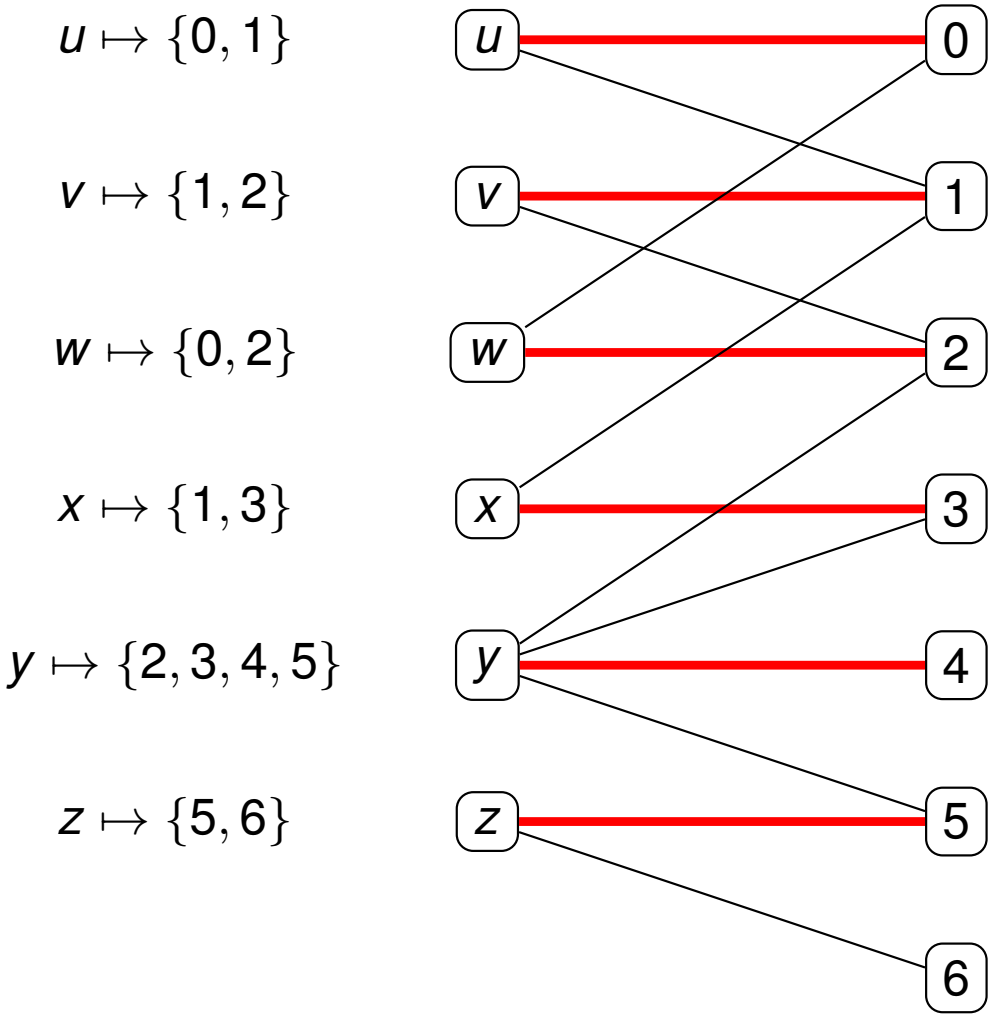
Efficient DC propagator (Régin, 1994) (Costa, 1994):

Start from a maximum matching, and orient all edges: if in matching, then from variable to value, else the other way.



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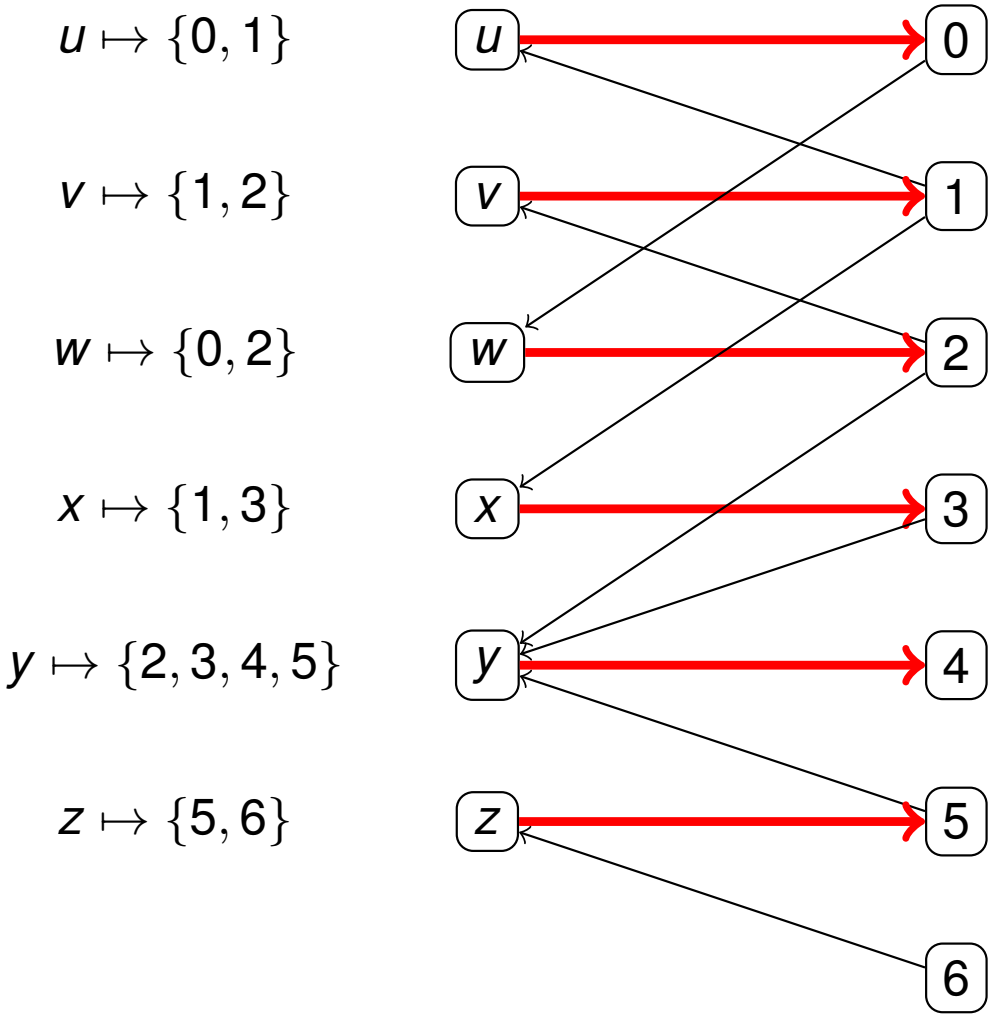
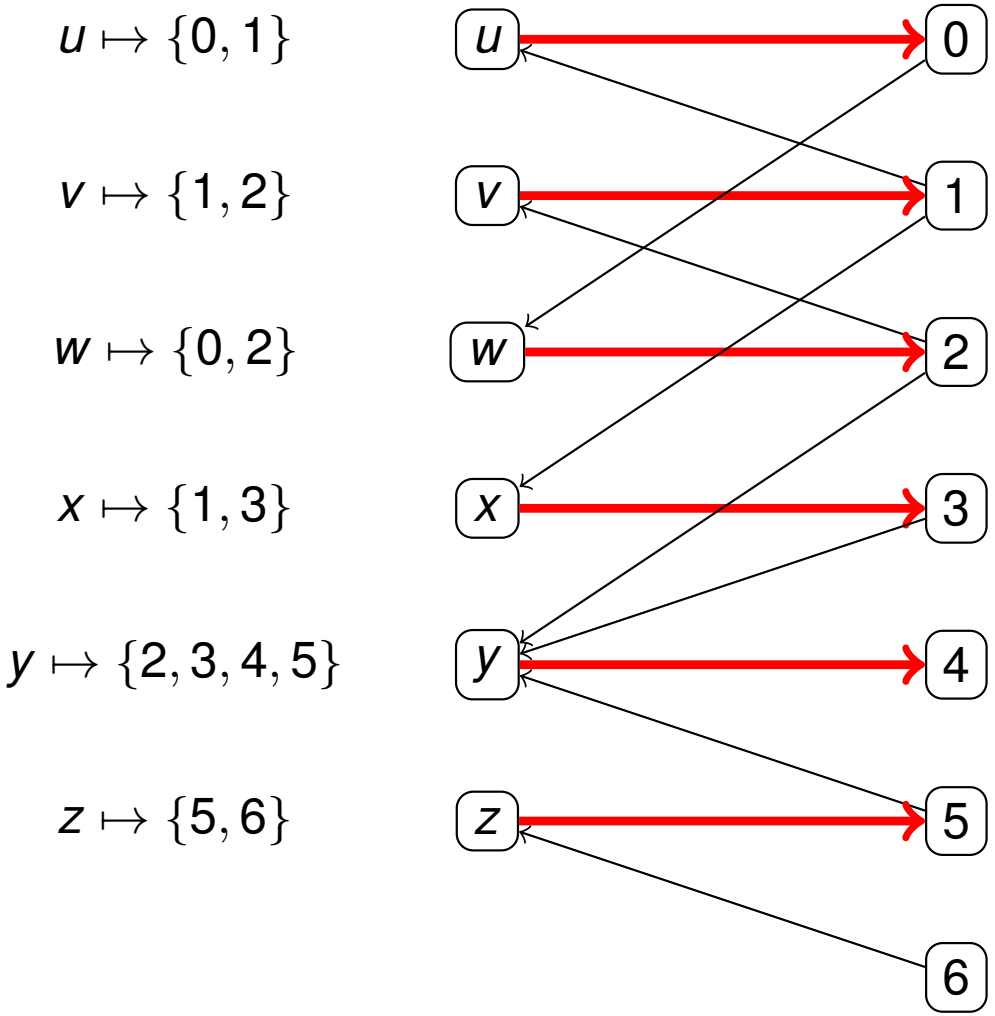


Efficient DC propagator (Régin, 1994) (Costa, 1994):

Start from all unmatched vertices (necessarily values here) and mark all arcs on all simple paths: arcs can be flipped.



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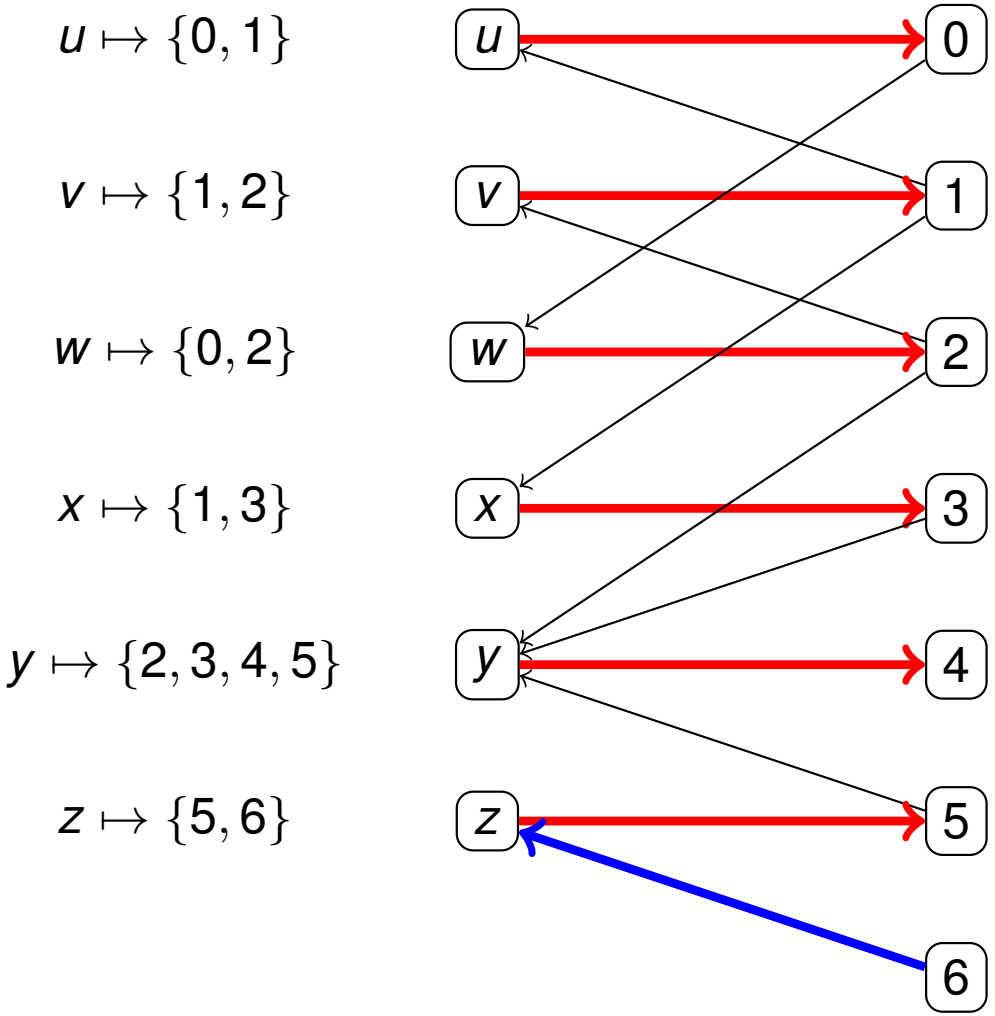

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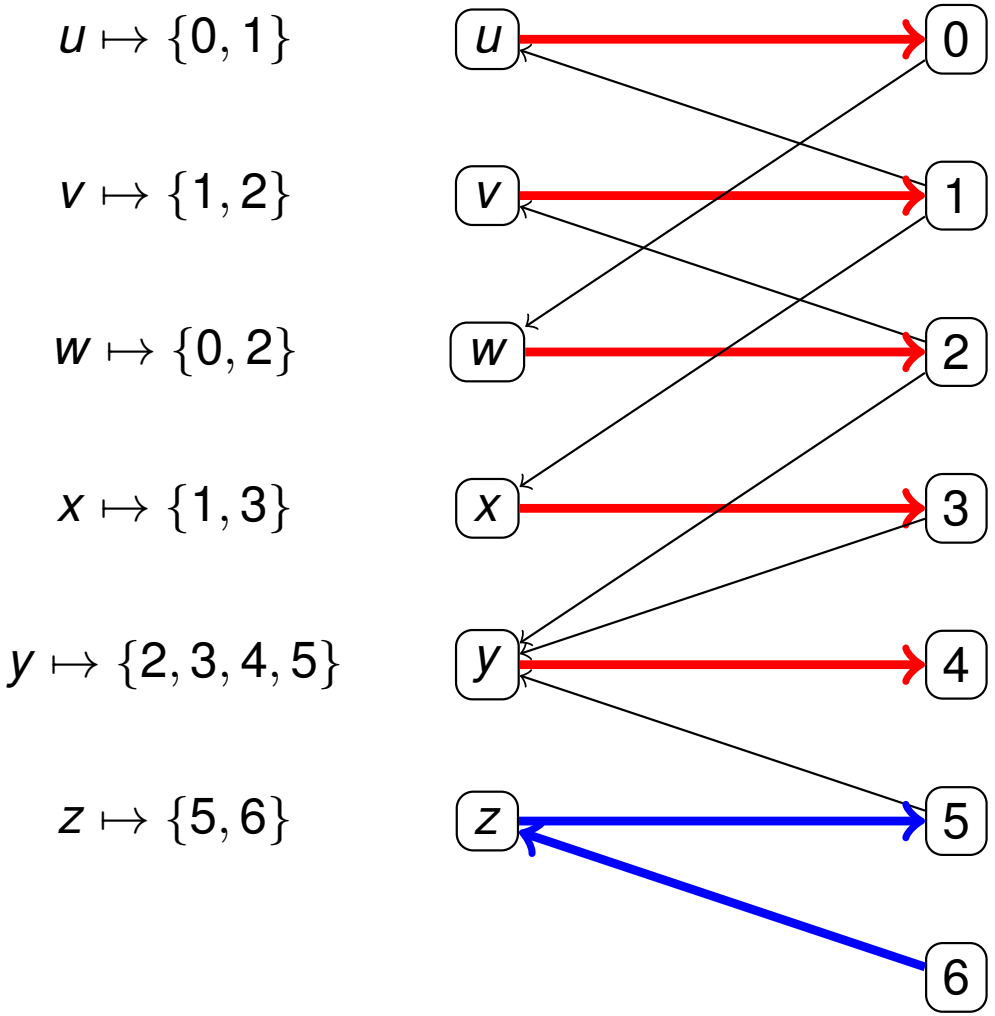

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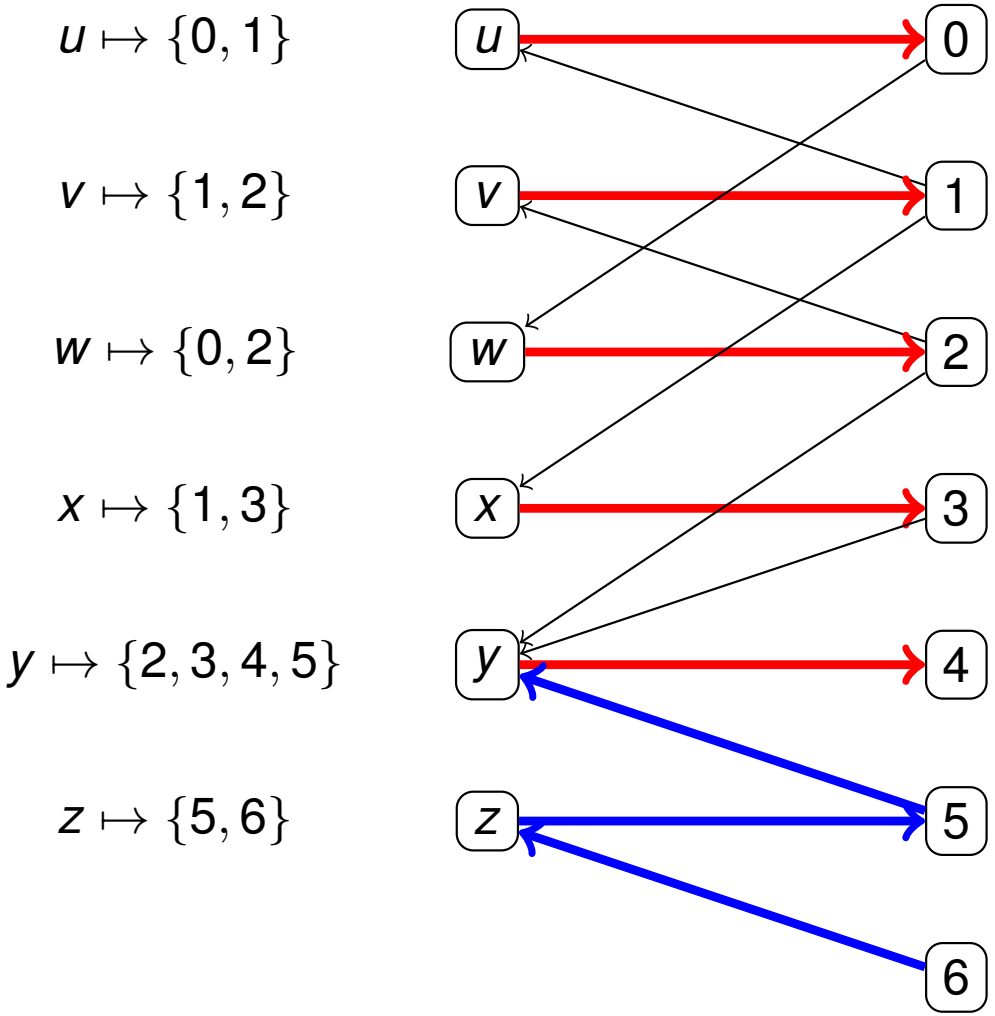

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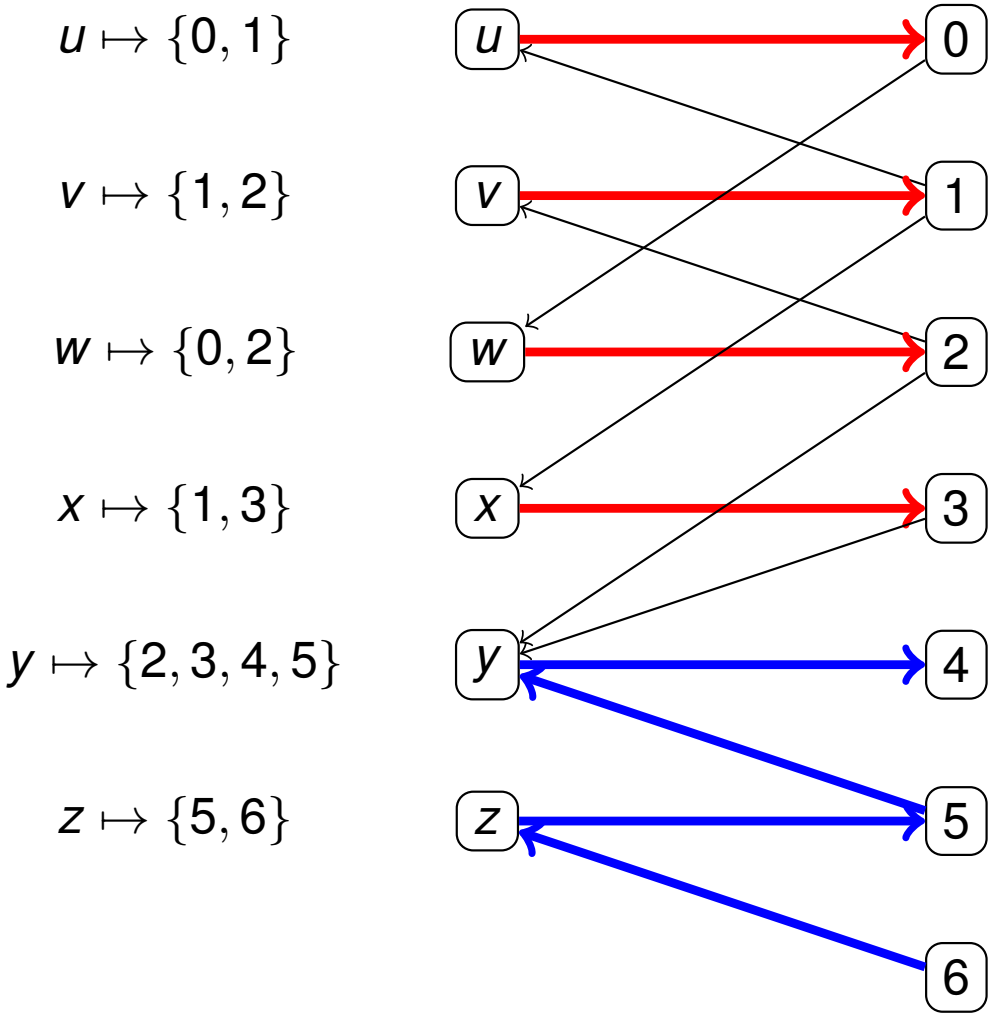
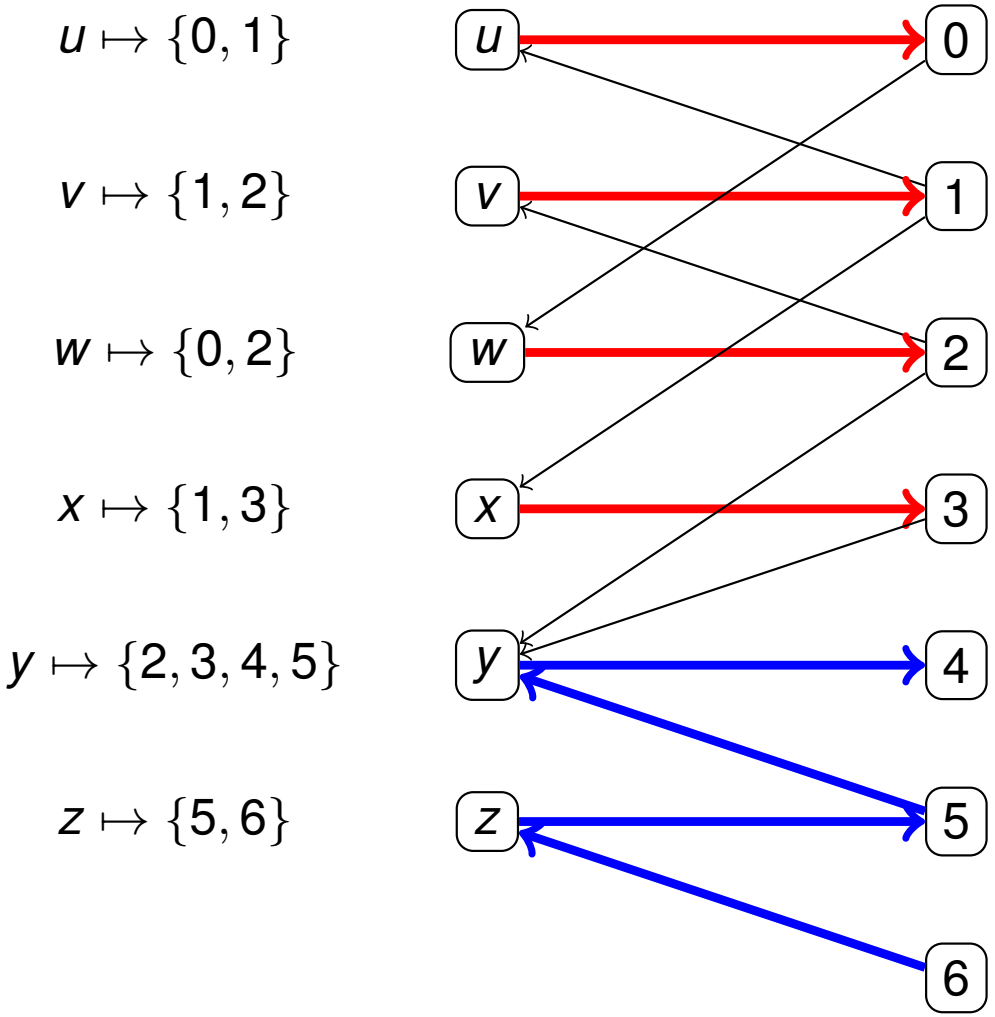


Efficient DC propagator (Régin, 1994) (Costa, 1994):

Mark all arcs in all strongly connected components (SCCs): the variables of an SCC take all the values of that SCC.



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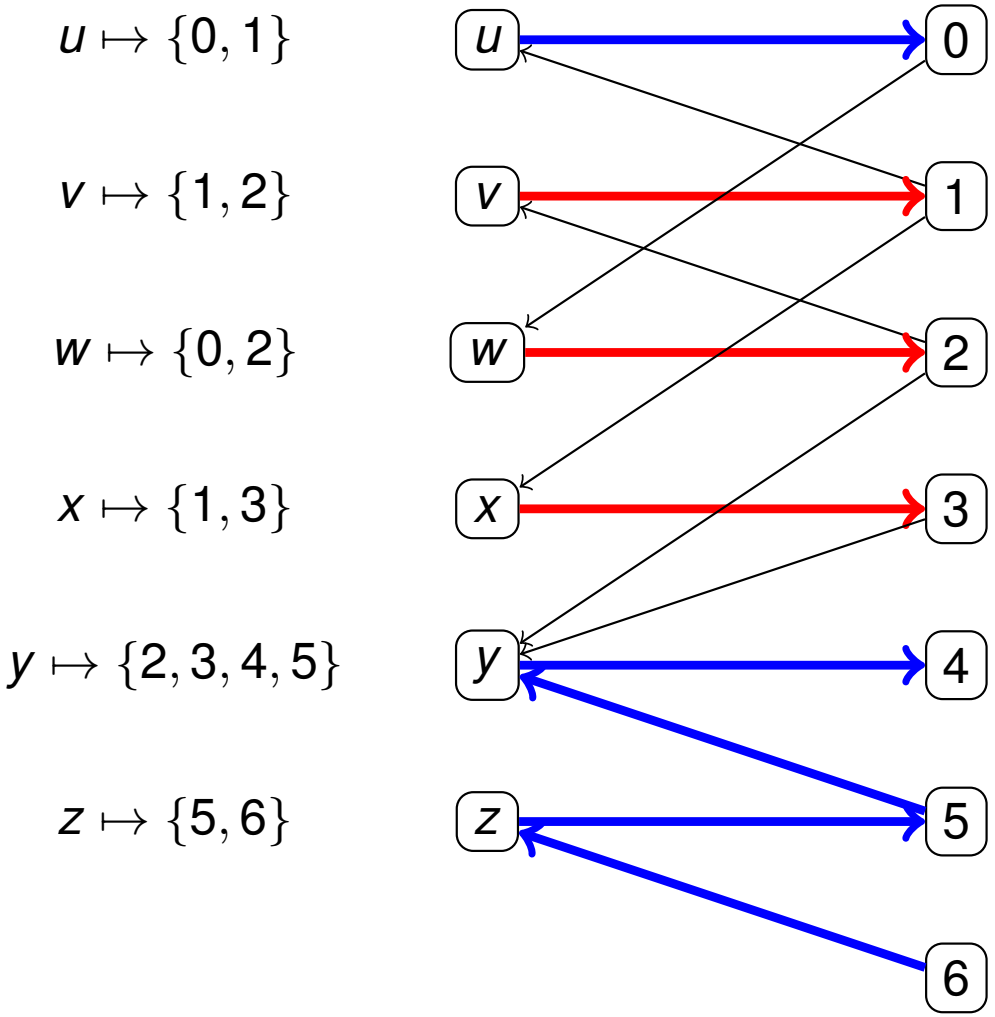

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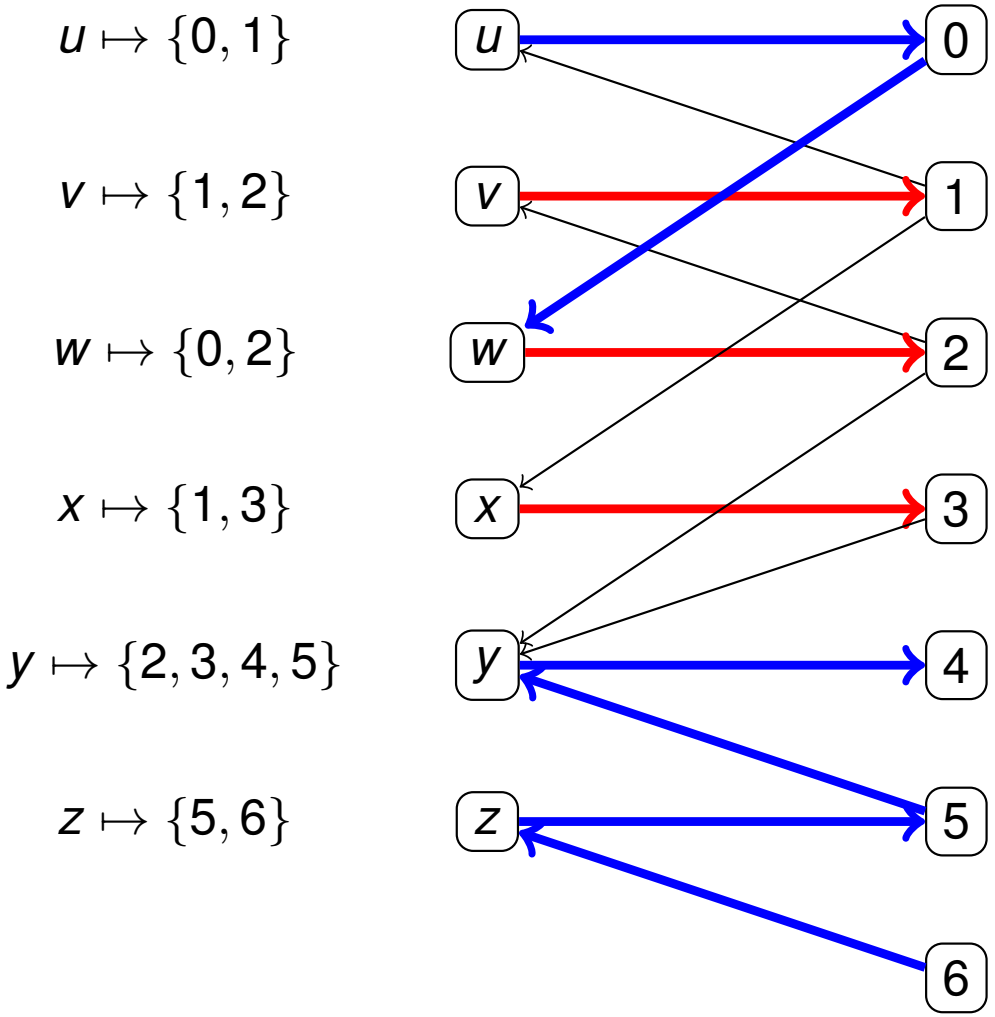

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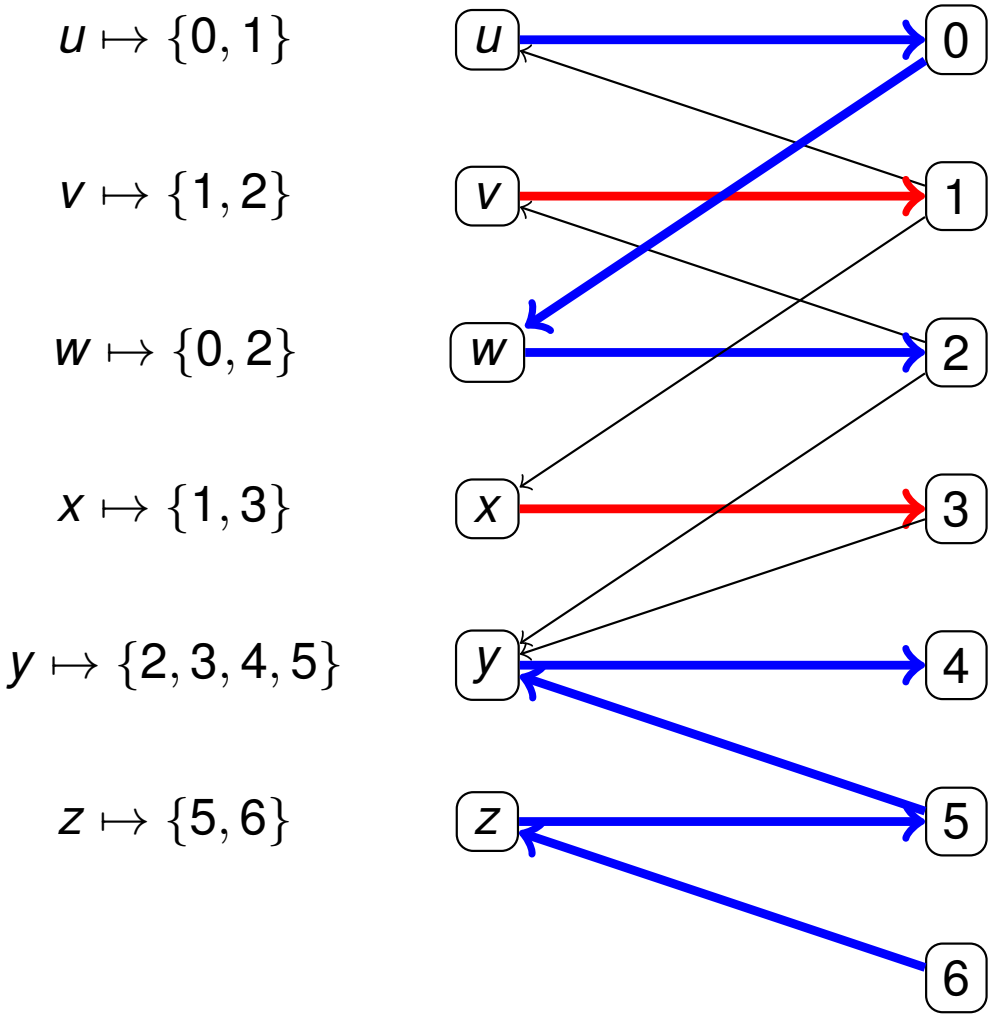

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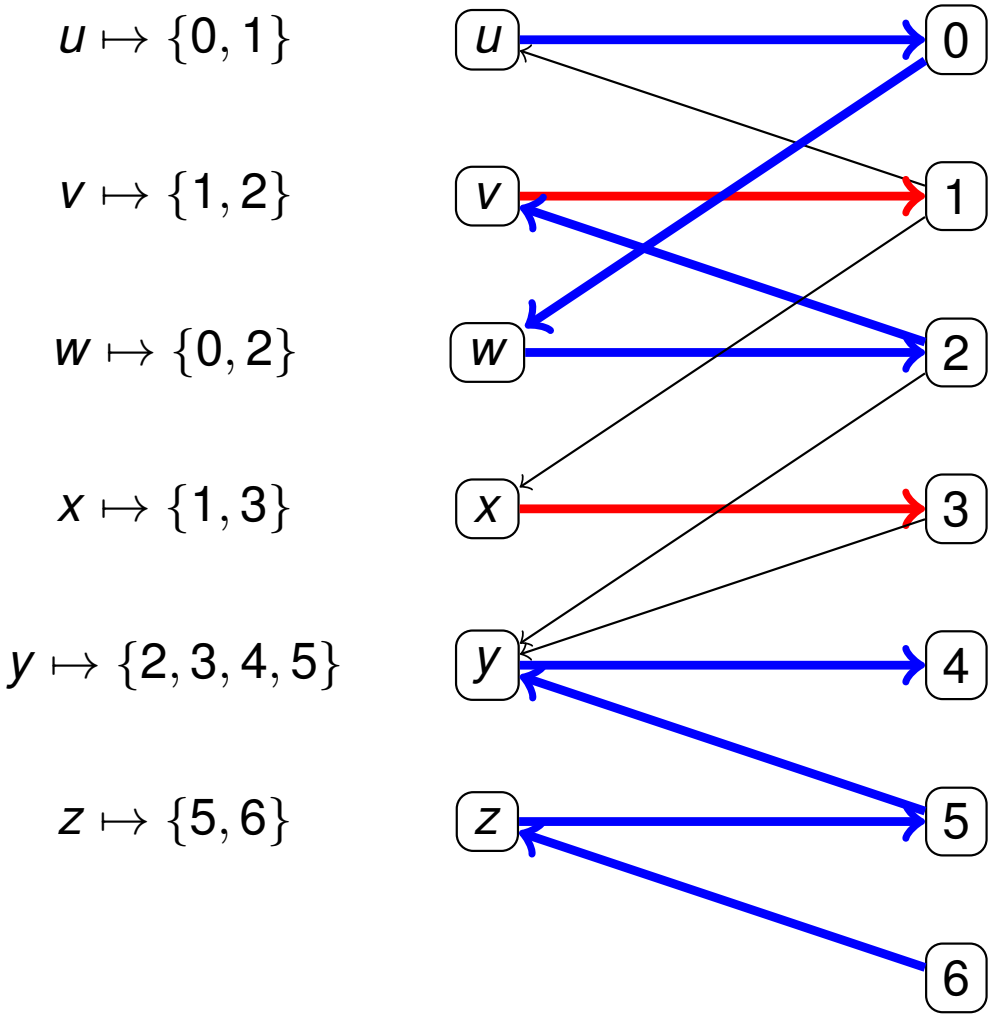

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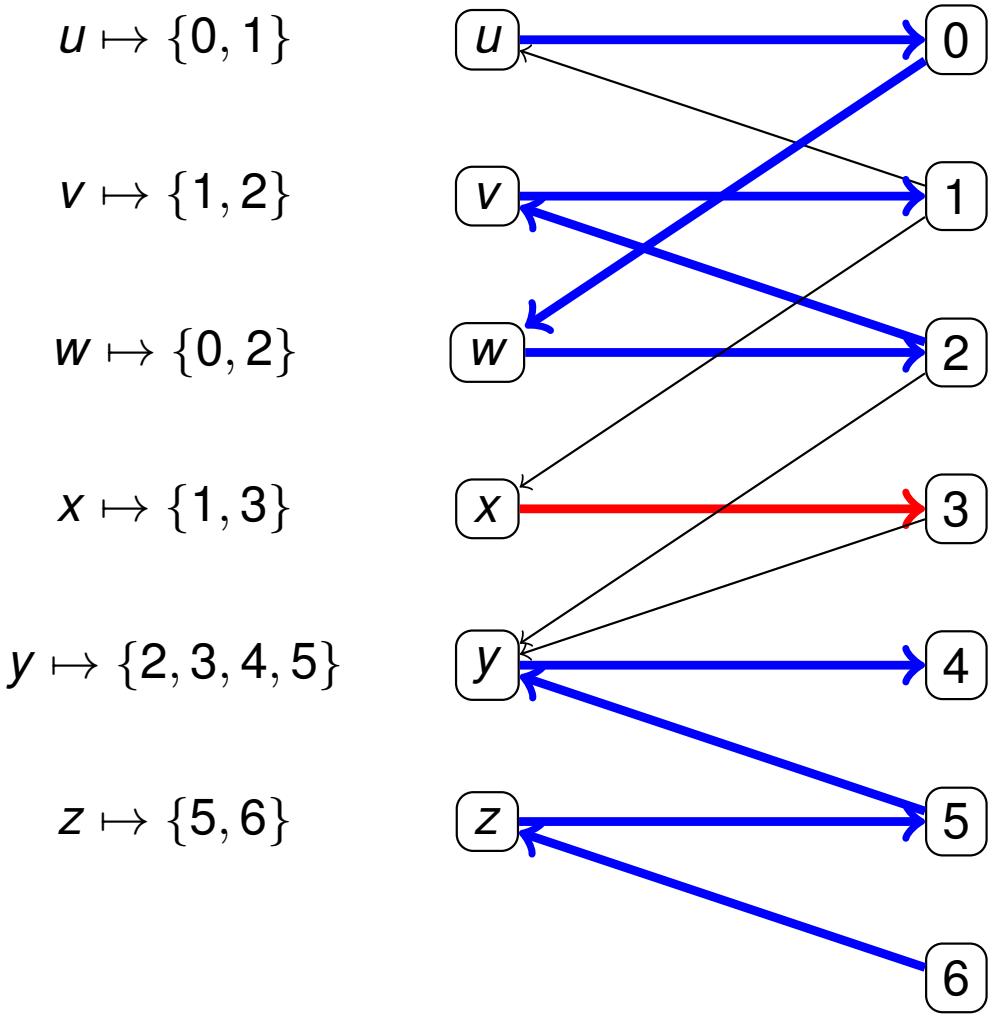

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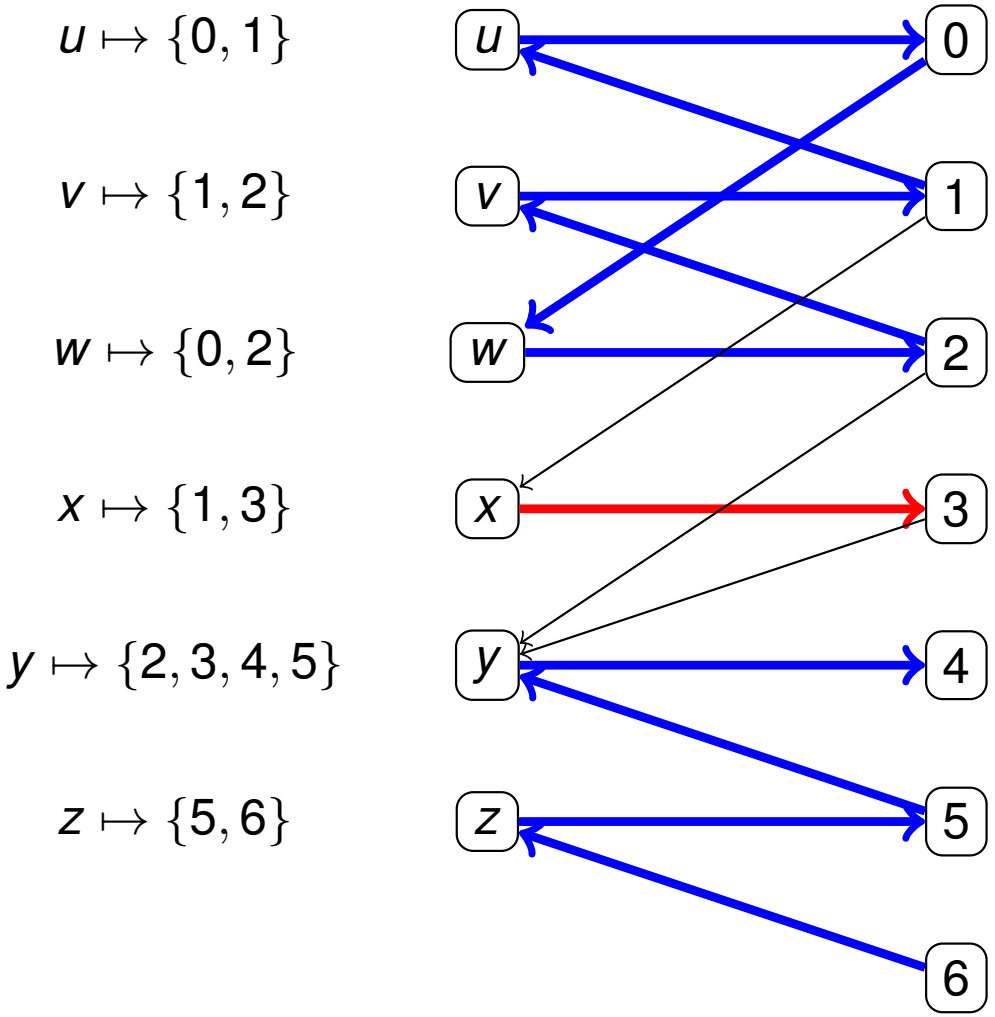
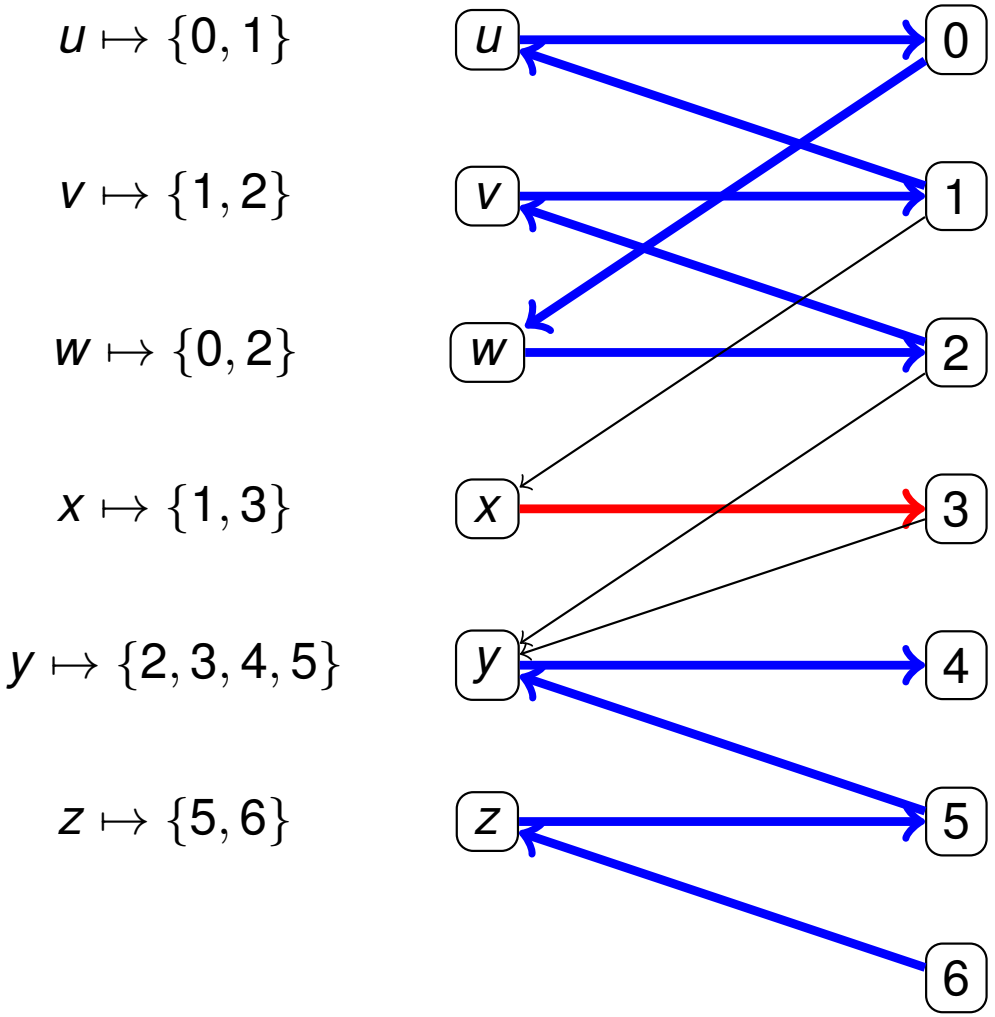


Efficient DC propagator (Régin, 1994) (Costa, 1994):

Every arc that is neither in the chosen maximum matching nor marked is in *no* maximum matching: prune accordingly.



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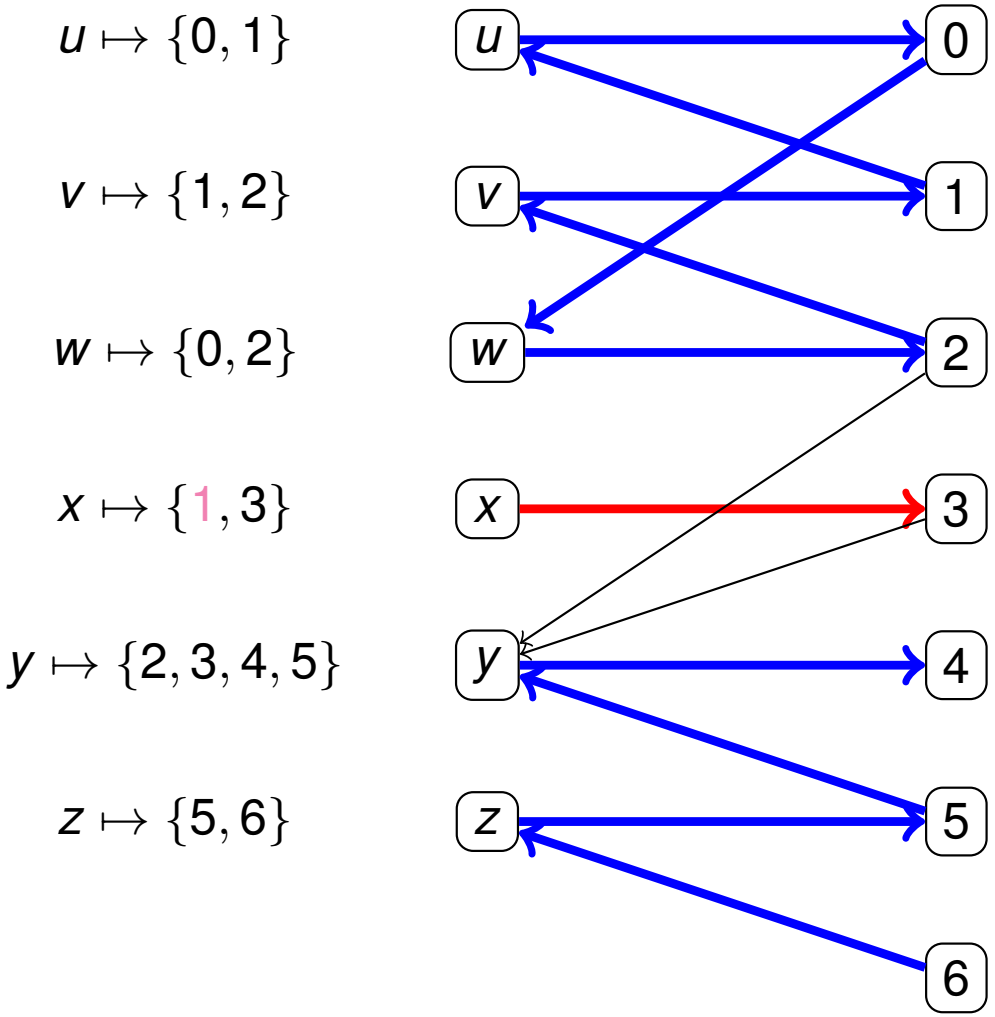

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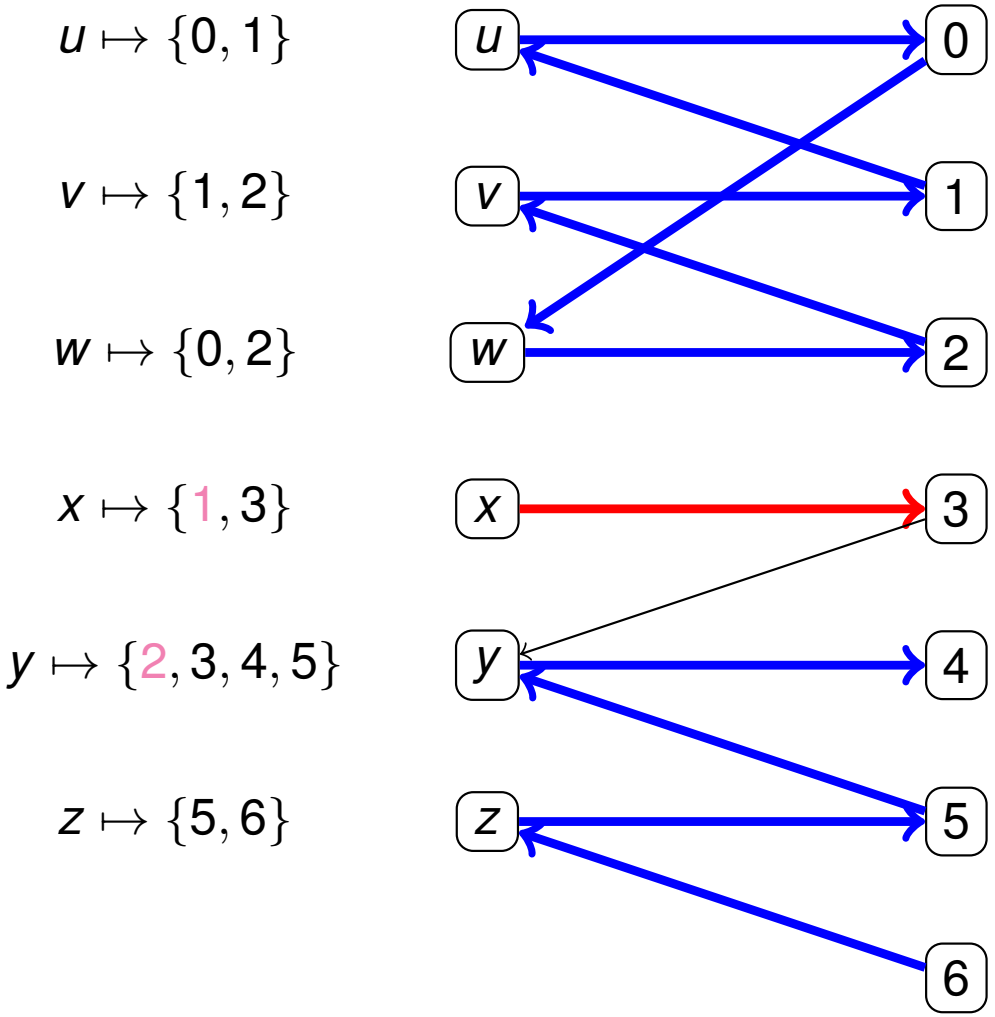

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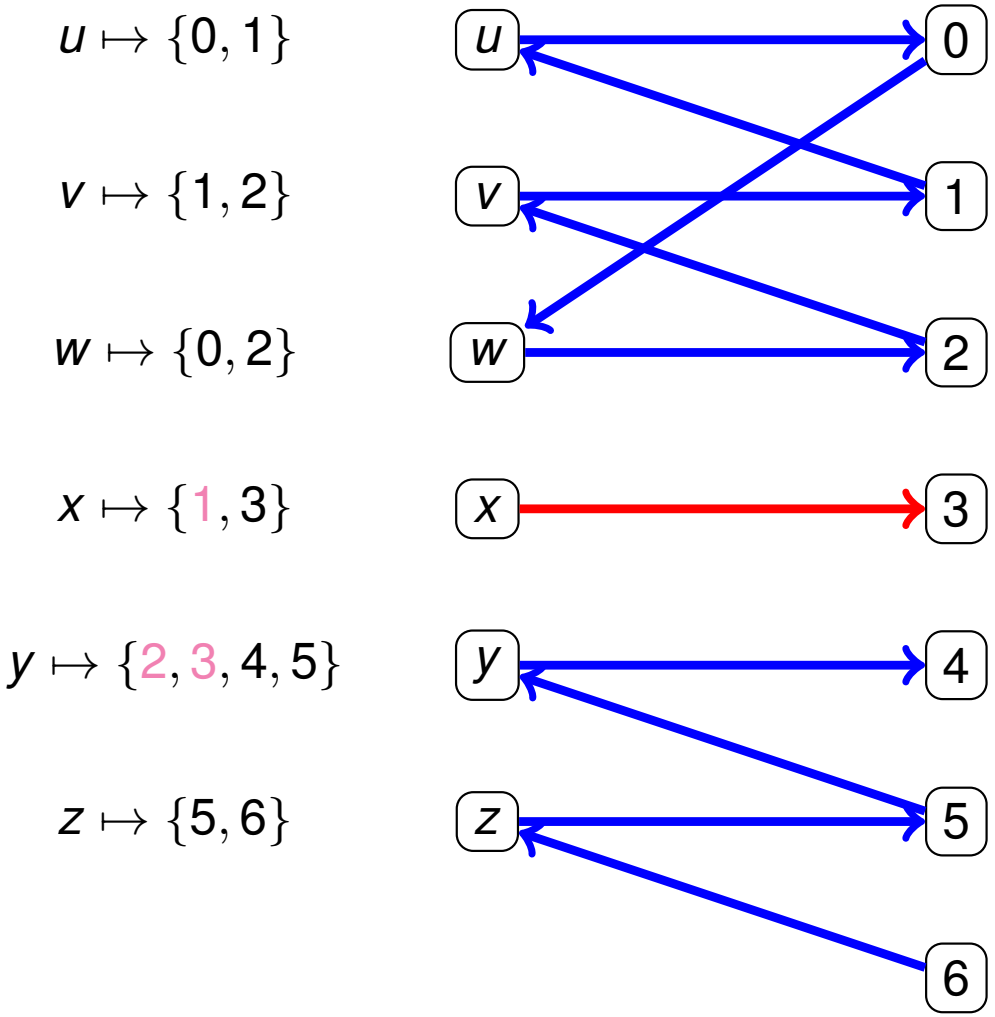
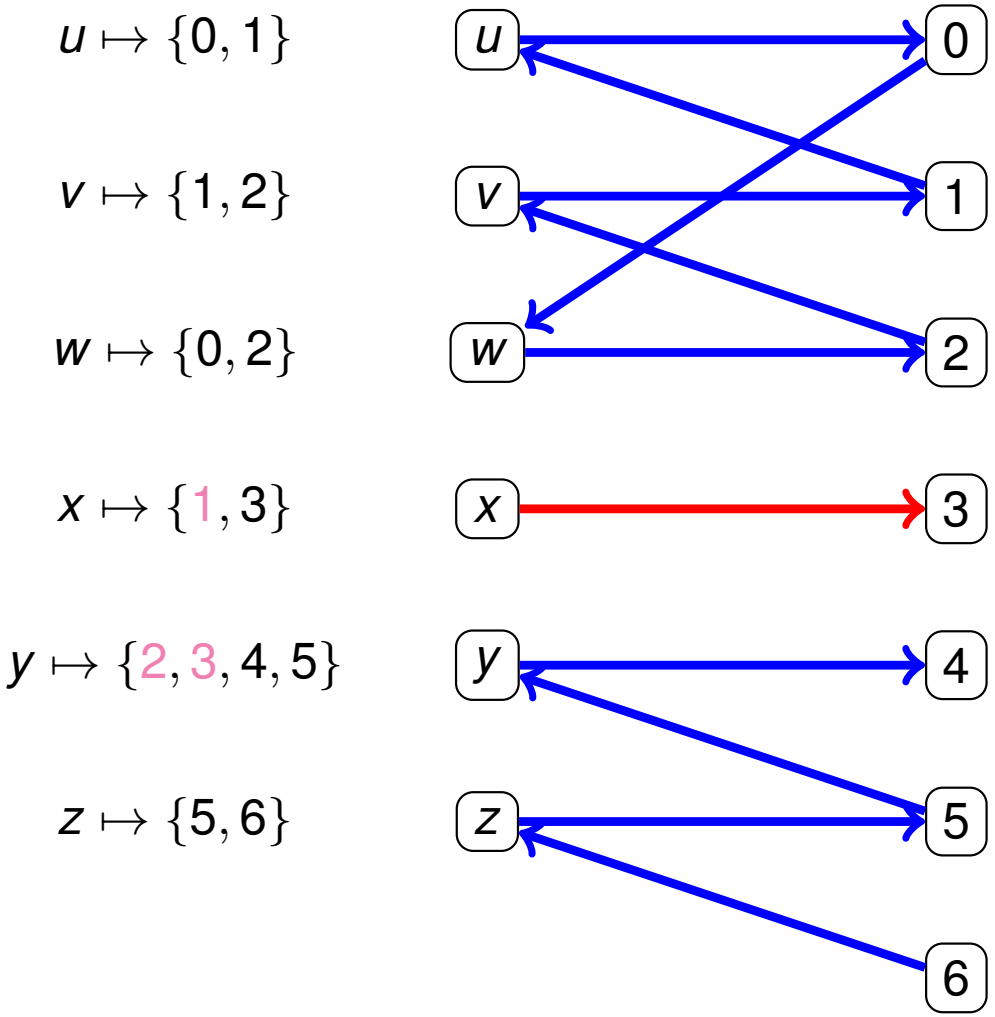


Efficient DC propagator (Régin, 1994) (Costa, 1994):

Every arc that is in the chosen maximum matching but not marked is in *every* maximum matching: fixed variable.



Underlying Theorem from Matching Theory

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Theorem (Berge, 1970) (Petersen, 1891)

Edge e belongs to some maximum matching if and only if, for an arbitrarily chosen maximum matching M:

- e belongs to a path of an even number of edges that starts at some node that is not incident to an edge of M and that alternates between edges in M and edges not in M.

or

- e belongs to a cycle of an even number of edges that alternates between edges in M and edges not in M (that is, the arc corresponding to e belongs to an SCC).

Complexity and Incrementality

Complexity:

The described DC propagator takes

$$\mathcal{O}\left(m \cdot \sqrt{n}\right) \text{ time and } \mathcal{O}\left(m \cdot n\right) \text{ space}$$

for n decision variables and $m \geq n$ values in their domains.

Incrementality via stateful propagator:

Keep the variable-value graph between invocations.
When the propagator is re-invoked:

1. Remove marks on arcs.
2. Remove arcs that no longer correspond to the store.
3. If an arc of the old maximum matching was removed, then first compute a new maximum matching.
4. Mark and prune.

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Is BC Needed for DISTINCT?

Propagation to BC often suffices for DISTINCT.

Example

Propagation to BC suffices to infer unsatisfiability for DISTINCT($\{x, y, z\}$) from the store $\{x, y, z \mapsto \{4, 5\}\}$.

Efficient BC propagators:

There are BC propagators that take $\mathcal{O}(n \cdot \lg n)$ time:

- Puget @ AAAI 1998
- Mehlhorn and Thiel @ CP 2000
- López-Ortiz, Quimper, Tromp, van Beek @ IJCAI 2003

The latter two run in $\mathcal{O}(n)$ time if sorting can be avoided, say when there are as many values as decision variables.

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