Topic 7: Solving Technologies
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Course 1DL442:
Combinatorial Optimisation and Constraint Programming,
whose part 1 is Course 1DL451:
Modelling for Combinatorial Optimisation

Outline

1. The MiniZinc Toolchain
2. Comparison Criteria
3. SAT
4. SMT & OMT
5. IP & MIP
6. CP
7. LS & CBLS
8. Hybrid Technologies
9. Case Study
10. Choosing a Technology and Backend

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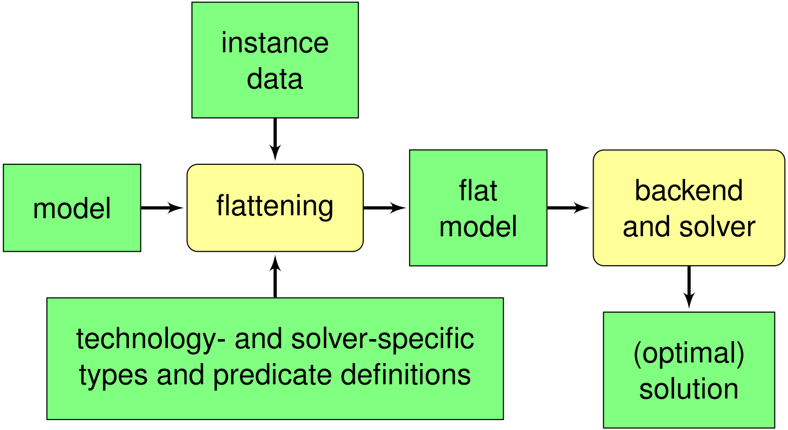
MiniZinc: Model Once, Solve Everywhere!



From a single language, one has access transparently to a wide range of solving technologies from which to choose.

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Objectives

An overview of some solving technologies:

- to understand their advantages and limitations;
- to help you choose a technology for a particular model;
- to help you adapt a model to a particular technology.

Examples (Solving technologies)

With general-purpose solvers, taking model and data as input:

- Boolean satisfiability (SAT)
- SAT (resp. optimisation) modulo theories (SMT and OMT)
- (Mixed) integer linear programming (IP and MIP)
- Constraint programming (CP) ☞ part 2 of 1DL442
- ...
- Hybrid technologies (LCG $=$ CP $+$ SAT, ...)

Methodologies, *usually without* modelling and solvers:

- Dynamic programming (DP)
- Greedy algorithms
- Approximation algorithms
- Local search (LS)
- ...

How to Compare Solving Technologies?

Modelling Language:

- What types of decision variables are available?
- Which constraint predicates are available?
- Can there be an objective function?

Guarantees:

- Are its solvers exact, given enough time:
  will they find all solutions, prove optimality, and prove unsatisfiability?
- If not, is there an approximation ratio for the solution quality?

Features:

- Can the modeller guide the solving? If yes, then how?
- In which application areas has the technology been successfully used?
- How do solvers work?

How Do Solvers Work? (Hooker, 2012)

Definition (Solving = Search + Inference + Relaxation)

- Search: Explore the space of candidate solutions.
- Inference: Reduce the space of candidate solutions.
- Relaxation: Exploit solutions to easier problems.

Definition (Systematic Search)

Progressively build a solution, and backtrack if necessary.
Use inference and relaxation to reduce the search effort.
It is used in most SAT, SMT, OMT, CP, LCG, and MIP solvers.

Definition (Local Search)

Start from a candidate solution and iteratively modify it a bit.
It is the basic idea behind LS and genetic algorithms (GA) technologies.

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Boolean Satisfiability Solving (SAT)

Modelling Language:

- Only Boolean decision variables.
- A conjunction (`/\`) of clauses. A clause is a disjunction (`\/`) of literals. A literal is a Boolean decision variable or its negation (`not`).
- Only for satisfaction problems; else: iterate over candidate obj. values.

Example (in MiniZinc syntax)

- Decision variables: `var bool: w, x, y, z;`
- Clauses:

```
constraint (not w \/ not y) /\ (not x \/ y)
        /\ (not w \/ x \/ not z)
        /\ (x \/ y \/ z) /\ (w \/ not z);
```

- A solution: `w=false`, `x=true`, `y=true`, `z=false`

The SAT Problem

Given a clause set, find a valuation, that is Boolean values for all the decision variables, so that all the clauses are satisfied.

- The decision version of this problem is NP-complete.
- Any combinatorial problem can be encoded into SAT.
  Careful: “encoded into” is not “reduced from”, but “reduced to”.
  There are recipes for clausifying non-Boolean constraints.
- There has been intensive research since the 1960s.
- We focus here on systematic search, namely DPLL [Davis-Putnam-Logemann-Loveland, 1962].

DPLL

Tree Search, upon starting from the empty valuation:

1. Perform inference (see below).
2. If some clause is unsatisfied, then backtrack.
3. If all decision variables have a value, then we have a solution.
4. Select an unvalued decision variable b and make two branches: one with $b = \texttt{true}$, and the other one with $b = \texttt{false}$.
5. Recursively explore each of the two branches.

Inference:

- Unit propagation: If all the literals in a clause evaluate to `false`, except one whose decision variable has no value yet, then that literal is made to evaluate to `true` so that the clause becomes satisfied.

Strategies and Improvements over DPLL

Search Strategies:

- On which decision variable to branch next?
- Which branch to explore next?
- Which search (depth-first, breadth-first, . . .) to use?

Improvements:

- Backjumping
- Clause learning
- Restarts
- A lot of implementation details
- . . .

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SAT Solving

- Guarantee: exact, given enough time.
- Mainly black-box: there are limited ways to guide the solving.
- It can scale to millions of decision variables and clauses.
- Encoding a problem can yield a huge SAT model.
- For model debugging purposes, solvers can extract an unsatisfiable core, that is a subset of the clauses that make the model unsatisfiable.
- It is mainly applied in hardware verification and software verification.

SAT @ MiniZinc and Uppsala University

- The MiniZinc toolchain was extended with the PicatSAT backend, which uses the SAT solver Plingeling.
- Several research groups at Uppsala University *use* SAT solvers, such as:
  - Algorithmic Program Verification
  - Embedded Systems
  - Programming Languages
  - Theory for Concurrent Systems
- My Algorithms & Datastructures 3 (1DL481) course explains SAT solving and has a homework where a model is generated and fed to a SAT solver.

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SAT Modulo Theories (SMT) and OMT

Modelling Language:

- Language of SAT: Boolean decision variables and clauses.
- Several theories extend the language, such as bit vectors, uninterpreted functions, or linear integer arithmetic.
- SMT is only for satisfaction problems.
- OMT (optimisation modulo theories) extends SMT.

Definition

A theory

- defines types for decision variables and defines constraint predicates;
- is associated with a sub-solver for any conjunction of its predicates.

Different SMT or OMT solvers may have different theories.

Example (Linear integer arithmetic; in MiniZinc syntax)

- Decision variables: `var int: x; var int: y;`
- Constraints:

```
constraint x >= 0; constraint y <= 0;
constraint x = y + 1 \/ x = 2 * y;
constraint x = 2 \/ y = -2 \/ x = y;
```

- Unique solution: `x = 0`, `y = 0`
- Decomposition:
  - Theory constraints, using reified constraints:

```
a <-> x >= 0;       b <-> y <= 0;
c <-> x = y + 1;    d <-> x = 2 * y;
e <-> x = 2;        f <-> y = -2;      g <-> x = y;
```

  - Boolean skeleton:

```
a /\ b /\ (c \/ d) /\ (e \/ f \/ g);
```

SMT Solving: DPLL(T)

Basic Idea:

- Separate the theory constraints and Boolean skeleton: each decision variable in the Boolean skeleton denotes whether a constraint holds or not.
- Use DPLL to solve the Boolean skeleton.
- If a constraint must hold as per DPLL, then submit it to the relevant theory solver.
- A theory solver operates on a constraint conjunction:
  - It checks whether the conjunction is satisfiable.
  - It tries to infer that other constraints must (respectively cannot) hold and it sets the corresponding Boolean variables to `true` (respectively `false`).

Strategies and Improvements

Search Strategies:

- On which decision variable to branch next?
- Which branch to explore next?
- Which strategy (depth-first, breadth-first, . . .) to use?

Improvements to SAT Solving:

- See slide 14.

Improvements to the Theory Solvers:

- More efficient inference algorithms: incrementality.
- Richer theories.
- . . .

SMT and OMT Solving

- Guarantee: exact, given enough time.
- Mainly black-box: there are limited ways to guide the solving.
- They are based on the very efficient SAT technology.
- They are mainly applied in hardware verification and software verification.

SMT and OMT @ MiniZinc and Uppsala University

- The MiniZinc toolchain was extended with:
  - fzn2smt: generates SMTlib models that can be fed to any SMT solver, such as CVC4, Yices 2, Z3, . . .
  - emzn2fzn + fzn2omt: generates models that can be fed to any OMT solver, such as OptiMathSAT, Z3, . . .
- Several research groups at Uppsala University *use* SMT solvers, such as:
  - Algorithmic Program Verification
  - Programming Languages
  - Theory for Concurrent Systems
- My Algorithms & Datastructures 3 (1DL481) course explains SMT solving and has a homework where a model is written and fed to an SMT solver.

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Integer (Linear) Programming (IP = ILP)

Modelling Language:

- Only integer decision variables.
- A set of linear equality and inequality constraints (note: no disequality $\neq$).
- Only for optimisation problems: linear objective function (else: a value).

Example (in MiniZinc syntax)

- Decision variables: `var int: p; var int: q;`
- Constraints:

```
constraint p >= 0; constraint q >= 0;
constraint     p + 2 * q <= 5;
constraint 3 * p + 2 * q <= 9;
```

- Objective: `solve maximize 3 * p + 4 * q;`
- Unique (in this case) optimal solution: `p = 1`, `q = 2`

Mathematical Programming

- 0-1 linear programming:
  linear (in)equalities over decision variables over the domain $\{0, 1\}$.
- Linear programming (LP):
  linear (in)equalities over floating-point decision variables.
- Mixed integer (linear) programming (MIP):
  linear (in)equalities over floating-point and integer decision variables.
- Quadratic programming (QP):
  quadratic objective function.
- …

There has been intensive research since the 1940s.

IP Solving

Basic Idea = Relaxation:

- Polytime algorithms (such as the interior-point method and the ellipsoid method) and exponential-time but practical algorithms (such as the simplex method) exist for solving LP models very efficiently.
- Use them for IP by occasionally relaxing an IP model via dropping its integrality requirement on the decision variables.

Implementations:

- Branch and bound = relaxation + search.
- Cutting-plane algorithms = relaxation + inference.
- Branch and cut = relaxation + search + inference.

Branch and Bound

Tree Search, upon initialising the incumbent to $\pm\infty$:

1. Relax the IP model into an LP model, and solve it.
2. If the LP model is unsatisfiable, then backtrack.
3. If all the decision variables have an integer value in the optimal LP solution, then backtrack upon updating, if need be, the incumbent to the objective value of that IP solution.
4. If the objective value of the optimal LP solution is no better than the incumbent, then backtrack.
5. Otherwise, some decision variable v has a non-integer value ρ. Make two branches: one with $v \leq \lfloor \rho \rfloor$, and the other one with $v \geq \lceil \rho \rceil$.
6. Recursively explore each of the two branches.

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Strategies and Improvements

Search Strategies:

- On which decision variable to branch next?
- Which branch to explore next?
- Which search (depth-first, breadth-first, . . .) to use?

Improvements:

- Cutting planes: Add implied linear constraints that improve the objective value of the LP relaxation.
- Decomposition: Split into a master problem and a subproblem, such as by the Benders decomposition.
- Solving the LP relaxation:
  - Primal-dual methods.
  - Efficient algorithms for special cases, such as flows.
  - Incremental solving.
- . . .

IP Solving

- Guarantee: exact, given enough time.
- Mainly black-box: limited ways to guide the solving.
- It scales well.
- Any combinatorial problem can be encoded into IP.
  There are recipes for linearising non-linear constraints.
- Advantages:
  - Provides both a lower bound and an upper bound on the objective value of optimal solutions, if stopped early.
  - Naturally extends to MIP solving.
  - ...
- Central method of operations research (OR),
  applied in production planning, vehicle routing, ...

MIP @ MiniZinc and Uppsala University

- The MiniZinc toolchain comes bundled with a backend that can be hooked to the following MIP solvers:
  - Cbc (open-source, bundled);
  - CPLEX Optimizer (commercial: requires a license);
  - FICO Xpress Solver (commercial: requires a license);
  - Gurobi Optimizer (commercial: requires a license);
  - HiGHS (open-source, bundled).
- The Optimisation research group at Uppsala University *uses* MIP solvers for 4G / 5G network planning and optimisation, etc.
- My Algorithms & Datastructures 3 (1DL481) course explains MIP solving and has a homework where a model is designed and fed to a MIP solver.

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Constraint Programming (CP)

Modelling Language = full MiniZinc:

- Any combination of Boolean, integer, enumeration, set decision variables.
- Constraints based on a large vocabulary of predicates.
- For satisfaction problems and optimisation problems.

Many solvers:

- There will be no standard for what is to be supported:
  different CP solvers may have different sets of types for decision variables and different constraint predicates (under different names).
- Some solvers support even higher-level types for decision variables, such as graphs and strings, and associated predicates.

Domains

Definition

The domain of a decision variable v, denoted here by $\text{dom}(v)$, is the set of values that v can still take during search:

- The domains of the decision variables are reduced by search and by inference (see the next two slides).
- A decision variable is said to be fixed if its domain is a singleton.
- Unsatisfiability occurs if the domain of a decision variable goes empty.

Note the difference between:

- a domain as a technology-independent declarative entity when modelling;
- a domain as a CP-technology procedural data structure when solving.

CP Solving

Tree Search, upon initialising each domain as in the model:

Satisfaction problem:

1. Perform inference (see the next slide).
2. If the domain of some decision variable is empty, then backtrack.
3. If all decision variables are fixed, then we have a solution.
4. Select a non-fixed decision variable v,
   partition its domain into two parts π_1 and π_2, and make two branches:
   one with $v \in \pi_1$, and the other one with $v \in \pi_2$.
5. Recursively explore each of the two branches.

Optimisation problem: when a feasible solution is found at step 3, first add the constraint that the next solution must be better and then backtrack.

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CP Inference

Definition

A propagator for a predicate γ deletes from the domains of the variables of a γ-constraint ~~the~~ values that cannot be in a solution to that constraint.
The propagator of a constraint is active as long as the Cartesian product of the domains of its variables is not known to contain only solutions to the constraint.

Examples

- For `x < y`: when dom(`x`) = `1..4` and dom(`y`) = `-1..3`, delete `3..4` from dom(`x`) and `-1..1` from dom(`y`). The propagator remains active.
- For `all_different([x,y,z])`: when dom(`x`) = `{1,3}` = dom(`y`) and dom(`z`) = `1..4`, delete `1` and `3` from dom(`z`) so that it becomes the non-range `{2,4}`. The propagator becomes inactive after dom(`x`) loses `1`.

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Strategies and Improvements

Search Strategies:

- On which decision variable to branch next?
- How to partition the domain of the chosen decision variable?
- Which search (depth-first, breadth-first, . . .) to use?

Improvements:

- Propagators, including for the predicates in Topic 3: Constraint Predicates. Not all impossible domain values need to be deleted: there is a compromise between algorithm complexity and achieved inference.
- Partition the chosen domain into at least two parts.
- Domain representations.
- Order in which propagators are executed.
- . . .

CP Solving

- Guarantee: exact, given enough time.
- White-box: one can design one's own propagators and search strategies, and choose among predefined ones.
- The higher-level modelling languages enable (for details, see Topic 8: Inference & Search in CP & LCG):
  - inference at a higher level;
  - search strategies stated in terms of problem concepts.

  They inspired the MiniZinc modelling language.
- Successful application areas:
  - Configuration
  - Personnel rostering
  - Scheduling and timetabling
  - Vehicle routing
  - …

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The MiniZinc Toolchain

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CP @ MiniZinc and Uppsala University

- The MiniZinc toolchain was extended with backends for many CP solvers, such as Gecode (bundled), Choco, JaCoP, Mistral, SICStus Prolog, . . .
- The Optimisation research group at Uppsala University contributes to the *design* of CP solvers and *uses* them, say for air traffic management, the configuration of wireless sensor networks, robot task sequencing, etc.
- Part 2 of my Combinatorial Optimisation and Constraint Programming (1DL442) course covers CP in depth.

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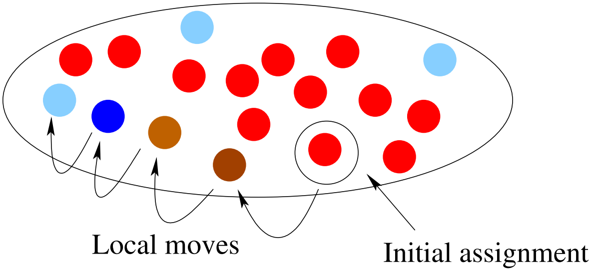
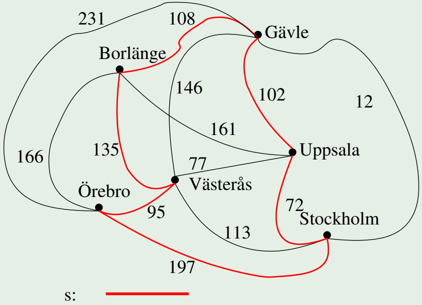
Local Search (LS)

- Each decision variable is fixed all the time.
- Search proceeds by moves: each move modifies the values of a few decision variables in the current assignment, and is selected upon probing the cost impacts of several candidate moves, called the neighbourhood.
- Stop when either a good enough assignment was found, or when an allocated resource was exhausted, such as time spent or iterations made.



Example (Travelling Salesperson, TSP)

- **Problem:** Given a set of cities with connecting roads, find a Hamiltonian circuit (tour) visiting each city exactly once, with minimal travel distance.
- **Representation:** We see the cities as vertices V and the roads as edges E of a (not necessarily complete) undirected graph $G = (V, E)$.
- **Example:**



We now design a local-search heuristic for TSP, without a modelling language.

Example (Travelling Salesperson: Algorithmic Choices)

We must define:

1. The **initial assignment**:

2. The **neighbourhood** of candidate moves:

3. The **cost** of a current assignment s (not necessarily its objective value):

4. The **neighbour selector**:

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The MiniZinc Toolchain

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Example (Travelling Salesperson: Algorithmic Choices)

We must define:

1. The **initial assignment**:
   An edge set $s \subseteq E$ that forms a tour
2. The **neighbourhood** of candidate moves:
3. The **cost** of a current assignment s (not necessarily its objective value):
4. The **neighbour selector**:

Example (Travelling Salesperson: Algorithmic Choices)

We must define:

1. The **initial assignment**:
   An edge set $s \subseteq E$ that forms a tour: NP-hard!

2. The **neighbourhood** of candidate moves:

3. The **cost** of a current assignment s (not necessarily its objective value):

4. The **neighbour selector**:

Example (Travelling Salesperson: Algorithmic Choices)

We must define:

1. The **initial assignment**:
   An edge set $s \subseteq E$ that forms a tour: NP-hard!
   Complete E by adding infinite-distance edges:
   any permutation of V yields a tour as an initial assignment.
2. The **neighbourhood** of candidate moves:
3. The **cost** of a current assignment s (not necessarily its objective value):
4. The **neighbour selector**:

Example (Travelling Salesperson: Algorithmic Choices)

We must define:

1. The **initial assignment**:
   An edge set $s \subseteq E$ that forms a tour: NP-hard!
   Complete E by adding infinite-distance edges:
   any permutation of V yields a tour as an initial assignment.
2. The **neighbourhood** of candidate moves:
   Replace two edges on the tour s by two other edges so that s is still a tour.
3. The **cost** of a current assignment s (not necessarily its objective value):
4. The **neighbour selector**:

Example (Travelling Salesperson: Algorithmic Choices)

We must define:

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   An edge set $s \subseteq E$ that forms a tour: NP-hard!
   Complete E by adding infinite-distance edges:
   any permutation of V yields a tour as an initial assignment.
2. The **neighbourhood** of candidate moves:
   Replace two edges on the tour s by two other edges so that s is still a tour.
3. The **cost** of a current assignment s (not necessarily its objective value):
   The sum of all distances on the tour:
   $f(s) = \sum_{(a,b)\in s} \text{Distance}(a, b)$, as no constraint violation can happen.
4. The **neighbour selector**:

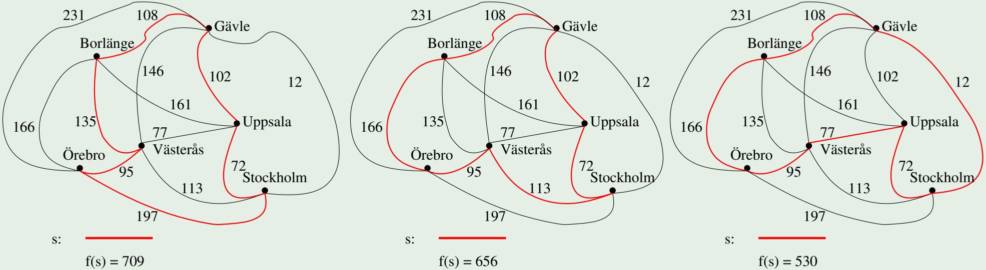
Example (Travelling Salesperson: Algorithmic Choices)

We must define:

1. The **initial assignment**:
   An edge set $s \subseteq E$ that forms a tour: NP-hard!
   Complete E by adding infinite-distance edges:
   any permutation of V yields a tour as an initial assignment.
2. The **neighbourhood** of candidate moves:
   Replace two edges on the tour s by two other edges so that s is still a tour.
3. The **cost** of a current assignment s (not necessarily its objective value):
   The sum of all distances on the tour:
   $f(s) = \sum_{(a,b) \in s} \text{Distance}(a, b)$, as no constraint violation can happen.
4. The **neighbour selector**:
   Select a random best neighbour.

Example (Travelling Salesperson: Sample Run)

Three consecutive improving current assignments:



This section is so far based on material by Magnus Rattfeldt.

Heuristics drive the search to (good enough) solutions:

- Which decision variables are modified in a move?
- Which new values do they get in the move?

Metaheuristics drive the search to global optima of the cost:

- Avoid cycles of moves and escape local optima of the cost.
- Explore many parts of the search space.
- Focus on promising parts of the search space.

Examples (Metaheuristics)

- Tabu search (1986):
  forbid recent moves from being done again.
- Simulated annealing (1983):
  perform random moves and accept degrading ones with a probability that decreases over time.
- Genetic algorithms (1975):
  use a pool of current assignments and cross them.

Systematic Search (as in SAT, SMT, OMT, MIP, and CP):

+ Will find an (optimal) solution, if one exists.
+ Will give a proof of unsatisfiability, otherwise.
– May take a long time to complete.
– Sometimes does not scale well to large instances.
– May need a lot of tweaking: search strategies, . . .

Local Search: (Hoos and Stützle, 2004)

+ May find an (optimal) solution, if one exists.
– Can rarely give a proof of unsatisfiability, otherwise.
– Can rarely guarantee that a found solution is optimal.
+ Often scales well to large instances.
– May need a lot of tweaking: (meta)heuristics, . . .

Local search trades solution quality for speed!

Constraint-Based Local Search (CBLS)

- MiniZinc-style modelling language:
  - Any combination of Boolean, integer, enumeration set decision variables.
  - Constraints based on a large vocabulary of predicates.
  - Three sorts of constraints: see the next three slides.
  - For satisfaction problems and optimisation problems.
- Fairly recent: around the year 2000.
- Guarantee: inexact on most instances (that is: there is no promise to find all solutions, to prove optimality, or to prove unsatisfiability), without an approximation ratio.
- White-box: one must design a search algorithm, which probes the cost impacts for guidance.
- More scalable than systematic technologies.

Definition

Each constraint predicate has a violation function: the violation of a constraint is zero if it is satisfied in the current assignment, else a positive value proportional to its dissatisfaction.

Example

For `a <= b`, let α and β be the current values of `a` and `b`: define the violation to be $\alpha - \beta$ if $\alpha \not\leq \beta$, and 0 otherwise.

Definition

A constraint with violation is explicit in a CBLS model and soft: it can be violated during search but ought to be satisfied in a solution.

The constraint violations are queried during search.

Definition

A one-way constraint is explicit in a CBLS model and hard:
it is kept satisfied during search by keeping the value of a decision variable equal to a total function on its other decision variables.

Example

For `p = a * b`, whenever either the value α of `a`, or the value β of `b`, or both are modified by a move, the value of `p` is automatically modified by the solver so as to remain equal to $\alpha \cdot \beta$.

CBLS solvers offer a special syntax for one-way constraints,
such as `p <== a * b` in OscaR.cbls,
but MiniZinc does not make such a syntactic distinction.

Definition

An implicit constraint is not in a CBLS model but hard: it is kept satisfied during search by choosing a feasible initial assignment and only making satisfaction-preserving moves, by the use of a constraint-specific neighbourhood.

Example

For `all_different(...)` when the number of decision variables is equal to the number of their domain values: the initial assignment has distinct values for all decision variables, and the neighbourhood only has moves that swap the values of two decision variables.

When building a CBLS model, a MiniZinc backend must:

- Aptly assort the otherwise all explicit and all soft constraints.
- Add a suitable heuristic and meta-heuristic.

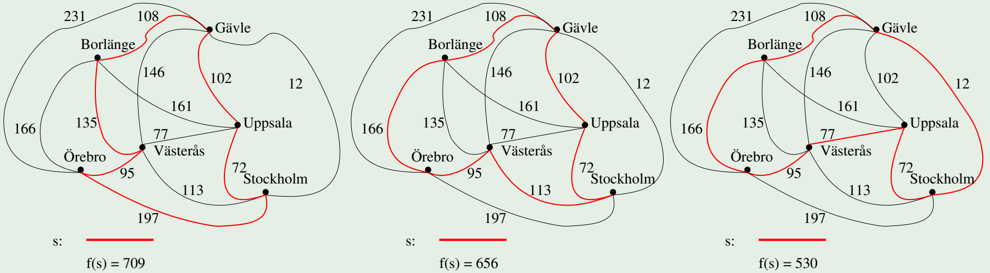
This is much more involved than just flattening and solving.

Example (Travelling Salesperson: Model and Solve)

Recall the model, from Topic 1: Introduction,
with a decision variable `Next[c]` for each city `c`:

```
solve minimize sum(c in Cities)(Distance[c,Next[c]]);
constraint circuit(Next); % ideally made implicit
```

Three consecutive current assignments, preserving the satisfaction of the `circuit(Next)` constraint and improving the objective value:



(CB)LS @ MiniZinc and Uppsala University

- The MiniZinc toolchain was extended with:
  - our fzn-oscar-cbls backend to the OscaR.cbls solver;
  - our Atlantis CBLS backend;
  - the Yuck CBLS backend.
- The Optimisation research group at Uppsala University contributes to the *design* of CBLS solvers.
- Several courses at Uppsala University discuss (CB)LS:
  - My Algorithms & Datastructures 3 (1DL481) course explains LS and has a homework where an LS program is written.
  - Artificial Intelligence (1DL340) discusses LS.
  - Machine Learning (1DT071) discusses LS.

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Outline

1. The MiniZinc Toolchain
2. Comparison Criteria
3. SAT
4. SMT & OMT
5. IP & MIP
6. CP
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8. **Hybrid Technologies**
9. Case Study
10. Choosing a Technology and Backend

Crossfertilisation

- Each technology has advantages and drawbacks.
- Good ideas from one technology can be applied to another technology.
- A hybrid technology combines several technologies.
- This can yield new advantages with fewer drawbacks.
- Some hybrid technologies are loosely coupled:
  separate solvers or sub-solvers cooperate.
- Other hybrid technologies are tightly coupled:
  a single solver handles the whole model.

Example (Loose hybrid technology)

Logic-based Benders decomposition: divide the problem into two parts:
a master problem, solved by IP, and a subproblem, solved by CP.

Tight Hybrid Technologies: Examples

Example (Lazy clause generation, LCG)

Use CP propagators to generate clauses in a SAT solver.

Example (Large-neighbourhood search, LNS on a COP)

Find a first solution by PC and then follow an LS procedure, where each move is performed by:

1. Restoring the domains for a subset of the decision variables.
2. Using a CP solver to find an (optimal) solution to the subproblem.

Example (Constrained integer programming, CIP)

Use CP propagators in an IP solver in order to generate linear inequalities for non-linear constraints.

Hybrids @ MiniZinc and Uppsala University

- The MiniZinc toolchain was extended with:
  - LCG backends: Chuffed (bundled) and Google CP-SAT;
  - a CIP backend: SCIP;
  - LNS backends: the solvers of the Gecode (bundled) and Google CP-SAT backends can perform LNS (prescribed via MiniZinc annotations).
- The Optimisation research group at Uppsala University contributes to the *design* of hybrid solvers and *uses* them (see slide 39).

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Example: Pigeonhole Problem

Example (Pigeonhole)

Place n pigeons into $n-1$ pigeonholes so that all pigeons are placed and no two pigeons are placed in the same pigeonhole.

This problem is trivially unsatisfiable, but is a popular benchmark for solvers.

We will use this problem to show:

- how solvers may use different definitions of the same constraint predicate;
- it is often important for efficiency to use pre-defined constraint predicates.

Pigeonhole: Models

Using all_different

```
int: n; % the number of pigeons
% Hole[p] = the hole of pigeon p:
array[1..n] of var 1..(n-1): Hole;
constraint all_different(Hole);
solve satisfy;
```

Using !=

```
constraint forall(i,j in 1..n where i < j)
  (Hole[i] != Hole[j]);
```

Constraint Predicate Definitions

Built-in `all_different` for probably all CP solvers

```
predicate all_different_int(array[int] of var int: X);

predicate int_ne(var int: x, var int: y);
```

Non-built-in `all_different` for SMT solvers

```
predicate all_different_int(array[int] of var int: X) =
  forall(i,j in index_set(X) where i < j)(X[i] != X[j]);

predicate int_ne(var int: x, var int: y);
```

Boolean-isation for SAT solvers

```
predicate all_different_int(array[int] of var int: X) =
  let {
    array[int,int] of var bool: Y = int2bools(X);
    array[...,...] of var bool: A;
  } in forall(i in ..., j in ...)
          ((A[i-1,j] -> A[i,j])
           /\ (Y[i,j] <-> (not A[i-1,j] /\ A[i,j])));
function array[int,int] of var bool: int2bools
                (array[int] of var int: X) = [...];
```

When `X` has n decision variables over domains of size m, this ladder encoding yields the two arrays `Y` and `A` of $n \cdot m$ Boolean decision variables (where `Y[i,v]=true` iff `X[i]=v`, and `A[i,v]=true` iff `v in X[1..i]`) as well as $\mathcal{O}(n^2)$ clauses of 2 or 3 literals. This is more compact and usually more efficient than the direct encoding, with $\mathcal{O}(n^3)$ clauses of 2 literals over only `Y`.

Linearisation for MIP solvers: Cbc, CPLEX, Gurobi, HiGHS, ...

```
predicate all_different_int(array[int] of var int: X) =
  let {array[int,int] of var 0..1: Y = eq_encode(X)
  } in forall(d in index_set_2of2(Y))
              (sum(i in index_set_1of2(Y))(Y[i,d]) <= 1);

predicate int_ne(var int: x, var int: y) =
  let {var 0..1: p}
  in x - y + 1 <= ub(x - y + 1) * (1 - p)
  /\ y - x + 1 <= ub(y - x + 1) * p;

% ... continued on next slide ...
```

Linearisation for MIP solvers (end)

```
% ... continued from previous slide ...

function array[int,int] of var int:
  eq_encode(array[int] of var int: X) =
  [... equality_encoding(...) ...]

predicate equality_encoding(var int: x,
                            array[int] of var 0..1: Y) =
  x in index_set(Y)
  /\
  sum(d in index_set(Y))(Y[d]) = 1
  /\
  sum(d in index_set(Y))(d * Y[d]) = x;
```

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Pigeonhole: Experimental Comparison

Time, in seconds, to prove unsatisfiability:

n	backend	`all_different`	`!=`
10	`mzn-gecode`	< 1	< 1
10	`mzn-gurobi`	< 1	58
11	`mzn-gecode`	< 1	9
11	`mzn-gurobi`	< 1	285
12	`mzn-gecode`	< 1	113
12	`mzn-gurobi`	< 1	3704
100	`mzn-gecode`	< 1	time-out
100	`mzn-gurobi`	< 1	time-out
300	`mzn-gecode`	< 1	time-out
300	`mzn-gurobi`	24	time-out
100,000	`mzn-gecode`	< 1	time-out
1,000,000	`mzn-gecode`	5	time-out

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Some Questions for Guidance

- Do you need guarantees that a found solution is optimal, that all solutions are found, and that unsatisfiability is provable?
- What types of decision variables are in your model?
- What constraint predicates are in your model?
- Does your problem look like a well-known problem?
- How do backends perform on easy problem instances?
- What is your favourite technology or backend?

Some Caveats

- Each problem can be modelled in many different ways.
- Different models of the same problem suit better for different backends.
- Performance on small instances does not always scale to larger instances.
- Sometimes, a good search strategy is more important than a good model (see Topic 8: Inference & Search in CP & LCG).
- Not all backends of the same technology have comparable performance.
- Some pure problems can be solved by specialist solvers, such as Concorde for the travelling salesperson problem, but real-life side constraints often make them inapplicable.
- Some problems are maybe even solvable in polynomial time and space.

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Take-Home Message:

- There are many solving technologies and backends.
- It is useful to highlight the commonalities and differences.
- No solving technology or backend can be universally better than all the others, unless P $=$ NP.

☞ Try them!

To go further:

John N. Hooker.
Integrated Methods for Optimization.
2nd edition, Springer, 2012.