ID2204: Constraint Programming

Introduction & Overview

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Lecture Overview

- What is Constraint Programming?

Sudoku is Constraint Programming

- ... more later

Sudoku

...is Constraint Programming!

Sudoku

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			

- Assign blank fields digits such that:
  digits distinct per rows, columns, blocks

Sudoku

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			

- Assign blank fields digits such that:
  digits distinct per **rows**, columns, blocks

Sudoku

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	8		4			1		
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Sudoku

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6		9						1
	8		4			1		
	6	3					8	
			6		8			

- Assign blank fields digits such that:
  digits distinct per rows, columns, **blocks**

Block Propagation

	8	
	6	3

- No field in block can take digits 3,6,8

Block Propagation

1,2,4,5,7,9	**8**	1,2,4,5,7,9
1,2,4,5,7,9	**6**	**3**
1,2,4,5,7,9	1,2,4,5,7,9	1,2,4,5,7,9

- No field in block can take digits 3,6,8
  - propagate to other fields in block
- Rows and columns: likewise

Propagation

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			

1,2,3,4,5,6,7,8,9

- Prune digits from fields such that:
  digits distinct per rows, columns, blocks

Propagation

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			

1,3,5,6,7,8

- Prune digits from fields such that:
  digits distinct per **rows**, columns, blocks

Propagation

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			

1,3,6,7

- Prune digits from fields such that:
  digits distinct per rows, **columns**, blocks

Propagation

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			

1,3,6

- Prune digits from fields such that:
  digits distinct per rows, columns, **blocks**

Iterated Propagation

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			

This particular Sudoku instance can be solved by propagation only, that is without backtracking search!

- Iterate propagation for rows, columns, blocks
- What if no assignment: search... later

Sudoku is Constraint Programming

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			

- Variables: fields
  - take values: digits
  - maintain set of possible values
- Constraints: distinct
  - relation among variables

- Modelling: variables, values, constraints
- Solving: propagation, search

Constraint Programming

- Variable domains
  - finite domain integer, finite sets, multisets, intervals, ...
- Constraints
  - distinct, arithmetic, scheduling, graphs, ...
- Solving
  - propagation, branching, exploration, ...
- Modelling
  - variables, values, constraints, heuristics, symmetries, ...

Plan of Lecture

- Introduction
  - what is constraint programming?
  - principles and applications
- Overview
  - course content
  - course goal
- Organizational

What Is Constraint Programming?

Running Example: SMM

- Find distinct digits for letters, such that

```
      SEND
+     MORE
----------
=    MONEY
```

The unique solution is 9567+1085=10652.

Constraint Model for SMM

- Variables:

  $S,E,N,D,M,O,R,Y \in \{0,\ldots,9\}$
- Constraints:

```
  distinct(S,E,N,D,M,O,R,Y)
```

$$
\begin{array}{rl}
 & 1000\times S+100\times E+10\times N+D \\
+ & 1000\times M+100\times O+10\times R+E \\
= & 10000\times M+1000\times O+100\times N+10\times E+Y
\end{array}
$$

$S\neq 0 \qquad M\neq 0$

Solving SMM

- Find values for variables

    such that

  all constraints satisfied

Finding a Solution

- Compute with possible values
  - rather than enumerating assignments
- Prune inconsistent values
  - constraint propagation
- Search
  - branch: define search tree
  - explore: explore search tree for solution

Constraint Propagation

Important Concepts

- Constraint store
- Propagator
- Constraint propagation

Constraint Store

$x \in \{3,4,5\}$ $y \in \{3,4,5\}$

- Maps variables to possible values
  - stores basic constraints

Constraint Store

finite domain constraints

$x \in \{3,4,5\}$ $y \in \{3,4,5\}$

- Maps variables to possible values
- Others: finite sets, intervals, trees, ...

Propagators

- Implement (non-basic) constraints

  $\text{distinct}(x_1,\ldots,x_n)$

  $x + 2*y = z$

Propagators

$x \geq y$ $\quad\quad$ $y > 3$

$x \in \{3,4,5\} \quad y \in \{3,4,5\}$

- Amplify store by constraint propagation

Strengthen: eliminate impossible values

Propagators

$x \geq y$ $\quad$ **$y > 3$**

$x \in \{3,4,5\}$ $\quad$ $y \in \{3,4,5\}$

- Amplify store by constraint propagation

Propagators

$x \geq y$ $\qquad$ $y > 3$

$x \in \{3,4,5\}$ $\quad$ $\mathbf{y \in \{4,5\}}$

- Amplify store by constraint propagation

Propagators

$\mathbf{x \geq y}$ $\quad$ $y>3$

$x \in \{3,4,5\}$ $\quad$ $\mathbf{y \in \{4,5\}}$

- Amplify store by constraint propagation

Propagators

$x \geq y$ $y > 3$

$\mathbf{x \in \{4,5\}}$ $y \in \{4,5\}$

- Amplify store by constraint propagation

Propagators

$x \geq y$ $\qquad$ $y > 3$

$x \in \{4,5\}$ $\quad$ $y \in \{4,5\}$

- Amplify store by constraint propagation
- Disappear when done (subsumed, entailed)
    - no more propagation possible

Propagators

$x \geq y$

$x \in \{4,5\}$ $y \in \{4,5\}$

- Amplify store by constraint propagation
- Disappear when done (subsumed, entailed)
  - no more propagation possible

Propagation for SMM

- Results in store

  $S \in \{9\}$ $E \in \{4, \ldots, 7\}$ $N \in \{5, \ldots, 8\}$ $D \in \{2, \ldots, 8\}$

  $M \in \{1\}$ $O \in \{0\}$ $R \in \{2, \ldots, 8\}$ $Y \in \{2, \ldots, 8\}$

- Propagation **alone** not sufficient!
  - create simpler sub-problems
  - branching

Constraints and Propagators

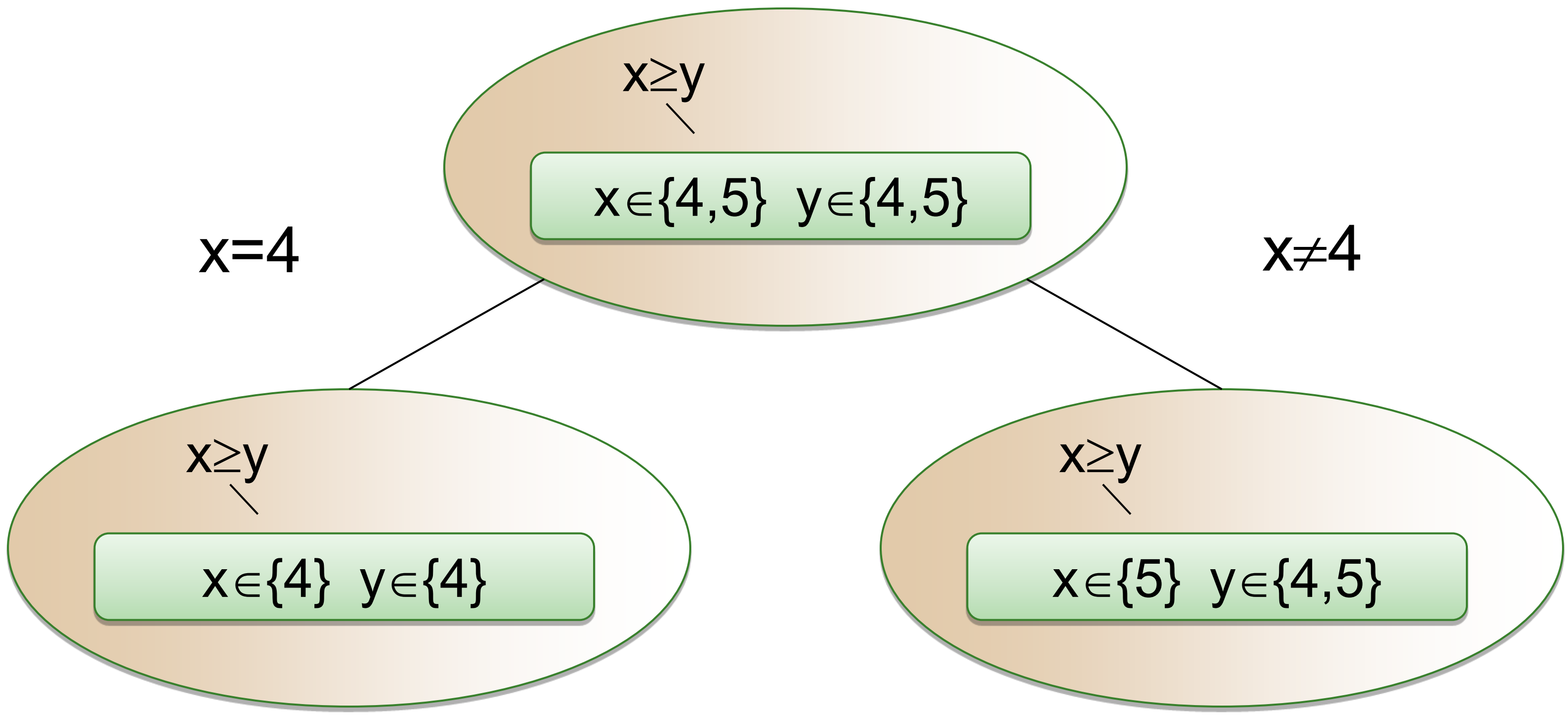
- Constraints state relations among variables
  - which value combinations satisfy constraint
- Propagators implement constraints
  - prune values in conflict with constraint
- Constraint propagation drives propagators for several constraints

Search

Important Concepts

- Branching
- Exploration
- Branching heuristics
- Best-solution search

Search: Branching



- Create subproblems with additional information
  - enable further constraint propagation

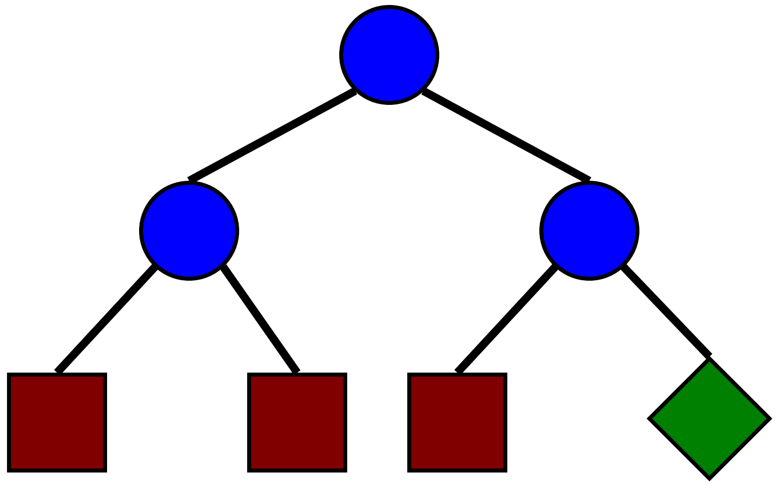
Example Branching Strategy

- Pick variable x with at least two values
- Pick value n from domain of x
- Branch with

  $x=n$ and $x \neq n$

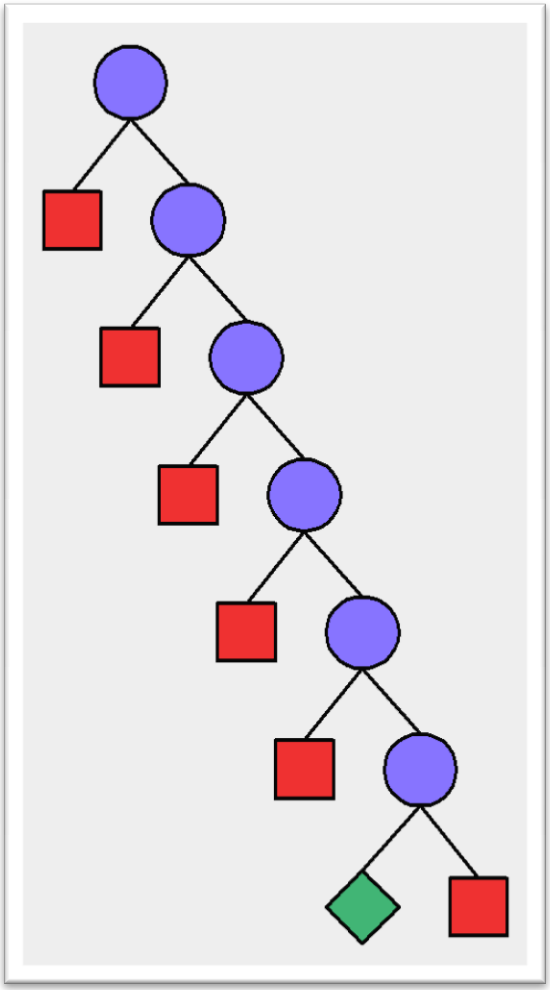
- Part of model

Search: Exploration



- Iterate propagation and branching
- Orthogonal: branching ⇆ exploration
- Nodes:
  - **Unsolved**
  - **Failed**
  - **Succeeded**

SMM: Unique Solution



```
   SEND
+  MORE
-------
= MONEY

   9567
+  1085
-------
= 10652
```

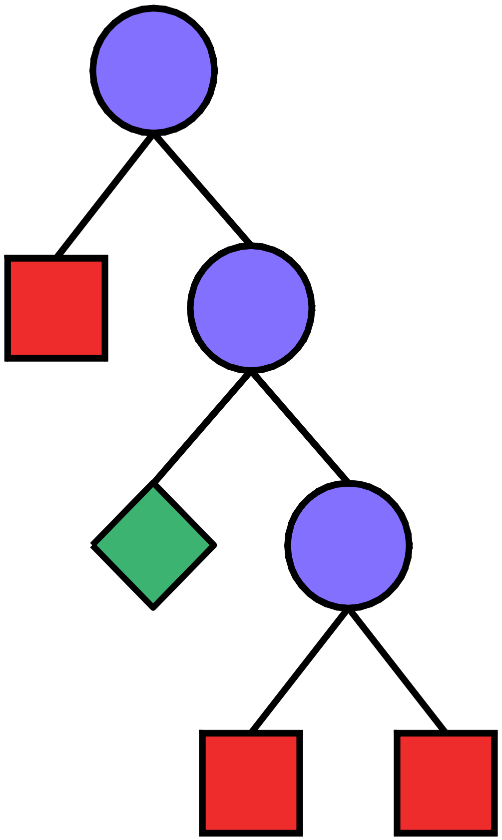
Heuristics for Branching

- Which variable
  - least possible values (first-fail)
  - application dependent heuristic
- Which value
  - minimum, median, maximum

    $x=m$ or $x \neq m$
  - split with median m

    $x<m$ or $x \geq m$
- Problem specific

SMM: Solution With First-fail



```
    SEND
+   MORE
--------
=  MONEY

    9567
+   1085
--------
=  10652
```

Send Most Money (SMM++)

- Find distinct digits for letters, such that

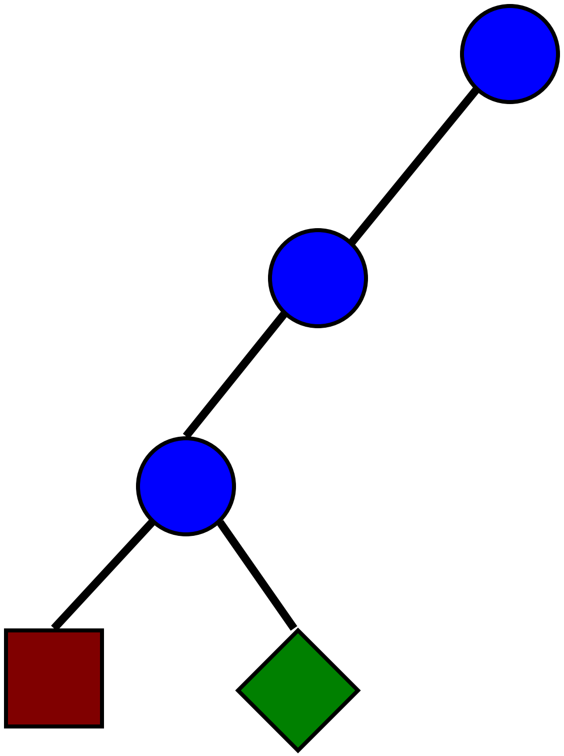
```
     SEND
+    MOST
---------
=   MONEY
```

and **MONEY** maximal

Best Solution Search

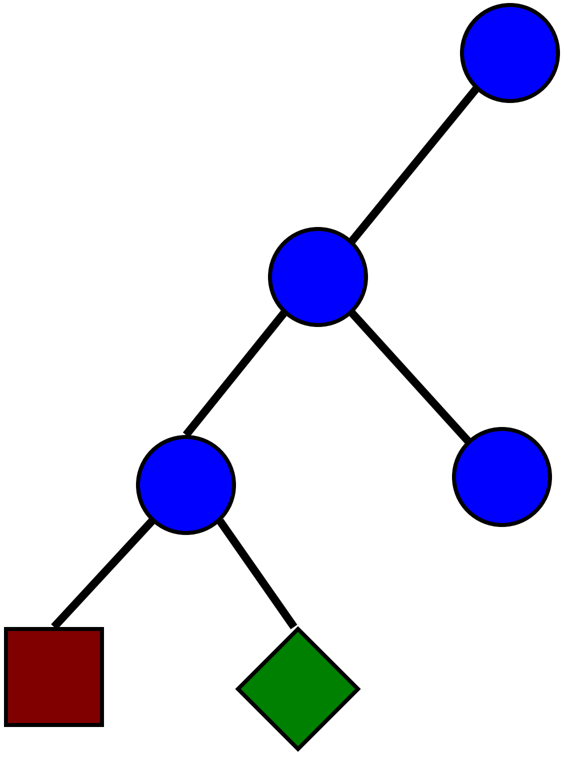
- Naïve approach:
  - compute all solutions
  - choose best
- Branch-and-bound approach:
  - compute first solution
  - add “betterness” constraint to open nodes
  - next solution will be “better”
  - prunes search space

Branch-and-bound Search



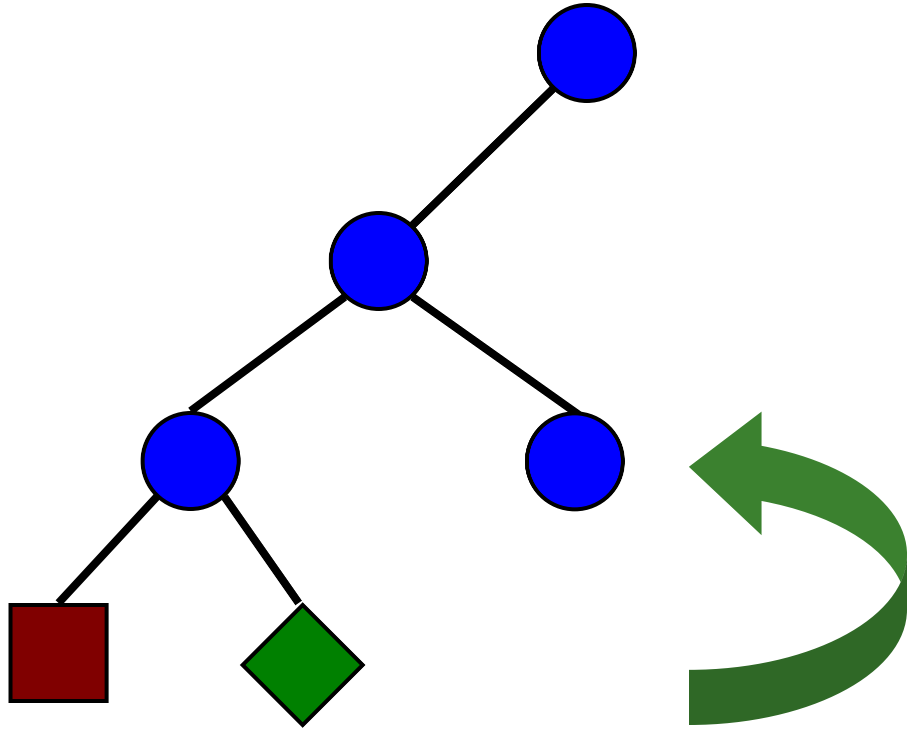
- Find first solution

Branch-and-bound Search



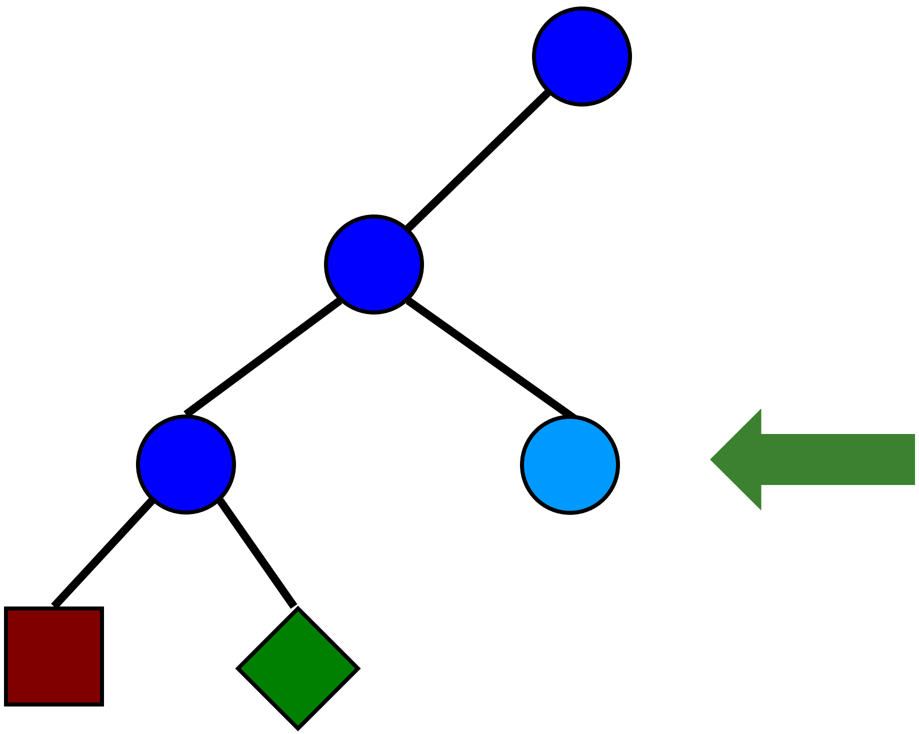
- Explore with additional constraint

Branch-and-bound Search



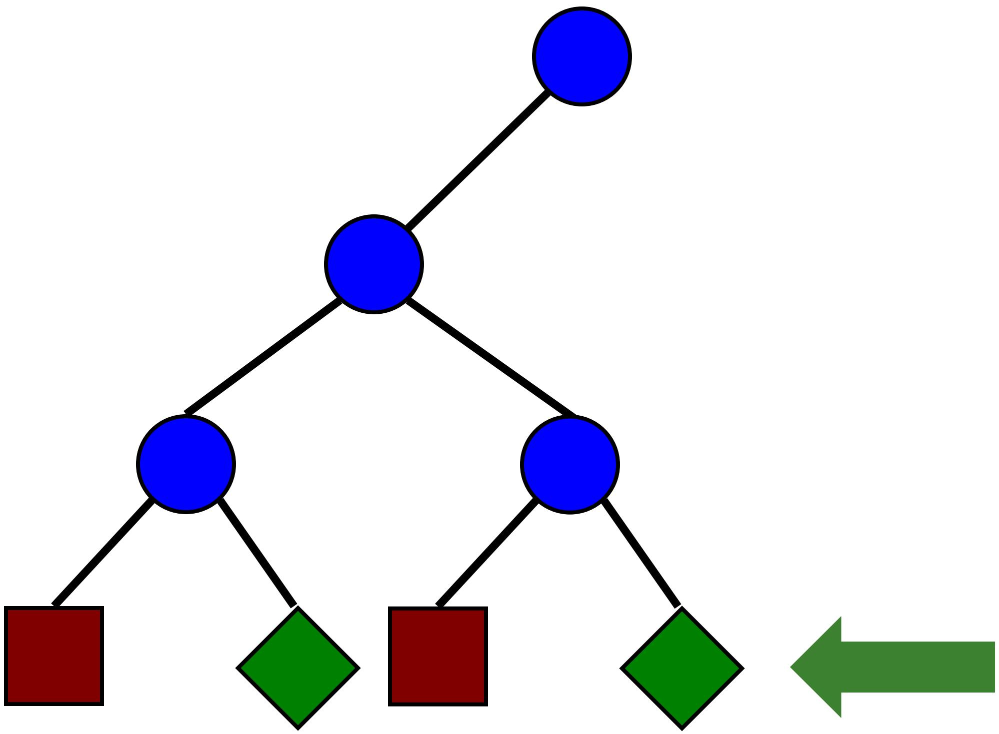
- Explore with additional constraint

Branch-and-bound Search



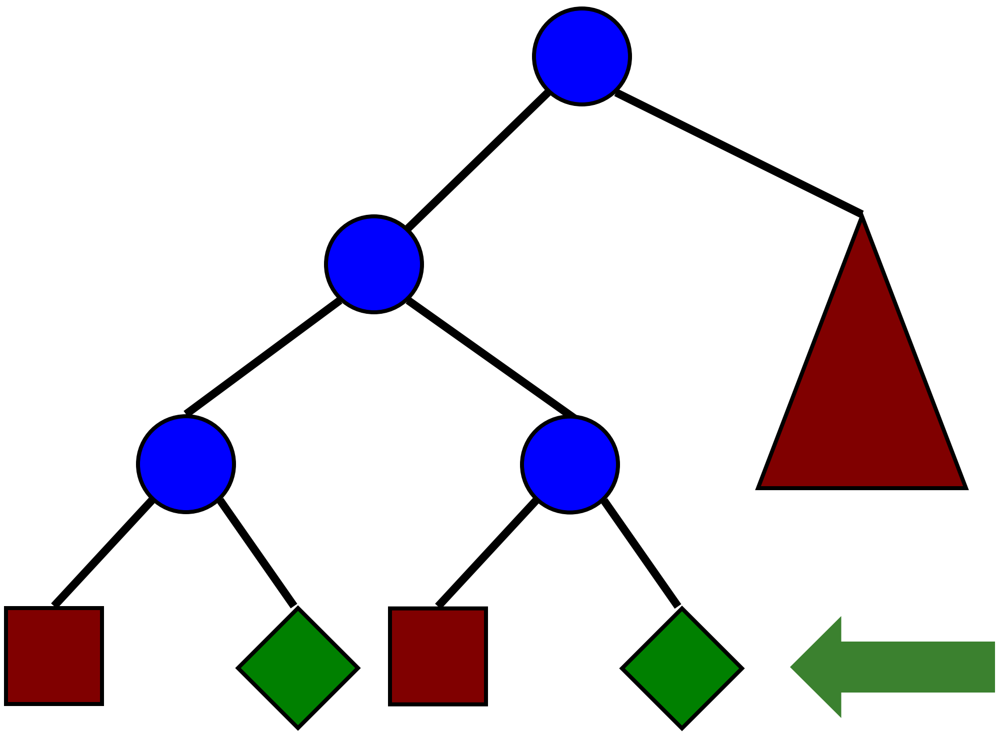
- Guarantees better solutions

Branch-and-bound Search



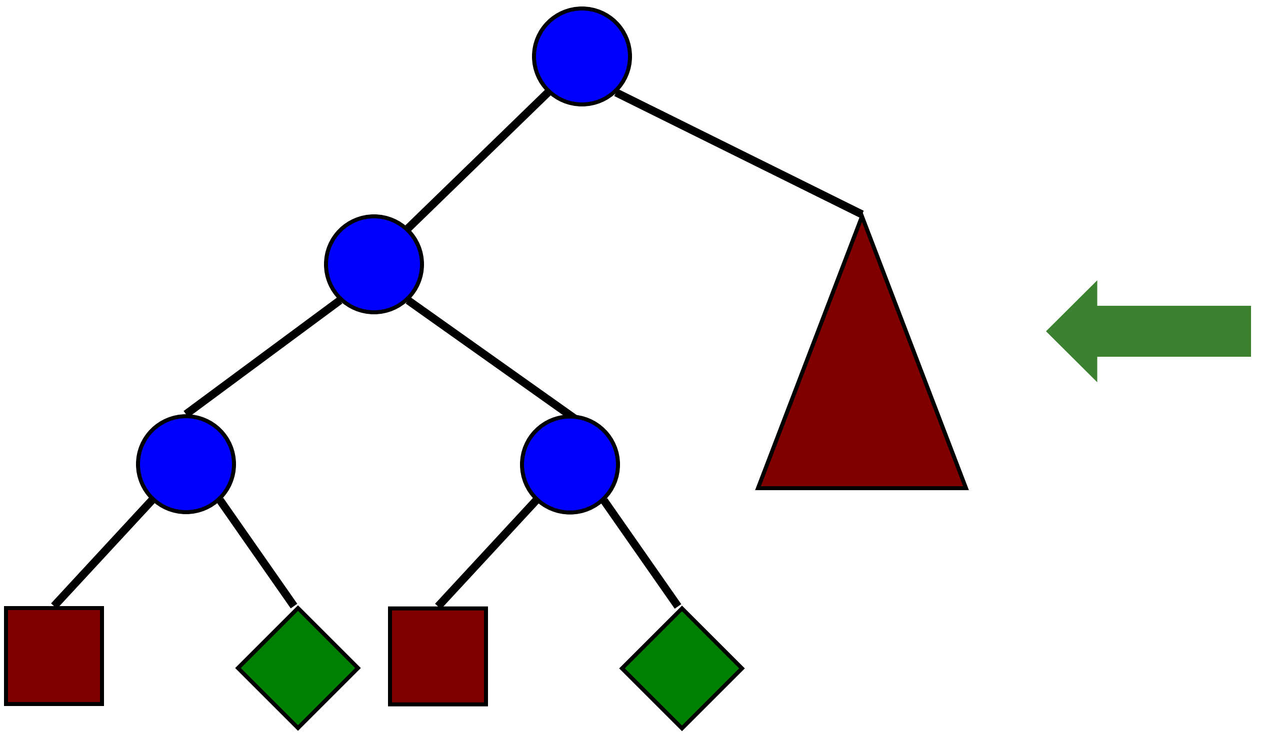
- Guarantees better solutions

Branch-and-bound Search



- Last solution best

Branch-and-bound Search



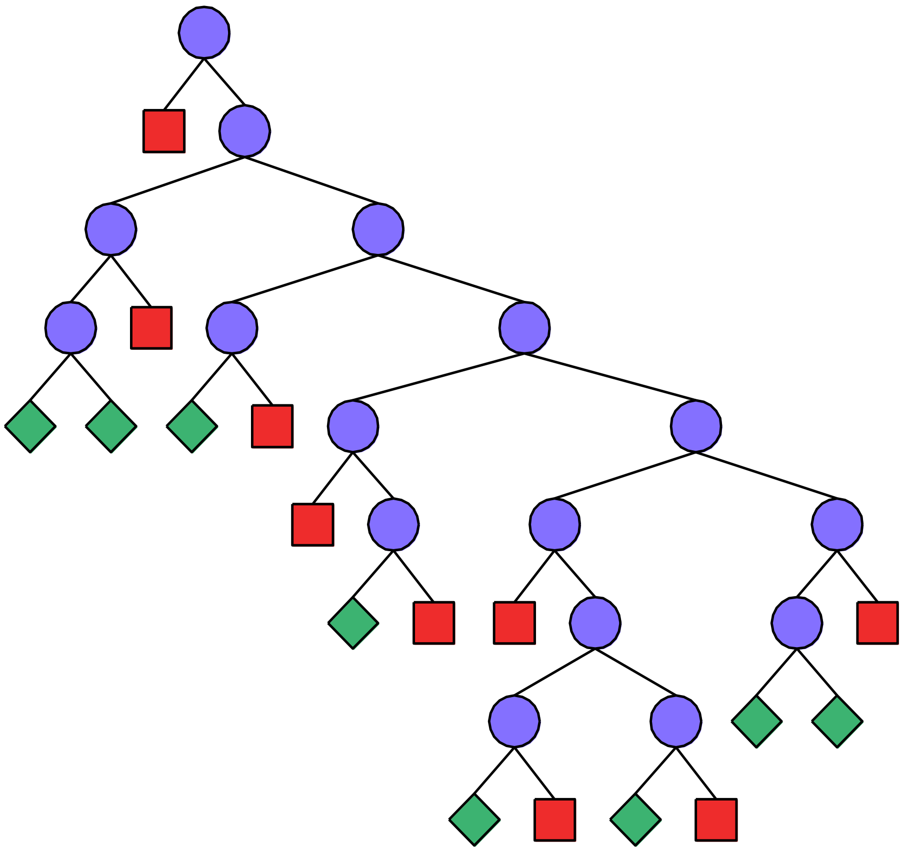
- Proof of optimality

Modelling SMM++

- Constraints and branching as before
- Order among solutions with constraints
  - so-far-best solution `S,E,N,D,M,O,T,Y`
  - current node `S,E,N,D,M,O,T,Y`
  - constraint added

    $$10000 \times M + 1000 \times O + 100 \times N + 10 \times E + Y$$
    $$<$$
    $$10000 \times M + 1000 \times O + 100 \times N + 10 \times E + Y$$

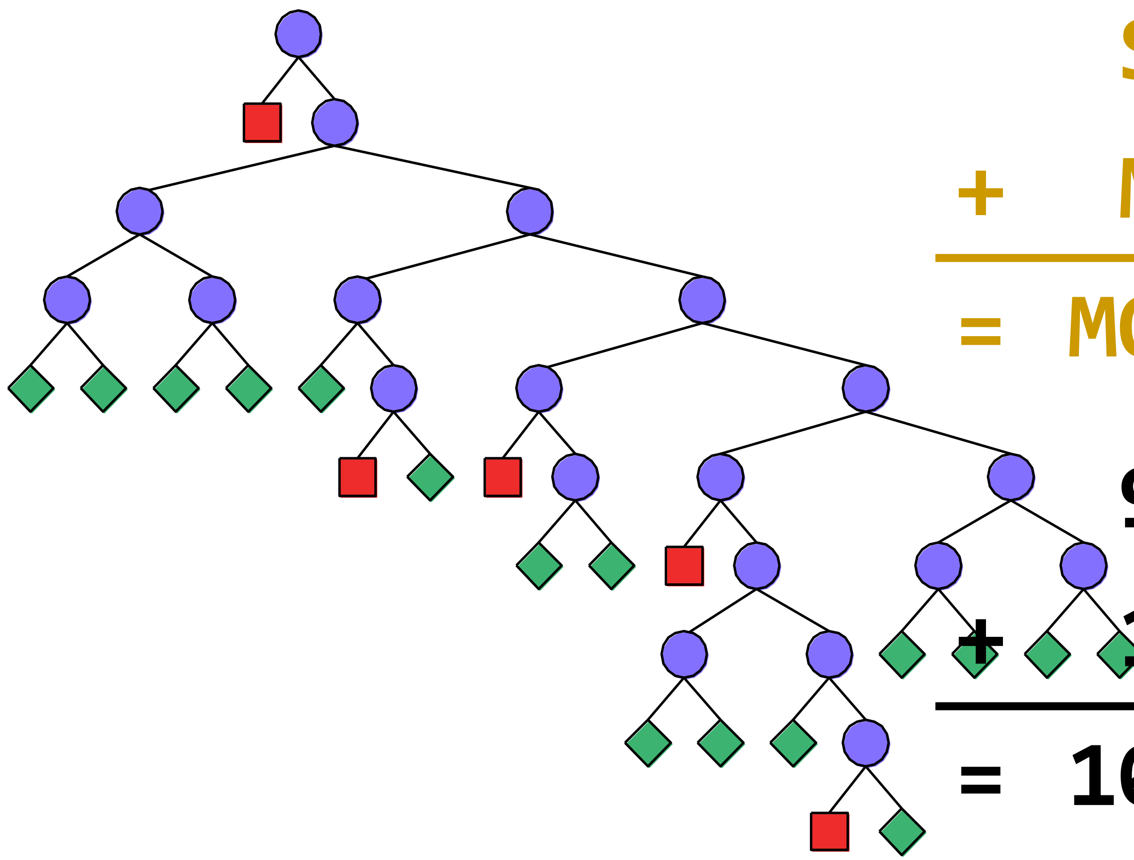
SMM++: Branch-and-bound



```
    SEND
+   MOST
= MONEY

    9782
+   1094
= 10876
```

SMM++: All Solution Search



```
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= MONEY

   9782
+  1094
-------
= 10876
```

Summary

Summary: Key Ideas and Principles

- Modelling
  - variables with domain
  - constraints to state relations
  - branching strategy
  - solution ordering
- Solving
  - constraint propagation
  - constraint branching
  - search tree exploration

applications

principles

Widely Applicable

- Timetabling
- Scheduling
- Crew rostering
- Resource allocation
- Workflow planning and optimization
- Gate allocation at airports
- Sports-event scheduling
- Railroad: track allocation, train allocation, schedules
- Automatic composition of music
- Genome sequencing
- Frequency allocation
- …

Draws on Variety of Techniques

- Artificial intelligence
  - basic idea, search, ...
- Operations research
  - scheduling, flow, ...
- Algorithms
  - graphs, matching, networks, ...
- Programming languages
  - programmability, extensionability, ...

Essential Aspect

- Compositional middleware for combining
  - smart algorithmic
  - problem substructures

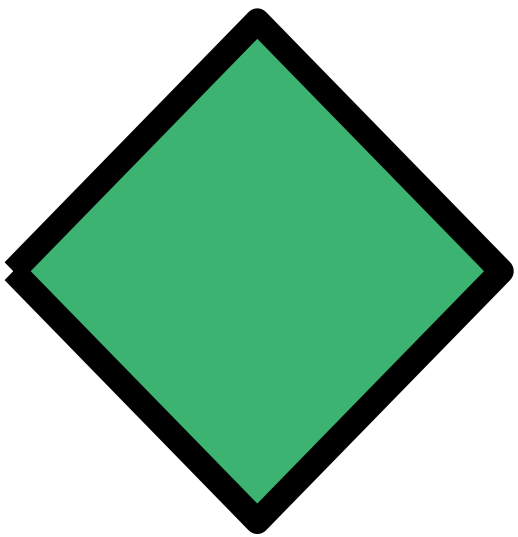
  components (propagators)
  - scheduling
  - graphs
  - flows
  - …

  plus
  - essential extra constraints

Principles

- Models for constraint propagation
    - properties and guarantees
- Strong constraint propagation
    - global constraints with strong algorithmic methods
    - mantra: search kills, search kills, search k…
- Branching strategies
- Exploration strategies

SMM: Strong Propagation



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Significance

- Constraint programming identified as a strategic direction in computer science research

    [ACM Computing Surveys, December 1996]

Course Overview

Content Overview

- As to be expected, no surprises:
  - **applications**
  - **principles**
  - pragmatics
  - limitations

Modeling with CP

- Basic solving methods
  - constraint propagation
  - search
- Typical techniques for modeling in different application areas
  - redundant constraints, symmetry elimination
- Refining models by strong algorithmic methods
- Heuristic search methods
- Application to hard real-size problems

Principles Underlying CP

- Models for
  - propagation
  - search

  and their essential properties
- Different levels of consistency (propagation strength)
- Different constraint domains
  - finite domains, finite sets, ...

Strong Algorithmic Methods

- Régin's distinct algorithm
- Edge-finding
- Integration
  - achieving required properties for propagation

Relation to Other Techniques

- Integer programming
- Local search
- Discussion of merits and weaknesses
- Hybrid approaches

Acknowledgments

- I am grateful to Pierre Flener for helpful comments and bugreports on these slides