Data Structures I, II, III, and IV

I. Amortized Analysis
II. Binary and Binomial Heaps
III. Fibonacci Heaps
IV. Union–Find
Data structures

Static problems. Given an input, produce an output.
Data structures

**Static problems.** Given an input, produce an output.

**Ex.** Sorting, FFT, edit distance, shortest paths, MST, max-flow, ...
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Data structures

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Data structure. Way to store and organize data.
Ex. Array, linked list, binary heap, binary search tree, hash table, ...
**Goal.** Design a data structure to support all operations in $O(1)$ time.

- **INIT($n$):** create and return an *initialized* array (all zero) of length $n$.
- **READ($A, i$):** return $i^{th}$ element of array.
- **WRITE($A, i, value$):** set $i^{th}$ element of array to $value$. 
Appetizer

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Assumptions.

- Can **malloc** an uninitialized array of length $n$ in $O(1)$ time.
- Given an array, can read or write $i^{th}$ element in $O(1)$ time.

*true in C or C++, but not Java*
**Appetizer**

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- Can **malloc** an uninitialized array of length $n$ in $O(1)$ time.
- Given an array, can read or write $i^{th}$ element in $O(1)$ time.

**Remark.** An array does **INIT** in $O(n)$ time and **READ** and **WRITE** in $O(1)$ time.
Appetizer

Appetizer

**Data structure.** Three arrays $A[1..n]$, $B[1..n]$, and $C[1..n]$, and an integer $k$.

- $A[i]$ stores the current value for READ (if initialized).

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- $A[i]$ stores the current value for READ (if initialized).
- $k =$ number of initialized entries.

```
1  2  3  4  5  6  7  8
A[]  ? 22 55 99  ? 33  ?  ?

k = 4
```
**Appetizer**

**Data structure.** Three arrays $A[1..n]$, $B[1..n]$, and $C[1..n]$, and an integer $k$.
- $A[i]$ stores the current value for READ (if initialized).
- $k =$ number of initialized entries.
- $C[j] =$ index of $j^{th}$ initialized entry for $j = 1, \ldots, k$.

\begin{align*}
C[1..8] &= \begin{bmatrix} 4 & 6 & 2 & 3 & ? & ? & ? & ? \end{bmatrix}
\end{align*}

$k = 4$

**Appetizer**

**Data structure.** Three arrays $A[1..n]$, $B[1..n]$, and $C[1..n]$, and an integer $k$.

- $A[i]$ stores the current value for READ (if initialized).
- $k$ = number of initialized entries.
- $C[j]$ = index of $j^{th}$ initialized entry for $j = 1, \ldots, k$.
- If $C[j] = i$, then $B[i] = j$ for $j = 1, \ldots, k$.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

$k = 4$


- $A[i]$ stores the current value for READ (if initialized).
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- $C[j] =$ index of $j^{th}$ initialized entry for $j = 1, \ldots, k$.
- If $C[j] = i$, then $B[i] = j$ for $j = 1, \ldots, k$.

Theorem. $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

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$k = 4$

Data structure. Three arrays \( A[1..n] \), \( B[1..n] \), and \( C[1..n] \), and an integer \( k \).

- \( A[i] \) stores the current value for \texttt{READ} (if initialized).
- \( k \) = number of initialized entries.
- \( C[j] \) = index of \( j^{th} \) initialized entry for \( j = 1, \ldots, k \).
- If \( C[j] = i \), then \( B[i] = j \) for \( j = 1, \ldots, k \).

Theorem. \( A[i] \) is initialized iff both \( 1 \leq B[i] \leq k \) and \( C[B[i]] = i \).

Pf. Ahead.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

\( k = 4 \)

Appetizer
**INIT** \((A, n)\)

\[ k \leftarrow 0. \]

\[ A \leftarrow \text{MALLOC}(n). \]

\[ B \leftarrow \text{MALLOC}(n). \]

\[ C \leftarrow \text{MALLOC}(n). \]
**Appetizer**

**INIT** $(A, n)$

$k \leftarrow 0.$

$A \leftarrow \text{MALLOC}(n).$

$B \leftarrow \text{MALLOC}(n).$

$C \leftarrow \text{MALLOC}(n).$

**READ** $(A, i)$

**IF** **INITIALIZED** $(A[i])$

**RETURN** $A[i].$

**ELSE**

**RETURN** $0.$
**Appetizer**

**INIT** \((A, n)\)

\[ k \leftarrow 0. \]

\[ A \leftarrow \text{MALLOC}(n). \]

\[ B \leftarrow \text{MALLOC}(n). \]

\[ C \leftarrow \text{MALLOC}(n). \]

**READ** \((A, i)\)

**IF** \((\text{INITIALIZED} (A[i]))\)

**RETURN** \(A[i]\).

**ELSE**

**RETURN** \(0\).

**WRITE** \((A, i, value)\)

**IF** \((\text{INITIALIZED} (A[i]))\)

\[ A[i] \leftarrow value. \]

**ELSE**

\[ k \leftarrow k + 1. \]

\[ A[i] \leftarrow value. \]

\[ B[i] \leftarrow k. \]

\[ C[k] \leftarrow i. \]
### Appetizer

#### INIT \( (A, n) \)

\[
k \leftarrow 0.
\]

\[
A \leftarrow \text{MALLOC}(n).
\]

\[
B \leftarrow \text{MALLOC}(n).
\]

\[
C \leftarrow \text{MALLOC}(n).
\]

#### READ \( (A, i) \)

\[
\text{IF } (\text{INITIALIZED } (A[i]))
\]

\[
\quad \text{RETURN } A[i].
\]

\[
\text{ELSE}
\]

\[
\quad \text{RETURN } 0.
\]

#### WRITE \( (A, i, \text{value}) \)

\[
\text{IF } (\text{INITIALIZED } (A[i]))
\]

\[
\quad A[i] \leftarrow \text{value}.
\]

\[
\text{ELSE}
\]

\[
\quad k \leftarrow k + 1.
\]

\[
\quad A[i] \leftarrow \text{value}.
\]

\[
\quad B[i] \leftarrow k.
\]

\[
\quad C[k] \leftarrow i.
\]

#### INITIALIZED \( (A, i) \)

\[
\text{IF } (1 \leq B[i] \leq k) \text{ and } (C[B[i]] = i)
\]

\[
\quad \text{RETURN } \text{true}.
\]

\[
\text{ELSE}
\]

\[
\quad \text{RETURN } \text{false}.
\]
**Theorem.** $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$. 

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$k = 4$

Theorem. $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

Pf. $\Rightarrow$

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<td>$A[]$</td>
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<tr>
<td>$B[]$</td>
<td>?</td>
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<td>4</td>
<td>1</td>
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<td>$C[]$</td>
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$k = 4$

**Theorem.**  $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

**Pf.**  $\Rightarrow$

- Suppose $A[i]$ is the $j^{th}$ entry to be initialized.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
A[] & \text{?} & 22 & 55 & 99 & \text{?} & 33 & \text{?} & \text{?} \\
B[] & \text{?} & 3 & 4 & 1 & \text{?} & 2 & \text{?} & \text{?} \\
C[] & 4 & 6 & 2 & 3 & \text{?} & \text{?} & \text{?} & \text{?} \\
\hline
\end{array}
\]

$k = 4$

Theorem. $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

Pf. $\Rightarrow$

- Suppose $A[i]$ is the $j^{th}$ entry to be initialized.
- Then $C[j] = i$ and $B[i] = j$.
Theorem. \( A[i] \) is initialized iff both \( 1 \leq B[i] \leq k \) and \( C[B[i]] = i \).

Pf. \( \Rightarrow \)
- Suppose \( A[i] \) is the \( j^{th} \) entry to be initialized.
- Then \( C[j] = i \) and \( B[i] = j \).
- Thus, \( C[B[i]] = i \).

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\text{A[ ]} & \text{?} & 22 & 55 & 99 & \text{?} & 33 & \text{?} & \text{?} \\
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\hline
\end{array}
\]

\( k = 4 \)

Theorem. \( A[i] \) is initialized iff both \( 1 \leq B[i] \leq k \) and \( C[B[i]] = i \).

Pf. \( \Leftarrow \)

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
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\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

\( k = 4 \)

**Theorem.** \( A[i] \) is initialized iff both \( 1 \leq B[i] \leq k \) and \( C[B[i]] = i \).

**Pf.** \( \iff \)

- Suppose \( A[i] \) is uninitialized.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

\( k = 4 \)

Theorem. $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

Pf. $\Leftarrow$

- Suppose $A[i]$ is uninitialized.
- If $B[i] < 1$ or $B[i] > k$, then $A[i]$ clearly uninitialized.

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$k = 4$

Theorem. \( A[i] \) is initialized iff both \( 1 \leq B[i] \leq k \) and \( C[B[i]] = i \).

Pf. \( \Leftarrow \)

- Suppose \( A[i] \) is uninitialized.
- If \( B[i] < 1 \) or \( B[i] > k \), then \( A[i] \) clearly uninitialized.
- If \( 1 \leq B[i] \leq k \) by coincidence, then we still can't have \( C[B[i]] = i \) because none of the entries \( C[1..k] \) can equal \( i \).

\[ A[] \]

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

\[ B[] \]

\[
\begin{array}{cccccccc}
? & 3 & 4 & 1 & ? & 2 & ? & ?
\end{array}
\]

\[ C[] \]

\[
\begin{array}{cccccccc}
\end{array}
\]

\( k = 4 \)

Amortized Analysis

- binary counter
- multipop stack
- dynamic table
Amortized analysis

**Worst-case analysis.** Determine worst-case running time of a data structure operation as function of the input size.
Amortized analysis

**Worst-case analysis.** Determine worst-case running time of a data structure operation as function of the input size.

can be too pessimistic if the only way to encounter an expensive operation is if there were lots of previous cheap operations
Amortized analysis

**Worst-case analysis.** Determine worst-case running time of a data structure operation as function of the input size.

*can be too pessimistic if the only way to encounter an expensive operation is if there were lots of previous cheap operations*

**Amortized analysis.** Determine worst-case running time of a sequence of data structure operations as a function of the input size.
Amortized analysis

**Worst-case analysis.** Determine worst-case running time of a data structure operation as function of the input size.

Amortized analysis. Determine worst-case running time of a sequence of data structure operations as a function of the input size.

**Ex.** Starting from an empty stack implemented with a dynamic table, any sequence of $n$ push and pop operations takes $O(n)$ time in the worst case.
Amortized analysis: applications

- Splay trees.
- Dynamic table.
- Fibonacci heaps.
- Garbage collection.
- Move-to-front list updating.
- Push-relabel algorithm for max flow.
- Path compression for disjoint-set union.
- Structural modifications to red-black trees.
- Security, databases, distributed computing, ...
Amortized analysis: applications

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AMORTIZED COMPUTATIONAL COMPLEXITY*

ROBERT ENDRE TARJAN†

Abstract. A powerful technique in the complexity analysis of data structures is amortization, or averaging over time. Amortized running time is a realistic but robust complexity measure for which we can obtain surprisingly tight upper and lower bounds on a variety of algorithms. By following the principle of designing algorithms whose amortized complexity is low, we obtain “self-adjusting” data structures that are simple, flexible and efficient. This paper surveys recent work by several researchers on amortized complexity.

ASM(MOS) subject classifications. 68C25, 68E05
Chapter 17

Amortized Analysis

- binary counter
- multipop stack
- dynamic table
Binary counter

**Goal.** Increment a $k$-bit binary counter (mod $2^k$).

**Representation.** $a_j = j^{th}$ least significant bit of counter.

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**Cost model.** Number of bits flipped.
Binary counter

**Goal.** Increment a \( k \)-bit binary counter (mod \( 2^k \)).

**Representation.** \( a_j = j^{th} \) least significant bit of counter.

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</table>

**Theorem.** Starting from the zero counter, a sequence of \( n \) INCREMENT operations flips \( O(nk) \) bits.
**Binary counter**

**Goal.** Increment a \( k \)-bit binary counter (mod \( 2^k \)).

**Representation.** \( a_j = j^{th} \) least significant bit of counter.

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**Theorem.** Starting from the zero counter, a sequence of \( n \) INCREMENT operations flips \( O(nk) \) bits.

**Pf.** At most \( k \) bits flipped per increment. ■
Aggregate method (brute force)

**Aggregate method.** Sum up sequence of operations, weighted by their cost.

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Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:
Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:

- Bit 0 flips $n$ times.
Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:

- Bit 0 flips $n$ times.
- Bit 1 flips $\lfloor n / 2 \rfloor$ times.
Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:

- Bit 0 flips $n$ times.
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**Theorem.** Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(n)$ bits.
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- Bit $j$ flips $\lfloor n/2^j \rfloor$ times.
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- The total number of bits flipped is

$$
\sum_{j=0}^{k-1} \left\lfloor \frac{n}{2^j} \right\rfloor < n \sum_{j=0}^{\infty} \frac{1}{2^j}
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$\sum_{j=0}^{k-1} \lfloor \frac{n}{2^j} \rfloor < n \sum_{j=0}^{\infty} \frac{1}{2^j}$

$= 2n \quad \blacksquare$
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  \[
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  \]

**Remark.** Theorem may be false if initial counter is not zero.
Accounting method (banker's method)
Accounting method (banker's method)

Assign (potentially) different charges to each operation.
Accounting method (banker's method)

Assign (potentially) different charges to each operation.

- \( D_i = \) data structure after \( i^{th} \) operation.
Accounting method (banker's method)

Assign (potentially) different charges to each operation.

- $D_i = \text{data structure after } i^{th} \text{ operation.}$
- $c_i = \text{actual cost of } i^{th} \text{ operation.}$
Accounting method (banker's method)

Assign (potentially) different charges to each operation.

- $D_i$ = data structure after $i^{th}$ operation.
- $c_i$ = actual cost of $i^{th}$ operation.
- $\hat{c_i}$ = amortized cost of $i^{th}$ operation = amount we charge operation $i$. 

(can be more or less than actual cost)
Accounting method (banker's method)

Assign (potentially) different charges to each operation.

- $D_i =$ data structure after $i^{th}$ operation.
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- $\hat{c}_i =$ amortized cost of $i^{th}$ operation = amount we charge operation $i$.
- When $\hat{c}_i > c_i$, we store credits in data structure $D_i$ to pay for future ops; when $\hat{c}_i < c_i$, we consume credits in data structure $D_i$. 

\begin{itemize}
  \item can be more or less than actual cost
\end{itemize}
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(can be more or less than actual cost)
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• Initial data structure \( D_0 \) starts with zero credits.

Key invariant. The total number of credits in the data structure \( \geq 0 \).

\[
\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0
\]
Accounting method (banker's method)

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Theorem. Starting from the initial data structure \( D_0 \), the total actual cost of any sequence of \( n \) operations is at most the sum of the amortized costs.
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**Theorem.** Starting from the initial data structure $D_0$, the total actual cost of any sequence of $n$ operations is at most the sum of the amortized costs.

**Pf.** The amortized cost of the sequence of operations is: $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$. ■
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Pf. The amortized cost of the sequence of operations is: \( \sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i \).

Intuition. Measure running time in terms of credits (time = money).
Binary counter: accounting method

**Credits.** One credit pays for a bit flip.
Binary counter: accounting method

**Credits.** One credit pays for a bit flip.

**Invariant.** Each 1 bit has one credit; each 0 bit has zero credits.

<table>
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Binary counter: accounting method

**Credits.** One credit pays for a bit flip.

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**Accounting.**

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$$\begin{array}{cccccccc}
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}$$
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increment

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![Binary counter example](image)
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![Increment Image]
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• The invariant is maintained. \( \Rightarrow \) number of credits in each bit \( \geq 0 \).
Potential method (physicist's method)

Potential function.  \( \Phi(D_i) \) maps each data structure \( D_i \) to a real number s.t.:

- \( \Phi(D_0) = 0 \).
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- \( c_i = \) actual cost of \( i^{th} \) operation.
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= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0) \\
\geq \sum_{i=1}^{n} c_i \quad \blacksquare
\]
## Binary counter: potential method

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**Potential function.** Let $\Phi(D) =$ number of 1 bits in the binary counter $D$.

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increment

```
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<td>0</td>
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</tr>
</tbody>
</table>
**Binary counter: potential method**

**Potential function.** Let $\Phi(D) =$ number of 1 bits in the binary counter $D$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(n)$ bits.
Binary counter: potential method

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**Pf.**
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Pf.
- Suppose that the $i^{th}$ increment operation flips $t_i$ bits from 1 to 0.
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Pf.

- Suppose that the $i^{th}$ increment operation flips $t_i$ bits from 1 to 0.
- The actual cost $c_i \leq t_i + 1$.  
  
  operation sets one bit to 1 (unless counter resets to zero)
Binary counter: potential method

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- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$
Binary counter: potential method

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  [Operation sets one bit to 1 (unless counter resets to zero)]
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$
  \[\leq c_i + 1 - t_i\]
Binary counter: potential method

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**Pf.**

- Suppose that the $i^{th}$ increment operation flips $t_i$ bits from 1 to 0.
- The actual cost $c_i \leq t_i + 1$. \(\text{operation sets one bit to 1 (unless counter resets to zero)}\)
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$
  $\leq c_i + 1 - t_i$
  $\leq 2$.  \(\blacksquare\)
Famous potential functions
Famous potential functions

**Fibonacci heaps.** \( \Phi(H) = 2 \text{trees}(H) + 2 \text{marks}(H) \)
Famous potential functions

**Fibonacci heaps.**  \( \Phi(H) = 2 \text{trees}(H) + 2 \text{marks}(H) \)

**Splay trees.**  \( \Phi(T) = \sum_{x \in T} \lfloor \log_2 \text{size}(x) \rfloor \)
Famous potential functions

**Fibonacci heaps.** $\Phi(H) = 2 \text{trees}(H) + 2 \text{marks}(H)$

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**Move-to-front.** $\Phi(L) = 2 \text{inversions}(L, L^*)$
Famous potential functions

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Famous potential functions

**Fibonacci heaps.** $\Phi(H) = 2 \text{trees}(H) + 2 \text{marks}(H)$

**Splay trees.** $\Phi(T) = \sum_{x \in T} \lfloor \log_2 \text{size}(x) \rfloor$

**Move-to-front.** $\Phi(L) = 2 \text{inversions}(L, L^*)$

**Preflow–push.** $\Phi(f) = \sum_{v : \text{excess}(v) > 0} \text{height}(v)$

**Red-black trees.** $\Phi(T) = \sum_{x \in T} w(x)$

$$w(x) = \begin{cases} 
0 & \text{if } x \text{ is red} \\
1 & \text{if } x \text{ is black and has no red children} \\
0 & \text{if } x \text{ is black and has one red child} \\
2 & \text{if } x \text{ is black and has two red children}
\end{cases}$$
Amortized Analysis

- binary counter
- multipop stack
- dynamic table

Section 17.4
Multipop stack

**Goal.** Support operations on a set of elements:
Multipop stack

**Goal.** Support operations on a set of elements:

- **PUSH(S, x):** push object \( x \) onto stack \( S \).
- **POP(S):** remove and return the most-recently added object.
Multipop stack

**Goal.** Support operations on a set of elements:

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- **MULTIPOP(S, k):** remove the most-recently added \( k \) objects.

\[
\text{MULTIPOP} \ (S, \ k) \\
\text{FOR } i = 1 \text{ TO } k \\
\text{POP} \ (S).
\]
**Multipop stack**

**Goal.** Support operations on a set of elements:
- **PUSH**(S, x): push object x onto stack S.
- **POP**(S): remove and return the most-recently added object.
- **MULTIPOP**(S, k): remove the most-recently added k objects.

```
MULTIPOP (S, k)
FOR  i = 1 TO k
    POP (S).
```

**Exceptions.** We assume POP throws an exception if stack is empty.
**Multipop stack**

**Goal.** Support operations on a set of elements:
- \( \text{PUSH}(S, x) \): push object \( x \) onto stack \( S \).
- \( \text{POP}(S) \): remove and return the most-recently added object.
- \( \text{MULTIPOP}(S, k) \): remove the most-recently added \( k \) objects.

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) \( \text{PUSH}, \text{POP}, \) and \( \text{MULTIPOP} \) operations takes \( O(n^2) \) time.
**Goal.** Support operations on a set of elements:

- **PUSH(S, x):** push object \( x \) onto stack \( S \).
- **POP(S):** remove and return the most-recently added object.
- **MULTIPOP(S, k):** remove the most-recently added \( k \) objects.

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) PUSH, POP, and MULTIPOP operations takes \( O(n^2) \) time.

**Pf.**
Multipop stack

**Goal.** Support operations on a set of elements:
- $\text{PUSH}(S, x)$: push object $x$ onto stack $S$.
- $\text{POP}(S)$: remove and return the most-recently added object.
- $\text{MULTIPOP}(S, k)$: remove the most-recently added $k$ objects.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$
$\text{PUSH}$, $\text{POP}$, and $\text{MULTIPOP}$ operations takes $O(n^2)$ time.

**Pf.**
- Use a singly-linked list.
Multipop stack

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- \( \text{PUSH}(S, x) \): push object \( x \) onto stack \( S \).
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**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) \( \text{PUSH} \), \( \text{POP} \), and \( \text{MULTIPOP} \) operations takes \( O(n^2) \) time.

**Pf.**
- Use a singly-linked list.
- \( \text{POP} \) and \( \text{PUSH} \) take \( O(1) \) time each.
**Multipop stack**

**Goal.** Support operations on a set of elements:
- \( \text{PUSH}(S, x) \): push object \( x \) onto stack \( S \).
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**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \)
\( \text{PUSH} \), \( \text{POP} \), and \( \text{MULTIPOP} \) operations takes \( O(n^2) \) time.

**Pf.**
- Use a singly-linked list.
- \( \text{POP} \) and \( \text{PUSH} \) take \( O(1) \) time each.
- \( \text{MULTIPOP} \) takes \( O(n) \) time.

![Diagram of a stack with elements 1, 4, 1, 3 and top marker]
Multipop stack

**Goal.** Support operations on a set of elements:
- **PUSH**(*S*, *x*): push object *x* onto stack *S*.
- **POP**(*S*): remove and return the most-recently added object.
- **MULTIPOP**(*S*, *k*): remove the most-recently added *k* objects.

**Theorem.** Starting from an empty stack, any intermixed sequence of *n* **PUSH**, **POP**, and **MULTIPOP** operations takes **O**(*n*²) time.

**Pf.**
- Use a singly-linked list.
- **POP** and **PUSH** take **O**(1) time each.
- **MULTIPOP** takes **O**(n) time.

---

overly pessimistic upper bound
Multipop stack: aggregate method

**Goal.** Support operations on a set of elements:
- $$\text{PUSH}(S, x)$$: push object $$x$$ onto stack $$S$$.
- $$\text{POP}(S)$$: remove and return the most-recently added object.
- $$\text{MULTIPOP}(S, k)$$: remove the most-recently added $$k$$ objects.

**Theorem.** Starting from an empty stack, any intermixed sequence of $$n$$ $$\text{PUSH}$$, $$\text{POP}$$, and $$\text{MULTIPOP}$$ operations takes $$O(n)$$ time.
Goal. Support operations on a set of elements:

- \( \text{PUSH}(S, x) \): push object \( x \) onto stack \( S \).
- \( \text{POP}(S) \): remove and return the most-recently added object.
- \( \text{MULTIPOP}(S, k) \): remove the most-recently added \( k \) objects.

Theorem. Starting from an empty stack, any intermixed sequence of \( n \) \( \text{PUSH} \), \( \text{POP} \), and \( \text{MULTIPOP} \) operations takes \( O(n) \) time.

Pf.
Multipop stack: aggregate method

**Goal.** Support operations on a set of elements:
- \(\text{PUSH}(S, x)\): push object \(x\) onto stack \(S\).
- \(\text{POP}(S)\): remove and return the most-recently added object.
- \(\text{MULTIPOP}(S, k)\): remove the most-recently added \(k\) objects.

**Theorem.** Starting from an empty stack, any intermixed sequence of \(n\) \(\text{PUSH}\), \(\text{POP}\), and \(\text{MULTIPOP}\) operations takes \(O(n)\) time.

**Pf.**
- An object is popped at most once for each time it is pushed onto stack.
Multipop stack: aggregate method

**Goal.** Support operations on a set of elements:
- \( \text{PUSH}(S, x) \): push object \( x \) onto stack \( S \).
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**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) \( \text{PUSH} \), \( \text{POP} \), and \( \text{MULTIPOP} \) operations takes \( O(n) \) time.

**Pf.**
- An object is popped at most once for each time it is pushed onto stack.
- There are \( \leq n \) \( \text{PUSH} \) operations.
Multipop stack: aggregate method

Goal. Support operations on a set of elements:
- \( \text{PUSH}(S, x) \): push object \( x \) onto stack \( S \).
- \( \text{POP}(S) \): remove and return the most-recently added object.
- \( \text{MULTIPOP}(S, k) \): remove the most-recently added \( k \) objects.

Theorem. Starting from an empty stack, any intermixed sequence of \( n \) \PUSH, \POP, and \MULTIPOP operations takes \( O(n) \) time.

Pf.
- An object is popped at most once for each time it is pushed onto stack.
- There are \( \leq n \) \PUSH operations.
- Thus, there are \( \leq n \) \POP operations
  (including those made within \MULTIPOP). ▪
Multipop stack: accounting method
Multipop stack: accounting method

**Credits.** One credit pays for a push or pop.
Multipop stack: accounting method

Credits. One credit pays for a push or pop.

Accounting.
Multipop stack: accounting method

Credits. One credit pays for a push or pop.

Accounting.
- \texttt{PUSH}(S, x): charge two credits.
  - use one credit to pay for pushing $x$ now
  - store one credit to pay for popping $x$ at some point in the future
Multipop stack: accounting method

**Credits.** One credit pays for a push or pop.

**Accounting.**

- **PUSH(S, x):** charge two credits.
  - use one credit to pay for pushing \( x \) now
  - store one credit to pay for popping \( x \) at some point in the future
- No other operation is charged a credit.
Multipop stack: accounting method

Credits. One credit pays for a push or pop.

Accounting.

• \textsc{PUSH}(S, x): charge two credits.
  - use one credit to pay for pushing \( x \) now
  - store one credit to pay for popping \( x \) at some point in the future
• No other operation is charged a credit.

Theorem. Starting from an empty stack, any intermixed sequence of \( n \) \textsc{PUSH}, \textsc{POP}, and \textsc{MULTIPOP} operations takes \( O(n) \) time.
Multipop stack: accounting method

Credits. One credit pays for a push or pop.

Accounting.
- \textbf{PUSH}(S, x): charge two credits.
  - use one credit to pay for pushing $x$ now
  - store one credit to pay for popping $x$ at some point in the future
- No other operation is charged a credit.

Theorem. Starting from an empty stack, any intermixed sequence of $n$
\textsc{Push}, \textsc{Pop}, and \textsc{MultiPop} operations takes $O(n)$ time.

\textbf{Pf.} The algorithm maintains the invariant that every object remaining on
the stack has 1 credit $\Rightarrow$ number of credits in data structure $\geq 0$. $\blacksquare$
Multipop stack: potential method
Potential function. Let $\Phi(D) = \text{number of objects currently on the stack}$. 

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$. 
Multipop stack: potential method

**Potential function.** Let $\Phi(D) =$ number of objects currently on the stack.

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**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTIPOP operations takes $O(n)$ time.
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Theorem. Starting from an empty stack, any intermixed sequence of $n$ \textsc{push}, \textsc{pop}, and \textsc{multiPop} operations takes $O(n)$ time.

Pf. [Case 1: push]
**Multipop stack: potential method**

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**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTIPOP operations takes $O(n)$ time.

**Pf.** [Case 1: push]
- Suppose that the $i^{th}$ operation is a PUSH.
Multipop stack: potential method

Potential function. Let \( \Phi(D) \) = number of objects currently on the stack.

- \( \Phi(D_0) = 0 \).
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Theorem. Starting from an empty stack, any intermixed sequence of \( n \) \texttt{PUSH}, \texttt{POP}, and \texttt{MULTIPOP} operations takes \( O(n) \) time.

Pf. [Case 1: push]

- Suppose that the \( i^{th} \) operation is a \texttt{PUSH}.
- The actual cost \( c_i = 1 \).
Multipop stack: potential method

Potential function. Let $\Phi(D) =$ number of objects currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTIPOP operations takes $O(n)$ time.

Pf. [Case 1: push]

- Suppose that the $i^{th}$ operation is a PUSH.
- The actual cost $c_i = 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2$. 
Multipop stack: potential method

**Potential function.** Let \( \Phi(D) = \) number of objects currently on the stack.
- \( \Phi(D_0) = 0. \)
- \( \Phi(D_i) \geq 0 \) for each \( D_i. \)

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) \texttt{PUSH}, \texttt{POP}, and \texttt{MULTIPOP} operations takes \( O(n) \) time.

**Pf.** [Case 2: pop]
Multipop stack: potential method

Potential function. Let $\Phi(D) =$ number of objects currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTIPOP operations takes $O(n)$ time.

Pf. [Case 2: pop]

- Suppose that the $i^{th}$ operation is a POP.
Multipop stack: potential method

**Potential function.** Let $\Phi(D) =$ number of objects currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ \texttt{PUSH}, \texttt{POP}, and \texttt{MULTIPOP} operations takes $O(n)$ time.

**Pf.** [Case 2: pop]

- Suppose that the $i^{th}$ operation is a \texttt{POP}.
- The actual cost $c_i = 1$. 
Multipop stack: potential method

**Potential function.** Let $\Phi(D) =$ number of objects currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ \textsc{push}, \textsc{pop}, and \textsc{multiPop} operations takes $O(n)$ time.

**Pf.** [Case 2: pop]

- Suppose that the $i^{th}$ operation is a \textsc{pop}.
- The actual cost $c_i = 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 - 1 = 0$. 
Multipop stack: potential method

**Potential function.** Let $\Phi(D) =$ number of objects currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTIPOP operations takes $O(n)$ time.

**Pf.** [Case 3: multipop]
Multipop stack: potential method

Potential function. Let $\Phi(D) =$ number of objects currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTIPOP operations takes $O(n)$ time.

Pf. [Case 3: multipop]

- Suppose that the $i^{th}$ operation is a MULTIPOP of $k$ objects.
Multipop stack: potential method

**Potential function.** Let \( \Phi(D) \) = number of objects currently on the stack.

- \( \Phi(D_0) = 0 \).
- \( \Phi(D_i) \geq 0 \) for each \( D_i \).

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) \textsc{push}, \textsc{pop}, and \textsc{multiPop} operations takes \( O(n) \) time.

**Pf.** [Case 3: multipop]

- Suppose that the \( i^{th} \) operation is a \textsc{multiPop} of \( k \) objects.
- The actual cost \( c_i = k \).
Multipop stack: potential method

**Potential function.** Let $\Phi(D) =$ number of objects currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ Push, Pop, and Multipop operations takes $O(n)$ time.

**Pf.** [Case 3: multipop]

- Suppose that the $i^{th}$ operation is a Multipop of $k$ objects.
- The actual cost $c_i = k$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = k - k = 0$. □
Amortized Analysis

- binary counter
- multipop stack
- dynamic table

Section 17.4
Dynamic table

**Goal.** Store items in a table (e.g., for hash table, binary heap).
- Two operations: **INSERT** and **DELETE**.
  - too many items inserted $\Rightarrow$ **expand** table.
  - too many items deleted $\Rightarrow$ **contract** table.
- Requirement: if table contains $m$ items, then space $= \Theta(m)$. 
Dynamic table

Goal. Store items in a table (e.g., for hash table, binary heap).
- Two operations: INSERT and DELETE.
  - too many items inserted ⇒ expand table.
  - too many items deleted ⇒ contract table.
- Requirement: if table contains $m$ items, then space = $\Theta(m)$.

Theorem. Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n^2)$ time.
Dynamic table

**Goal.** Store items in a table (e.g., for hash table, binary heap).

- Two operations: **INSERT** and **DELETE**.
  - too many items inserted ⇒ expand table.
  - too many items deleted ⇒ contract table.
- Requirement: if table contains \( m \) items, then space = \( \Theta(m) \).

**Theorem.** Starting from an empty dynamic table, any intermixed sequence of \( n \) **INSERT** and **DELETE** operations takes \( O(n^2) \) time.

**Pf.** A single **INSERT** or **DELETE** takes \( O(n) \) time. □
Dynamic table

**Goal.** Store items in a table (e.g., for hash table, binary heap).
- Two operations: **INSERT** and **DELETE**.
  - too many items inserted $\Rightarrow$ **expand** table.
  - too many items deleted $\Rightarrow$ **contract** table.
- Requirement: if table contains $m$ items, then space $= \Theta(m)$.

**Theorem.** Starting from an empty dynamic table, any intermixed sequence of $n$ **INSERT** and **DELETE** operations takes $O(n^2)$ time.

**Pf.** A single **INSERT** or **DELETE** takes $O(n)$ time. □
Dynamic table: insert only

- Initialize empty table of capacity 1.
- **INSERT**: if table is full, first copy all items to a table of twice the capacity.

<table>
<thead>
<tr>
<th>insert</th>
<th>old capacity</th>
<th>new capacity</th>
<th>insert cost</th>
<th>copy cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>8</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>8</td>
<td>1</td>
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<tr>
<td>9</td>
<td>8</td>
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<td>1</td>
<td>8</td>
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<td>⋮</td>
<td>⋮</td>
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<td>⋮</td>
</tr>
</tbody>
</table>

**Cost model.** Number of items written (due to insertion or copy).
Dynamic table: insert only (aggregate method)

**Theorem.** [via aggregate method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.
Dynamic table: insert only (aggregate method)

**Theorem.** [via aggregate method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** Let $c_i$ denote the cost of the $i^{th}$ insertion.

$$c_i = \begin{cases} 
  i & \text{if } i - 1 \text{ is an exact power of } 2 \\
  1 & \text{otherwise}
\end{cases}$$
Dynamic table: insert only (aggregate method)

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\]

Starting from empty table, the cost of a sequence of \( n \) INSERT operations is:
Dynamic table: insert only (aggregate method)

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Starting from empty table, the cost of a sequence of \( n \) \textsc{insert} operations is:

\[
\sum_{i=1}^{n} c_i \leq n + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j
\]
Dynamic table: insert only (aggregate method)

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\[
\sum_{i=1}^{n} c_i \leq n + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j < n + 2n
\]
Dynamic table: insert only (aggregate method)

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Starting from empty table, the cost of a sequence of $n$ INSERT operations is:

$$\sum_{i=1}^{n} c_i \leq n + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j$$

$$< n + 2n$$

$$= 3n \quad \blacksquare$$
Dynamic table: insert only (accounting method)

WLOG, can assume the table fills from left to right.

| 1 | 2 |   |
Dynamic table: insert only (accounting method)

WLOG, can assume the table fills from left to right.

| 1 | 2 | 3 |
Dynamic table: insert only (accounting method)

WLOG, can assume the table fills from left to right.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

"www."
Dynamic table: insert only (accounting method)

WLOG, can assume the table fills from left to right.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th></th>
<th></th>
</tr>
</thead>
</table>
Dynamic table: insert only (accounting method)

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<table>
<thead>
<tr>
<th>1</th>
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<th>3</th>
<th>4</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
</table>
Dynamic table: insert only (accounting method)

WLOG, can assume the table fills from left to right.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="images/dice.png" alt="Dice" /> <img src="images/dice.png" alt="Dice" /></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="images/dice.png" alt="Dice" /> <img src="images/dice.png" alt="Dice" /></td>
<td></td>
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</tr>
</tbody>
</table>
Dynamic table: insert only (accounting method)

WLOG, can assume the table fills from left to right.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</thead>
</table>

[Image of chips]

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
</table>

[Image of chips]
Dynamic table:  insert only (accounting method)

WLOG, can assume the table fills from left to right.
Dynamic table: insert only (accounting method)

WLOG, can assume the table fills from left to right.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
</table>

![Poker chips](image)

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
</table>

![Poker chips](image)

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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</thead>
</table>

![Poker chips](image)
Dynamic table: insert only (accounting method)

WLOG, can assume the table fills from left to right.

<table>
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<th>1</th>
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</tr>
</thead>
</table>

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<th>7</th>
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<tr>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
</table>
Dynamic table: insert only (accounting method)

WLOG, can assume the table fills from left to right.

<table>
<thead>
<tr>
<th>1</th>
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</thead>
</table>

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<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
</table>

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
WLOG, can assume the table fills from left to right.

Dynamic table: insert only (accounting method)

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
</table>

![Table with 1 to 4 tokens]

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
</table>

![Table with 1 to 8 tokens]

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
</table>

![Table with 1 to 11 tokens]
Dynamic table: insert only (accounting method)

WLOG, can assume the table fills from left to right.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1</th>
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<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
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</thead>
<tbody>
<tr>
<td><img src="image7.png" alt="Image" /></td>
<td><img src="image8.png" alt="Image" /></td>
<td><img src="image9.png" alt="Image" /></td>
<td><img src="image10.png" alt="Image" /></td>
<td><img src="image11.png" alt="Image" /></td>
<td><img src="image12.png" alt="Image" /></td>
<td><img src="image13.png" alt="Image" /></td>
<td><img src="image14.png" alt="Image" /></td>
<td><img src="image15.png" alt="Image" /></td>
<td><img src="image16.png" alt="Image" /></td>
<td><img src="image17.png" alt="Image" /></td>
<td><img src="image18.png" alt="Image" /></td>
</tr>
</tbody>
</table>
Dynamic table: insert only (accounting method)

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<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
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</table>

![Diagram of tokens]
WLOG, can assume the table fills from left to right.

<table>
<thead>
<tr>
<th>1</th>
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<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
</table>

Dynamic table: insert only (accounting method)
Dynamic table: insert only (accounting method)

WLOG, can assume the table fills from left to right.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
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<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Token Image" /></td>
<td><img src="image2.png" alt="Token Image" /></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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<th>1</th>
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<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image3.png" alt="Token Image" /></td>
<td><img src="image4.png" alt="Token Image" /></td>
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<td><img src="image6.png" alt="Token Image" /></td>
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Dynamic table: insert only (accounting method)

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<p>| | | | |</p>
<table>
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<tbody>
<tr>
<td>1</td>
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</tbody>
</table>

![Images of tokens]

<p>| | | | | | | | |</p>
<table>
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</table>

![Images of tokens]

<p>| | | | | | | | | | | | | | | | |</p>
<table>
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<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
</tbody>
</table>
Dynamic table: insert only (accounting method)

Accounting.
Dynamic table: insert only (accounting method)

Accounting.

- **INSERT:** charge 3 credits (use 1 credit to insert; save 2 with new item).
Dynamic table: insert only (accounting method)

Accounting.
• INSERT: charge 3 credits (use 1 credit to insert; save 2 with new item).

Theorem. [via accounting method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.
Dynamic table: insert only (accounting method)

**Accounting.**
- **INSERT:** charge 3 credits (use 1 credit to insert; save 2 with new item).

**Theorem.** [via accounting method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** The algorithm maintains the invariant that there are 2 credits with each item in right half of table.
Dynamic table: insert only (accounting method)

Accounting.
- **INSERT**: charge 3 credits (use 1 credit to insert; save 2 with new item).

**Theorem.** [via accounting method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** The algorithm maintains the invariant that there are 2 credits with each item in right half of table.
- When table doubles, one-half of the items in the table have 2 credits.
Dynamic table: insert only (accounting method)

Accounting.
• **INSERT**: charge 3 credits (use 1 credit to insert; save 2 with new item).

**Theorem.** [via accounting method] Starting from an empty dynamic table, any sequence of \( n \) **INSERT** operations takes \( O(n) \) time.

**Pf.** The algorithm maintains the invariant that there are 2 credits with each item in right half of table.
• When table doubles, one-half of the items in the table have 2 credits.
• This pays for the work needed to double the table. □
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

number of elements  
capacity of array
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

**Pf.** Let \( \Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i) \).

\[ \begin{align*}
&\text{number of elements} \\
&\text{capacity of array}
\end{align*} \]

**Case 1.** [does not trigger expansion] \( \text{size}(D_i) \leq \text{capacity}(D_{i-1}) \).
**Dynamic table: insert only (potential method)**

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

Case 1. [does not trigger expansion] $\text{size}(D_i) \leq \text{capacity}(D_{i-1})$.

- Actual cost $c_i = 1$. 
Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

Case 1. [does not trigger expansion] $\text{size}(D_i) \leq \text{capacity}(D_{i-1})$.

- Actual cost $c_i = 1$.
- $\Phi(D_i) - \Phi(D_{i-1}) = 2$. 
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

Case 1. [does not trigger expansion] $\text{size}(D_i) \leq \text{capacity}(D_{i-1})$.

- Actual cost $c_i = 1$.
- $\Phi(D_i) - \Phi(D_{i-1}) = 2$.
- Amortized costs $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 2 = 3$. 
Theorem. [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

**Case 1.** [does not trigger expansion] $\text{size}(D_i) \leq \text{capacity}(D_{i-1})$.
- Actual cost $c_i = 1$.
- $\Phi(D_i) - \Phi(D_{i-1}) = 2$.
- Amortized costs $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 2 = 3$.

**Case 2.** [triggers expansion] $\text{size}(D_i) = 1 + \text{capacity}(D_{i-1})$. 
Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of \( n \) \textsc{insert} operations takes \( O(n) \) time.

\textbf{Pf.} Let \( \Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i) \).

Case 1. [does not trigger expansion] \( \text{size}(D_i) \leq \text{capacity}(D_{i-1}) \).
  \begin{itemize}
  \item Actual cost \( c_i = 1 \).
  \item \( \Phi(D_i) - \Phi(D_{i-1}) = 2 \).
  \item Amortized costs \( \hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 2 = 3 \).
  \end{itemize}

Case 2. [triggers expansion] \( \text{size}(D_i) = 1 + \text{capacity}(D_{i-1}) \).
  \begin{itemize}
  \item Actual cost \( c_i = 1 + \text{capacity}(D_{i-1}) \).
  \end{itemize}
Theorem. [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

Case 1. [does not trigger expansion] $\text{size}(D_i) \leq \text{capacity}(D_{i-1})$.

- Actual cost $c_i = 1$.
- $\Phi(D_i) - \Phi(D_{i-1}) = 2$.
- Amortized costs $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 2 = 3$.

Case 2. [triggers expansion] $\text{size}(D_i) = 1 + \text{capacity}(D_{i-1})$.

- Actual cost $c_i = 1 + \text{capacity}(D_{i-1})$.
- $\Phi(D_i) - \Phi(D_{i-1}) = 2 - \text{capacity}(D_i) + \text{capacity}(D_{i-1}) = 2 - \text{capacity}(D_{i-1})$. 

---

number of elements  \[\uparrow\]  capacity of array
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of $n$ \textsc{insert} operations takes $O(n)$ time.

**Pf.** Let $\Phi(D_i) = 2 \ size(D_i) - \ capacity(D_i)$.

**Case 1.** [does not trigger expansion] \ $\ size(D_i) \leq \ capacity(D_{i-1})$.

- Actual cost $c_i = 1$.
- $\Phi(D_i) - \Phi(D_{i-1}) = 2$.
- Amortized costs $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 2 = 3$.

**Case 2.** [triggers expansion] \ $\ size(D_i) = 1 + \ capacity(D_{i-1})$.

- Actual cost $c_i = 1 + \ capacity(D_{i-1})$.
- $\Phi(D_i) - \Phi(D_{i-1}) = 2 - \ capacity(D_i) + \ capacity(D_{i-1}) = 2 - \ capacity(D_{i-1})$.
- Amortized costs $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 2 = 3$. ■
Dynamic table: doubling and halving

Thrashing.

- Initialize table to be of fixed capacity, say 1.
- INSERT: if table is full, expand to a table of twice the capacity.
- DELETE: if table is $\frac{1}{2}$-full, contract to a table of half the capacity.
Dynamic table: doubling and halving

Thrashing.
- Initialize table to be of fixed capacity, say 1.
- **INSERT**: if table is full, expand to a table of twice the capacity.
- **DELETE**: if table is \( \frac{1}{2} \)-full, contract to a table of half the capacity.

Efficient solution.
- Initialize table to be of fixed capacity, say 1.
- **INSERT**: if table is full, expand to a table of twice the capacity.
- **DELETE**: if table is \( \frac{1}{4} \)-full, contract to a table of half the capacity.
Dynamic table: doubling and halving

Thrashing.
- Initialize table to be of fixed capacity, say 1.
- **INSERT**: if table is full, expand to a table of twice the capacity.
- **DELETE**: if table is $\frac{1}{2}$-full, contract to a table of half the capacity.

Efficient solution.
- Initialize table to be of fixed capacity, say 1.
- **INSERT**: if table is full, expand to a table of twice the capacity.
- **DELETE**: if table is $\frac{1}{4}$-full, contract to a table of half the capacity.

Memory usage. A dynamic table uses $O(n)$ memory to store $n$ items.
Dynamic table: doubling and halving

Thrashing.
- Initialize table to be of fixed capacity, say 1.
- \texttt{INSERT}: if table is full, expand to a table of twice the capacity.
- \texttt{DELETE}: if table is $\frac{1}{2}$-full, contract to a table of half the capacity.

Efficient solution.
- Initialize table to be of fixed capacity, say 1.
- \texttt{INSERT}: if table is full, expand to a table of twice the capacity.
- \texttt{DELETE}: if table is $\frac{1}{4}$-full, contract to a table of half the capacity.

Memory usage. A dynamic table uses $O(n)$ memory to store $n$ items.
Pf. Table is always at least $\frac{1}{4}$-full (provided it is not empty). □
Dynamic table: insert and delete (accounting method)

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<th>7</th>
<th>8</th>
</tr>
</thead>
</table>
## Dynamic table: insert and delete (accounting method)

<p>| | | | | | | | |</p>
<table>
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</tbody>
</table>
Dynamic table: insert and delete (accounting method)

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**insert**
Dynamic table: insert and delete (accounting method)

<table>
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Dynamic table: insert and delete (accounting method)

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**insert**
### Dynamic table: insert and delete (accounting method)

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</table>

![Insert operation diagram]
## Dynamic table: insert and delete (accounting method)

### Insert

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### Delete

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</tbody>
</table>
# Dynamic table: insert and delete (accounting method)

## Insert

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</tr>
</thead>
</table>

## Delete

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|-----|-----|-----|


Dynamic table: insert and delete (accounting method)

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<thead>
<tr>
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<th>12</th>
</tr>
</thead>
</table>

**insert**

**delete**
Dynamic table: insert and delete (accounting method)

<table>
<thead>
<tr>
<th>insert</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<th>12</th>
</tr>
</thead>
</table>

delete

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |     |     |     |     |
Dynamic table: insert and delete (accounting method)

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</tbody>
</table>

**insert**

**delete**
Dynamic table: insert and delete (accounting method)

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<tbody>
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### Dynamic table: insert and delete (accounting method)

#### insert

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#### delete

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Dynamic table: insert and delete (accounting method)

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Dynamic table: insert and delete (accounting method)

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**insert**

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**delete**

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</table>
Dynamic table: insert and delete (accounting method)

**Insert**

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</tbody>
</table>

**Delete**

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<tbody>
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</table>

**Resize and Delete**

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</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
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</tbody>
</table>
Dynamic table: insert and delete (accounting method)

<table>
<thead>
<tr>
<th>insert</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>delete</th>
<th>1</th>
<th>2</th>
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</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>resize and delete</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
</table>
Dynamic table: insert and delete (accounting method)

Accounting.
Dynamic table: insert and delete (accounting method)

Accounting.

- **INSERT**: charge 3 credits (1 credit for insert; save 2 with new item).

discard any existing credits
Dynamic table: insert and delete (accounting method)

Accounting.

- **INSERT**: charge 3 credits (1 credit for insert; save 2 with new item).
- **DELETE**: charge 2 credits (1 credit to delete, save 1 in emptied slot).

discard any existing credits
Dynamic table: insert and delete (accounting method)

Accounting.

- **INSERT**: charge 3 credits (1 credit for insert; save 2 with new item).
- **DELETE**: charge 2 credits (1 credit to delete, save 1 in emptied slot).

![discard any existing credits]

**Theorem.** [via accounting method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.
Dynamic table: insert and delete (accounting method)

Accounting.
- **INSERT**: charge 3 credits (1 credit for insert; save 2 with new item).
- **DELETE**: charge 2 credits (1 credit to delete, save 1 in emptied slot).

Theorem. [via accounting method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.

Pf. The algorithm maintains the invariant that there are 2 credits with each item in the right half of table; 1 credit with each empty slot in the left half.
Dynamic table: insert and delete (accounting method)

Accounting.

- **INSERT**: charge 3 credits (1 credit for insert; save 2 with new item).
- **DELETE**: charge 2 credits (1 credit to delete, save 1 in emptied slot).

**Theorem.** [via accounting method] Starting from an empty dynamic table, any intermixed sequence of \(n\) **INSERT** and **DELETE** operations takes \(O(n)\) time.

**Pf.** The algorithm maintains the invariant that there are 2 credits with each item in the right half of table; 1 credit with each empty slot in the left half.

- When table doubles, each item in right half of table has 2 credits.
Dynamic table: insert and delete (accounting method)

Accounting.
- **INSERT**: charge 3 credits (1 credit for insert; save 2 with new item).
- **DELETE**: charge 2 credits (1 credit to delete, save 1 in emptied slot).

Theorem. [via accounting method] Starting from an empty dynamic table, any intermixed sequence of \( n \) INSERT and DELETE operations takes \( O(n) \) time.

Pf. The algorithm maintains the invariant that there are 2 credits with each item in the right half of table; 1 credit with each empty slot in the left half.
- When table doubles, each item in right half of table has 2 credits.
- When table halves, each empty slot in left half of table has 1 credit. ▪
Dynamic table: insert and delete (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.
Theorem. [via potential method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.

Pf sketch.
Dynamic table: insert and delete (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.

**Pf sketch.**

- Let $\alpha(D_i) = \frac{\text{size}(D_i)}{\text{capacity}(D_i)}$. 
Dynamic table: insert and delete (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any intermixed sequence of \( n \) INSERT and DELETE operations takes \( O(n) \) time.

**Pf sketch.**

- Let \( \alpha(D_i) = \text{size}(D_i) / \text{capacity}(D_i) \).

- Define \( \Phi(D_i) = \begin{cases} 
2 \text{ size}(D_i) - \text{capacity}(D_i) & \text{if } \alpha(D_i) \geq 1/2 \\
\frac{1}{2} \text{ capacity}(D_i) - \text{size}(D_i) & \text{if } \alpha(D_i) < 1/2 
\end{cases} \)
Dynamic table: insert and delete (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.

**Pf sketch.**

- Let $\alpha(D_i) = \text{size}(D_i) / \text{capacity}(D_i)$.

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2 \text{size}(D_i) - \text{capacity}(D_i) & \text{if } \alpha(D_i) \geq 1/2 \\
\frac{1}{2} \text{capacity}(D_i) - \text{size}(D_i) & \text{if } \alpha(D_i) < 1/2 
\end{cases}$

- When $\alpha(D) = 1/2$, $\Phi(D) = 0$. [zero potential after resizing]
Dynamic table: insert and delete (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.

Pf sketch.

• Let $\alpha(D_i) = \frac{\text{size}(D_i)}{\text{capacity}(D_i)}$.

• Define $\Phi(D_i) = \begin{cases} 2 \text{size}(D_i) - \text{capacity}(D_i) & \text{if } \alpha(D_i) \geq 1/2 \\ \frac{1}{2} \text{capacity}(D_i) - \text{size}(D_i) & \text{if } \alpha(D_i) < 1/2 \end{cases}$

• When $\alpha(D) = 1/2$, $\Phi(D) = 0$. [zero potential after resizing]
• When $\alpha(D) = 1$, $\Phi(D) = \text{size}(D_i)$. [can pay for expansion]
Dynamic table: insert and delete (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.

**Pf sketch.**

- Let $\alpha(D_i) = \frac{\text{size}(D_i)}{\text{capacity}(D_i)}$.

- Define $\Phi(D_i) = \begin{cases} 
  2\ \text{size}(D_i) - \text{capacity}(D_i) & \text{if } \alpha(D_i) \geq 1/2 \\
  \frac{1}{2} \ \text{capacity}(D_i) - \text{size}(D_i) & \text{if } \alpha(D_i) < 1/2 
\end{cases}$

- When $\alpha(D) = 1/2$, $\Phi(D) = 0$. [zero potential after resizing]
- When $\alpha(D) = 1$, $\Phi(D) = \text{size}(D_i)$. [can pay for expansion]
- When $\alpha(D) = 1/4$, $\Phi(D) = \text{size}(D_i)$. [can pay for contraction]

...