DATA STRUCTURES I, II, III, AND IV

I. Amortized Analysis
II. Binary and Binomial Heaps
III. Fibonacci Heaps
IV. Union–Find
Data structures

**Static problems.** Given an input, produce an output.
Data structures

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**Ex.** Sorting, FFT, edit distance, shortest paths, MST, max-flow, ...
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Ex. Array, linked list, binary heap, binary search tree, hash table, ...
Appetizer

**Goal.** Design a data structure to support all operations in $O(1)$ time.

- **INIT($n$):** create and return an *initialized* array (all zero) of length $n$.
- **READ($A$, $i$):** return element $i$ in array.
- **WRITE($A$, $i$, value):** set element $i$ in array to $value$. 
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**Assumptions.**

- Can **MALLOC** an uninitialized array of length $n$ in $O(1)$ time.
- Given an array, can read or write element $i$ in $O(1)$ time.
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Assumptions.
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Remark. An array does INIT in $\Theta(n)$ time and READ and WRITE in $\Theta(1)$ time.

- $A[i]$ stores the current value for READ (if initialized).

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<tr>
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<tbody>
<tr>
<td>A</td>
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Appetizer

**Data structure.** Three arrays $A[1..n]$, $B[1..n]$, and $C[1..n]$, and an integer $k$.

- $A[i]$ stores the current value for READ (if initialized).
- $k = \text{number of initialized entries}$.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\end{array}
\]

$k = 4$

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<tr>
<th>A[ ]</th>
<th>1</th>
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<th>C[ ]</th>
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$k = 4$


- $A[i]$ stores the current value for READ (if initialized).
- $k$ = number of initialized entries.
- $C[j]$ = index of $j^{th}$ initialized element for $j = 1, \ldots, k$.
- If $C[j] = i$, then $B[i] = j$ for $j = 1, \ldots, k$.

$$
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\hline
\end{array}
$$

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**Theorem.** $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

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<tr>
<td>$A[]$</td>
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<tr>
<td>$B[]$</td>
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<td>?</td>
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<tr>
<td>$C[]$</td>
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<td>?</td>
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$k = 4$


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Pf. Ahead.

\[
\begin{array}{cccccccc}
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\hline
\end{array}
\]

$k = 4$

Appetizer
**INIT** \((A, n)\)

\[
k \leftarrow 0.
\]

\[
A \leftarrow \text{MALLOC}(n).
\]

\[
B \leftarrow \text{MALLOC}(n).
\]

\[
C \leftarrow \text{MALLOC}(n).
\]
**Appetizer**

**INIT** \((A, n)\)

\[ k \leftarrow 0. \]
\[ A \leftarrow \text{ALLOC}(n). \]
\[ B \leftarrow \text{ALLOC}(n). \]
\[ C \leftarrow \text{ALLOC}(n). \]

**READ** \((A, i)\)

**IF** (IS-INITIALIZED \((A[i])\))

\[ \text{RETURN } A[i]. \]

**ELSE**

\[ \text{RETURN } 0. \]
**Appetizer**

**INIT \((A, n)\)**

\[
\begin{align*}
k & \leftarrow 0. \\
A & \leftarrow \text{MALLOC}(n). \\
B & \leftarrow \text{MALLOC}(n). \\
C & \leftarrow \text{MALLOC}(n).
\end{align*}
\]

**READ \((A, i)\)**

\[
\begin{align*}
\text{IF (IS-INITIALIZED \((A[i])\))} & \\
\text{RETURN } A[i]. \\
\text{ELSE} & \\
\text{RETURN } 0.
\end{align*}
\]

**WRITE \((A, i, value)\)**

\[
\begin{align*}
\text{IF (IS-INITIALIZED \((A[i])\))} & \\
A[i] & \leftarrow value. \\
\text{ELSE} & \\
k & \leftarrow k + 1. \\
A[i] & \leftarrow value. \\
B[i] & \leftarrow k. \\
C[k] & \leftarrow i.
\end{align*}
\]
\textbf{Appetizer}

\begin{align*}
\textbf{INIT} (A, n) \quad & k \leftarrow 0. \\
& A \leftarrow \text{MALLOC}(n). \\
& B \leftarrow \text{MALLOC}(n). \\
& C \leftarrow \text{MALLOC}(n). \\
\end{align*}

\begin{align*}
\textbf{READ} (A, i) \quad & \text{IF (IS-INITIALIZED} (A[i]) \text{)} \\
& \quad \text{RETURN} \ A[i]. \\
& \quad \text{ELSE} \\
& \quad \text{RETURN} \ 0. \\
\end{align*}

\begin{align*}
\textbf{WRITE} (A, i, value) \quad & \text{IF (IS-INITIALIZED} (A[i]) \text{)} \\
& \quad A[i] \leftarrow value. \\
& \quad \text{ELSE} \\
& \quad k \leftarrow k + 1. \\
& \quad A[i] \leftarrow value. \\
& \quad B[i] \leftarrow k. \\
& \quad C[k] \leftarrow i. \\
\end{align*}

\begin{align*}
\textbf{IS-INITIALIZED} (A, i) \quad & \text{IF} \ (1 \leq B[i] \leq k) \text{ and } (C[B[i]] = i) \\
& \quad \text{RETURN} \ true. \\
& \quad \text{ELSE} \\
& \quad \text{RETURN} \ false. \\
\end{align*}
Theorem. \( A[i] \) is initialized iff both \( 1 \leq B[i] \leq k \) and \( C[B[i]] = i \).

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
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\( k = 4 \)

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**Pf.** ⇒

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Theorem. $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

Pf. ⇒

- Suppose $A[i]$ is the $j^{th}$ entry to be initialized.

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Pf. \( \Rightarrow \)
- Suppose \( A[i] \) is the \( j^{th} \) entry to be initialized.
- Then \( C[j] = i \) and \( B[i] = j \).

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\begin{array}{cccccccc}
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\( k = 4 \)

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**Pf.** $\Rightarrow$

- Suppose $A[i]$ is the $j^{th}$ entry to be initialized.
- Then $C[j] = i$ and $B[i] = j$.
- Thus, $C[B[i]] = i$.

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**Theorem.** $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

**Pf.** $\Leftarrow$

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$k = 4$

**Theorem.** $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

**Pf.** $\iff$

- Suppose $A[i]$ is uninitialized.

\[\begin{array}{c@{}c@{}c@{}c@{}c@{}c@{}c@{}c@{}c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
A & \? & 22 & 55 & 99 & \? & 33 & \? & \?
\end{array}\]

\[\begin{array}{c@{}c@{}c@{}c@{}c@{}c@{}c@{}c@{}c}
& & & & & & & \\
\hline
B & \? & 3 & 4 & 1 & \? & 2 & \? & \?
\end{array}\]

\[\begin{array}{c@{}c@{}c@{}c@{}c@{}c@{}c}
4 & 6 & 2 & 3 & \? & \? & \?
\end{array}\]

$k = 4$

Theorem. $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

Pf. \( \Leftarrow \)

- Suppose $A[i]$ is uninitialized.
- If $B[i] < 1$ or $B[i] > k$, then $A[i]$ clearly uninitialized.

\[ k = 4 \]

\[
\begin{array}{cccccccc}
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\hline
\hline
\hline
\end{array}
\]

Theorem. \( A[i] \) is initialized iff both \( 1 \leq B[i] \leq k \) and \( C[B[i]] = i \).

Pf. \( \Leftarrow \)

- Suppose \( A[i] \) is uninitialized.
- If \( B[i] < 1 \) or \( B[i] > k \), then \( A[i] \) clearly uninitialized.
- If \( 1 \leq B[i] \leq k \) by coincidence, then we still can’t have \( C[B[i]] = i \) because none of the entries \( C[1..k] \) can equal \( i \). □

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

\( k = 4 \)

Amortized Analysis

- binary counter
- multi-pop stack
- dynamic table
Amortized analysis

**Worst-case analysis.** Determine worst-case running time of a data structure operation as function of the input size $n$. 
Amortized analysis

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_can be too pessimistic if the only way to encounter an expensive operation is when there were lots of previous cheap operations_
Amortized analysis

Worst-case analysis. Determine worst-case running time of a data structure operation as function of the input size $n$.

Amortized analysis. Determine worst-case running time of a sequence of $n$ data structure operations.

Can be too pessimistic if the only way to encounter an expensive operation is when there were lots of previous cheap operations.
Amortized analysis

Worst-case analysis. Determine worst-case running time of a data structure operation as function of the input size $n$.

Amortized analysis. Determine worst-case running time of a sequence of $n$ data structure operations.

Ex. Starting from an empty stack implemented with a dynamic table, any sequence of $n$ push and pop operations takes $O(n)$ time in the worst case.
Amortized analysis: applications

- Splay trees.
- Dynamic table.
- Fibonacci heaps.
- Garbage collection.
- Move-to-front list updating.
- Push–relabel algorithm for max flow.
- Path compression for disjoint-set union.
- Structural modifications to red–black trees.
- Security, databases, distributed computing, ...
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**AMORTIZED COMPUTATIONAL COMPLEXITY**

**ROBERT ENDRE TARJAN**

Abstract. A powerful technique in the complexity analysis of data structures is amortization, or averaging over time. Amortized running time is a realistic but robust complexity measure for which we can obtain surprisingly tight upper and lower bounds on a variety of algorithms. By following the principle of designing algorithms whose amortized complexity is low, we obtain “self-adjusting” data structures that are simple, flexible and efficient. This paper surveys recent work by several researchers on amortized complexity.

ASM(MOS) subject classifications. 68C25, 68E05
Amortized Analysis

- binary counter
- multi-pop stack
- dynamic table

Chapter 17
Binary counter

**Goal.** Increment a $k$-bit binary counter (mod $2^k$).

**Representation.** $a_j = j^{th}$ least significant bit of counter.

<table>
<thead>
<tr>
<th>Counter value</th>
<th>$A7$</th>
<th>$A6$</th>
<th>$A5$</th>
<th>$A4$</th>
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**Binary counter**

**Goal.** Increment a $k$-bit binary counter (mod $2^k$).

**Representation.** $a_j = j^{th}$ least significant bit of counter.

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**Cost model.** Number of bits flipped.
**Binary counter**

**Goal.** Increment a $k$-bit binary counter (mod $2^k$).

**Representation.** $a_j = j^{th}$ least significant bit of counter.

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**Theorem.** Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(nk)$ bits.
Binary counter

Goal. Increment a $k$-bit binary counter (mod $2^k$).

Representation. $a_j = j^{th}$ least significant bit of counter.

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Theorem. Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(nk)$ bits.

Pf. At most $k$ bits flipped per increment. □
Binary counter

**Goal.** Increment a $k$-bit binary counter (mod $2^k$).

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**Theorem.** Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(nk)$ bits.  

**Pf.** At most $k$ bits flipped per increment.  ■
Aggregate method (brute force)

**Aggregate method.** Analyze cost of a sequence of operations.

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</tbody>
</table>
Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:
Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:

- Bit 0 flips $n$ times.
Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:

- Bit 0 flips $n$ times.
- Bit 1 flips $\lfloor n / 2 \rfloor$ times.
Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:

- Bit 0 flips $n$ times.
- Bit 1 flips $\lfloor n / 2 \rfloor$ times.
- Bit 2 flips $\lfloor n / 4 \rfloor$ times.
Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:

- Bit 0 flips $n$ times.
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**Theorem.** Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(n)$ bits.
Binary counter: aggregate method

Starting from the zero counter, in a sequence of \( n \) INCREMENT operations:
- Bit 0 flips \( n \) times.
- Bit 1 flips \( \lfloor n/2 \rfloor \) times.
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**Theorem.** Starting from the zero counter, a sequence of \( n \) INCREMENT operations flips \( O(n) \) bits.

**Pf.**
Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:

- Bit 0 flips $n$ times.
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- ...

**Theorem.** Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(n)$ bits.

**Pf.**

- Bit $j$ flips $\lceil n / 2^j \rceil$ times.
Binary counter: aggregate method

Starting from the zero counter, in a sequence of \( n \) \textsc{increment} operations:

- Bit 0 flips \( n \) times.
- Bit 1 flips \( \lfloor n/2 \rfloor \) times.
- Bit 2 flips \( \lfloor n/4 \rfloor \) times.
- ...

\textbf{Theorem.} Starting from the zero counter, a sequence of \( n \) \textsc{increment} operations flips \( O(n) \) bits.

\textbf{Pf.}

- Bit \( j \) flips \( \lfloor n/2^j \rfloor \) times.
- The total number of bits flipped is \( \sum_{j=0}^{k-1} \lfloor n/2^j \rfloor \)
Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:

- Bit 0 flips $n$ times.
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- Bit $j$ flips $\lfloor n/2^j \rfloor$ times.
- The total number of bits flipped is

$$
\sum_{j=0}^{k-1} \lfloor \frac{n}{2^j} \rfloor < n \sum_{j=0}^{\infty} \frac{1}{2^j}
$$


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- The total number of bits flipped is 
  \[
  \sum_{j=0}^{k-1} \lfloor \frac{n}{2^j} \rfloor < n \sum_{j=0}^{\infty} \frac{1}{2^j} = 2n \]
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**Remark.** Theorem may be false if initial counter is not zero.
Accounting method (banker’s method)
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Assign (potentially) different charges to each operation.
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- $D_i = \text{data structure after } i^{th} \text{ operation.}$
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can be more or less than actual cost
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- When $\hat{c}_i > c_i$, we store credits in data structure $D_i$ to pay for future ops; when $\hat{c}_i < c_i$, we consume credits in data structure $D_i$. 

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- Initial data structure $D_0$ starts with 0 credits.

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Credit invariant. The total number of credits in the data structure $\geq 0$. 

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Credit invariant. The total number of credits in the data structure $\geq 0$.

$$\sum_{i=1}^{\infty} \hat{c}_i - \sum_{i=1}^{\infty} c_i \geq 0$$

our job is to choose suitable amortized costs so that this invariant holds
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Credit invariant. The total number of credits in the data structure $\geq 0$.

$$\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0$$

Theorem. Starting from the initial data structure $D_0$, the total actual cost of any sequence of $n$ operations is at most the sum of the amortized costs.
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**Credit invariant.** The total number of credits in the data structure $\geq 0$.

$$\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0$$

**Theorem.** Starting from the initial data structure $D_0$, the total actual cost of any sequence of $n$ operations is at most the sum of the amortized costs.

**Pf.** The amortized cost of the sequence of $n$ operations is:

$$\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i.$$
Accounting method (banker’s method)

Assign (potentially) different charges to each operation.

- \( D_i \) = data structure after \( i^{th} \) operation.
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- When \( \hat{c}_i > c_i \), we store credits in data structure \( D_i \) to pay for future ops; when \( \hat{c}_i < c_i \), we consume credits in data structure \( D_i \).
- Initial data structure \( D_0 \) starts with 0 credits.

Credit invariant. The total number of credits in the data structure \( D_i \) is at least 0.

\[
\sum_{i=1}^{\infty} \hat{c}_i - \sum_{i=1}^{\infty} c_i \geq 0
\]

Theorem. Starting from the initial data structure \( D_0 \), the total actual cost of any sequence of \( n \) operations is at most the sum of the amortized costs.

Pf. The amortized cost of the sequence of \( n \) operations is: \( \sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i \). □

Intuition. Measure running time in terms of credits (time = money).
Binary counter: accounting method

Credits. One credit pays for a bit flip.

<table>
<thead>
<tr>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
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</table>
**Binary counter: accounting method**

**Credits.** One credit pays for a bit flip.

**Invariant.** Each 1 bit has one credit; each 0 bit has zero credits.

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- Flip bit $j$ from 0 to 1: charge 2 credits (use one and save one in bit $j$).
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```
increment

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![Tokens]
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![Binary counter circuit diagram]
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Binary counter: accounting method

Credits. One credit pays for a bit flip.

Invariant. Each 1 bit has one credit; each 0 bit has zero credits.

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- Flip bit $j$ from 0 to 1: charge 2 credits (use one and save one in bit $j$).
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increment

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![Chip icons]
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- Each INCREMENT operation flips at most one 0 bit to a 1 bit, so the amortized cost per INCREMENT $\leq 2$. 

the rightmost 0 bit (unless counter overflows)
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- Total actual cost of $n$ operations $\leq$ sum of amortized costs $\leq 2n$. ▪

accounting method theorem

the rightmost 0 bit (unless counter overflows)
Potential method (physicist’s method)

Potential function. $\Phi(D_i)$ maps each data structure $D_i$ to a real number s.t.:

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each data structure $D_i$. 

\[ \Phi(D_i) \to \text{real number} \]
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\[
\geq \sum_{i=1}^{n} c_i \quad \blacksquare
\]
Binary counter: potential method

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  \[ \leq c_i + 1 - t_i \] \( \rightarrow \) potential decreases by 1 for \( t_i \) bits flipped from 1 to 0 and increases by 1 for bit flipped from 0 to 1
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  \[
  \leq c_i + 1 - t_i \quad \text{potential decreases by 1 for} \ t_i \text{bits flipped from 1 to 0 and increases by 1 for bit flipped from 0 to 1}
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- Total actual cost of \( n \) operations \( \leq \) sum of amortized costs \( \leq 2n \).
Famous potential functions
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Fibonacci heaps. \( \Phi(H) = 2 \text{trees}(H) + 2 \text{marks}(H) \)
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**Fibonacci heaps.** \( \Phi(H) = 2 \text{trees}(H) + 2 \text{marks}(H) \)

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Famous potential functions

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**Preflow-push.** $\Phi(f) = \sum_{v : \text{excess}(v) > 0} \text{height}(v)$
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**Splay trees.** \( \Phi(T) = \sum_{x \in T} \left\lfloor \log_2 \text{size}(x) \right\rfloor \)

**Move-to-front.** \( \Phi(L) = 2 \, \text{inversions}(L, L^*) \)

**Preflow–push.** \( \Phi(f) = \sum_{v : \text{excess}(v) > 0} \text{height}(v) \)

**Red–black trees.** \( \Phi(T) = \sum_{x \in T} w(x) \)

\[
w(x) = \begin{cases} 
0 & \text{if } x \text{ is red} \\
1 & \text{if } x \text{ is black and has no red children} \\
0 & \text{if } x \text{ is black and has one red child} \\
2 & \text{if } x \text{ is black and has two red children}
\end{cases}
\]
Amortized Analysis

- binary counter
- multi-pop stack
- dynamic table

Section 17.4
Multipop stack

**Goal.** Support operations on a set of elements:
Multipop stack

**Goal.** Support operations on a set of elements:

- **PUSH**($S, x$): add element $x$ to stack $S$.
- **POP**($S$): remove and return the most-recently added element.
Multipop stack

**Goal.** Support operations on a set of elements:

- **PUSH**($S, x$): add element $x$ to stack $S$.
- **POP**($S$): remove and return the most-recently added element.
- **MULTI-POP**($S, k$): remove the most-recently added $k$ elements.

```
MULTI-POP(S, k)

FOR i = 1 TO k
    POP(S).
```
Multipop stack

**Goal.** Support operations on a set of elements:

- **PUSH**(S, x): add element x to stack S.
- **POP**(S): remove and return the most-recently added element.
- **MULTI-POP**(S, k): remove the most-recently added k elements.

\[
\text{MULTI-POP}(S, k) \\
\text{FOR } i = 1 \text{ TO } k \\
\text{POP}(S).
\]

**Exceptions.** We assume **POP** throws an exception if stack is empty.
**Multipop stack**

**Goal.** Support operations on a set of elements:
- **PUSH(S, x):** add element $x$ to stack $S$.
- **POP(S):** remove and return the most-recently added element.
- **MULTI-POP(S, k):** remove the most-recently added $k$ elements.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$
**PUSH, POP, and MULTI-POP** operations takes $O(n^2)$ time.
**Multipop stack**

**Goal.** Support operations on a set of elements:
- **PUSH(S, x):** add element \( x \) to stack \( S \).
- **POP(S):** remove and return the most-recently added element.
- **MULTI-POP(S, k):** remove the most-recently added \( k \) elements.

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) PUSH, POP, and MULTI-POP operations takes \( O(n^2) \) time.

**Pf.**
Multipop stack

Goal. Support operations on a set of elements:
- **PUSH**(S, x): add element x to stack S.
- **POP**(S): remove and return the most-recently added element.
- **MULTI-POP**(S, k): remove the most-recently added k elements.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n^2)$ time.

**Pf.**
- Use a singly linked list.
Multipop stack

**Goal.** Support operations on a set of elements:
- \textsc{PUSH}(S, x): add element \(x\) to stack \(S\).
- \textsc{POP}(S): remove and return the most-recently-added element.
- \textsc{MULTI-POP}(S, k): remove the most-recently-added \(k\) elements.

**Theorem.** Starting from an empty stack, any intermixed sequence of \(n\) \textsc{PUSH}, \textsc{POP}, and \textsc{MULTI-POP} operations takes \(O(n^2)\) time.

**Pf.**
- Use a singly linked list.
- \textsc{POP} and \textsc{PUSH} take \(O(1)\) time each.
**Multipop stack**

**Goal.** Support operations on a set of elements:
- \( \text{PUSH}(S, x) \): add element \( x \) to stack \( S \).
- \( \text{POP}(S) \): remove and return the most-recently added element.
- \( \text{MULTI-POP}(S, k) \): remove the most-recently added \( k \) elements.

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) \( \text{PUSH} \), \( \text{POP} \), and \( \text{MULTI-POP} \) operations takes \( O(n^2) \) time.

**Pf.**
- Use a singly linked list.
- \( \text{POP} \) and \( \text{PUSH} \) take \( O(1) \) time each.
- \( \text{MULTI-POP} \) takes \( O(n) \) time.  ■
Multipop stack

**Goal.** Support operations on a set of elements:
- \( \text{PUSH}(S, x) \): add element \( x \) to stack \( S \).
- \( \text{POP}(S) \): remove and return the most-recently added element.
- \( \text{MULTI-POP}(S, k) \): remove the most-recently added \( k \) elements.

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) \( \text{PUSH} \), \( \text{POP} \), and \( \text{MULTI-POP} \) operations takes \( O(n^2) \) time.

**Pf.**
- Use a singly linked list.
- \( \text{POP} \) and \( \text{PUSH} \) take \( O(1) \) time each.
- \( \text{MULTI-POP} \) takes \( O(n) \) time.

\[ 
\begin{array}{c}
\text{top} & \rightarrow & 1 & \rightarrow & 4 & \rightarrow & 1 & \rightarrow & 3 & \rightarrow \\
\end{array}
\]

overly pessimistic upper bound
Multipop stack: aggregate method

**Goal.** Support operations on a set of elements:
- **PUSH**($S, x$): add element $x$ to stack $S$.
- **POP**($S$): remove and return the most-recently added element.
- **MULTI-POP**($S, k$): remove the most-recently added $k$ elements.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTI-POP operations takes $O(n)$ time.
Multipop stack: aggregate method

Goal. Support operations on a set of elements:
- \textsc{PUSH}(S, x): add element \(x\) to stack \(S\).
- \textsc{POP}(S): remove and return the most-recently added element.
- \textsc{MULTI-POP}(S, k): remove the most-recently added \(k\) elements.

Theorem. Starting from an empty stack, any intermixed sequence of \(n\) \textsc{PUSH}, \textsc{POP}, and \textsc{MULTI-POP} operations takes \(O(n)\) time.

Pf.
Multipop stack: aggregate method

**Goal.** Support operations on a set of elements:
- **PUSH(S, x):** add element \( x \) to stack \( S \).
- **POP(S):** remove and return the most-recently added element.
- **MULTI-POP(S, k):** remove the most-recently added \( k \) elements.

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) PUSH, POP, and MULTI-POP operations takes \( O(n) \) time.

**Pf.**
- An element is popped at most once for each time that it is pushed.
Multipop stack: aggregate method

**Goal.** Support operations on a set of elements:
- **PUSH(S, x):** add element \( x \) to stack \( S \).
- **POP(S):** remove and return the most-recently added element.
- **MULTI-POP(S, k):** remove the most-recently added \( k \) elements.

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) **PUSH, POP, and MULTI-POP** operations takes \( O(n) \) time.

**Pf.**
- An element is popped at most once for each time that it is pushed.
- There are \( \leq n \) **PUSH** operations.
Multipop stack: aggregate method

Goal. Support operations on a set of elements:
- \(\text{PUSH}(S, x)\): add element \(x\) to stack \(S\).
- \(\text{POP}(S)\): remove and return the most-recently added element.
- \(\text{MULTI-POP}(S, k)\): remove the most-recently added \(k\) elements.

Theorem. Starting from an empty stack, any intermixed sequence of \(n\) \(\text{PUSH}\), \(\text{POP}\), and \(\text{MULTI-POP}\) operations takes \(O(n)\) time.

Pf.
- An element is popped at most once for each time that it is pushed.
- There are \(\leq n\) \(\text{PUSH}\) operations.
- Thus, there are \(\leq n\) \(\text{POP}\) operations (including those made within \(\text{MULTI-POP}\)).  

Multipop stack: accounting method
Multipop stack: accounting method

**Credits.** 1 credit pays for either a **PUSH** or **POP**.

**Invariant.** Every element on the stack has 1 credit.
Multipop stack: accounting method

**Credits.** 1 credit pays for either a PUSH or POP.

**Invariant.** Every element on the stack has 1 credit.
Multipop stack: accounting method

Credits. 1 credit pays for either a PUSH or POP.

Invariant. Every element on the stack has 1 credit.

Accounting.
  • PUSH(S, x): charge 2 credits.
    - use 1 credit to pay for pushing x now
Multipop stack: accounting method

Credits. 1 credit pays for either a \texttt{PUSH} or \texttt{POP}.

Invariant. Every element on the stack has 1 credit.

Accounting.

- \texttt{PUSH}(S,x): charge 2 credits.
  - use 1 credit to pay for pushing $x$ now
  - store 1 credit to pay for popping $x$ at some point in the future
Multipop stack: accounting method

Credits. 1 credit pays for either a PUSH or POP.

Invariant. Every element on the stack has 1 credit.

Accounting.
- PUSH(S, x): charge 2 credits.
  - use 1 credit to pay for pushing x now
  - store 1 credit to pay for popping x at some point in the future
- POP(S): charge 0 credits.
Multipop stack: accounting method

**Credits.** 1 credit pays for either a PUSH or POP.

**Invariant.** Every element on the stack has 1 credit.

**Accounting.**
- **PUSH(S, x):** charge 2 credits.
  - use 1 credit to pay for pushing x now
  - store 1 credit to pay for popping x at some point in the future
- **POP(S):** charge 0 credits.

**Theorem.** Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes \( O(n) \) time.
Multipop stack: accounting method

Credits. 1 credit pays for either a \texttt{PUSH} or \texttt{POP}.

Invariant. Every element on the stack has 1 credit.

Accounting.
\begin{itemize}
\item \texttt{PUSH}(S, x): charge 2 credits.
  \begin{itemize}
  \item use 1 credit to pay for pushing \( x \) now
  \item store 1 credit to pay for popping \( x \) at some point in the future
  \end{itemize}
\item \texttt{POP}(S): charge 0 credits.
\end{itemize}

Theorem. Starting from an empty stack, any intermixed sequence of \( n \) \texttt{PUSH}, \texttt{POP}, and \texttt{MULTI-POP} operations takes \( O(n) \) time.

Pf.
Multipop stack: accounting method

Credits. 1 credit pays for either a PUSH or POP.

Invariant. Every element on the stack has 1 credit.

Accounting.
- \text{PUSH}(S, x): charge 2 credits.
  - use 1 credit to pay for pushing \( x \) now
  - store 1 credit to pay for popping \( x \) at some point in the future
- \text{POP}(S): charge 0 credits.

Theorem. Starting from an empty stack, any intermixed sequence of \( n \) \text{PUSH}, \text{POP}, and \text{MULTI-POP} operations takes \( O(n) \) time.

Pf.
- Invariant \( \Rightarrow \) number of credits in data structure \( \geq 0 \).
Multipop stack: accounting method

**Credits.** 1 credit pays for either a PUSH or POP.

**Invariant.** Every element on the stack has 1 credit.

**Accounting.**
- **PUSH(S, x):** charge 2 credits.
  - use 1 credit to pay for pushing x now
  - store 1 credit to pay for popping x at some point in the future
- **POP(S):** charge 0 credits.

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) PUSH, POP, and MULTI-POP operations takes \( O(n) \) time.

**Pf.**
- Invariant \( \Rightarrow \) number of credits in data structure \( \geq 0 \).
- Amortized cost per operation \( \leq 2 \).
Multipop stack: accounting method

Credits. 1 credit pays for either a PUSH or POP.

Invariant. Every element on the stack has 1 credit.

Accounting.

• PUSH(S, x): charge 2 credits.
  - use 1 credit to pay for pushing x now
  - store 1 credit to pay for popping x at some point in the future
• POP(S): charge 0 credits.

Theorem. Starting from an empty stack, any intermixed sequence of \( n \) PUSH, POP, and MULTI-POP operations takes \( O(n) \) time.

Pf.

• Invariant \( \Rightarrow \) number of credits in data structure \( \geq 0 \).
• Amortized cost per operation \( \leq 2 \).
• Total actual cost of \( n \) operations \( \leq \) sum of amortized costs \( \leq 2n. \) ■
Multipop stack: potential method
Multipop stack: potential method

Potential function. Let $\Phi(D) =$ number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$. 
Multipop stack: potential method

Potential function. Let $\Phi(D) =$ number of elements currently on the stack.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$ \textsc{push}, \textsc{pop}, and \textsc{multi-pop} operations takes $O(n)$ time.
**Multipop stack: potential method**

**Potential function.** Let $\Phi(D) =$ number of elements currently on the stack.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ **PUSH**, **POP**, and **MULTI-POP** operations takes $O(n)$ time.

**Pf.** [Case 1: push]
Potential function. Let $\Phi(D) =$ number of elements currently on the stack.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [Case 1: push]
- Suppose that the $i^{th}$ operation is a PUSH.
Multipop stack: potential method

**Potential function.** Let \( \Phi(D) \) = number of elements currently on the stack.

- \( \Phi(D_0) = 0 \).
- \( \Phi(D_i) \geq 0 \) for each \( D_i \).

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) \texttt{PUSH}, \texttt{POP}, and \texttt{MULTI-POP} operations takes \( O(n) \) time.

**Pf.** [Case 1: push]

- Suppose that the \( i^{th} \) operation is a \texttt{PUSH}.
- The actual cost \( c_i = 1 \).
**Multipop stack: potential method**

**Potential function.** Let $\Phi(D) =$ number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ **PUSH**, **POP**, and **MULTI-POP** operations takes $O(n)$ time.

**Pf.** [Case 1: push]

- Suppose that the $i^{th}$ operation is a **PUSH**.
- The actual cost $c_i = 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2$. 
Multipop stack: potential method

**Potential function.** Let $\Phi(D) =$ number of elements currently on the stack.

- $\Phi(D_0) = 0.$
- $\Phi(D_i) \geq 0$ for each $D_i.$

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ \textsc{push}, \textsc{pop}, and \textsc{multi-pop} operations takes $O(n)$ time.

**Pf.** [Case 2: pop]
Multipop stack: potential method

Potential function. Let $\Phi(D) =$ number of elements currently on the stack.
  
  - $\Phi(D_0) = 0$.
  - $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$ 
\textsc{Push}, \textsc{Pop}, and \textsc{Multi-Pop} operations takes $O(n)$ time.

Pf. [Case 2: \textsc{pop}]
  
  - Suppose that the $i^{th}$ operation is a \textsc{Pop}.
Potential function. Let $\Phi(D) =$ number of elements currently on the stack.
  - $\Phi(D_0) = 0$.
  - $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$
push, pop, and multi-pop operations takes $O(n)$ time.

Pf. [Case 2: pop]
  - Suppose that the $i^{th}$ operation is a pop.
  - The actual cost $c_i = 1$. 
Potential function. Let $\Phi(D) =$ number of elements currently on the stack.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$ \textsc{Push}, \textsc{Pop}, and \textsc{Multi-Pop} operations takes $O(n)$ time.

Pf. [Case 2: pop]
- Suppose that the $i^{th}$ operation is a \textsc{Pop}.
- The actual cost $c_i = 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 - 1 = 0$. 
Multipop stack: potential method

Potential function. Let \( \Phi(D) \) = number of elements currently on the stack.

- \( \Phi(D_0) = 0 \).
- \( \Phi(D_i) \geq 0 \) for each \( D_i \).

Theorem. Starting from an empty stack, any intermixed sequence of \( n \) \textsc{push}, \textsc{pop}, and \textsc{multi-pop} operations takes \( O(n) \) time.

Pf. [Case 3: multi-pop]
**Multipop stack: potential method**

**Potential function.** Let $\Phi(D) =$ number of elements currently on the stack.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

**Pf.** [Case 3: multi-pop]
- Suppose that the $i^{th}$ operation is a MULTI-POP of $k$ objects.
Multipop stack: potential method

**Potential function.** Let $\Phi(D) =$ number of elements currently on the stack.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ Push, Pop, and Multi-Pop operations takes $O(n)$ time.

**Pf.** [Case 3: multi-pop]
- Suppose that the $i^{th}$ operation is a Multi-Pop of $k$ objects.
- The actual cost $c_i = k$. 
Multipop stack: potential method

Potential function. Let $\Phi(D) = \text{number of elements currently on the stack.}$
- $\Phi(D_0) = 0.$
- $\Phi(D_i) \geq 0$ for each $D_i.$

Theorem. Starting from an empty stack, any intermixed sequence of $n$ \textsc{Push}, \textsc{Pop}, and \textsc{Multi-Pop} operations takes $O(n)$ time.

Pf. [Case 3: multi-pop]
- Suppose that the $i^{th}$ operation is a \textsc{Multi-Pop} of $k$ objects.
- The actual cost $c_i = k.$
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = k - k = 0.$ □
Multipop stack: potential method

**Potential function.** Let $\Phi(D) =$ number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ `PUSH`, `POP`, and `MULTI-POP` operations takes $O(n)$ time.

**Pf.** [putting everything together]
Multipop stack: potential method

Potential function. Let $\Phi(D) =$ number of elements currently on the stack.
  • $\Phi(D_0) = 0$.
  • $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [putting everything together]
  • Amortized cost $\hat{c}_i \leq 2$.  \[\text{2 for push; 0 for pop and multi-pop}\]
Potential function. Let $\Phi(D) =$ number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$ PUS H, POP, and MULTI-P OP operations takes $O(n)$ time.

Pf. [putting everything together]

- Amortized cost $\hat{c}_i \leq 2$.  
  2 for push; 0 for pop and multi-pop
- Sum of amortized costs $\hat{c}_i$ of the $n$ operations $\leq 2n$. 

Multipop stack: potential method
Multipop stack: potential method

Potential function. Let $\Phi(D) =$ number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [putting everything together]

- Amortized cost $\hat{c}_i \leq 2$. $\blacktriangleright$ 2 for push; 0 for pop and multi-pop
- Sum of amortized costs $\hat{c}_i$ of the $n$ operations $\leq 2n$.
- Total actual cost $\leq$ sum of amortized cost $\leq 2n$. $\blacksquare$

potential method theorem
Amortized Analysis

- binary counter
- multi-pop stack
- dynamic table
Dynamic table

**Goal.** Store items in a table (e.g., for hash table, binary heap).

- Two operations: **INSERT** and **DELETE**.
  - too many items inserted ⇒ **expand** table.
  - too many items deleted ⇒ **contract** table.
- Requirement: if table contains $m$ items, then space = $\Theta(m)$. 
Dynamic table

Goal. Store items in a table (e.g., for hash table, binary heap).

- Two operations: `INSERT` and `DELETE`.
  - too many items inserted $\Rightarrow$ expand table.
  - too many items deleted $\Rightarrow$ contract table.
- Requirement: if table contains $m$ items, then space $= \Theta(m)$.

Theorem. Starting from an empty dynamic table, any intermixed sequence of $n$ `INSERT` and `DELETE` operations takes $O(n^2)$ time.
Dynamic table

**Goal.** Store items in a table (e.g., for hash table, binary heap).

- Two operations: **INSERT** and **DELETE**.
  - too many items inserted $\Rightarrow$ **expand** table.
  - too many items deleted $\Rightarrow$ **contract** table.
- Requirement: if table contains $m$ items, then space $= \Theta(m)$.

**Theorem.** Starting from an empty dynamic table, any intermixed sequence of $n$ **INSERT** and **DELETE** operations takes $O(n^2)$ time.

**Pf.** Each **INSERT** or **DELETE** takes $O(n)$ time. •
Dynamic table

Goal. Store items in a table (e.g., for hash table, binary heap).
  • Two operations: INSERT and DELETE.
    - too many items inserted ⇒ expand table.
    - too many items deleted ⇒ contract table.
  • Requirement: if table contains \( m \) items, then space = \( \Theta(m) \).

Theorem. Starting from an empty dynamic table, any intermixed sequence of \( n \) INSERT and DELETE operations takes \( O(n^2) \) time.

Pf. Each INSERT or DELETE takes \( O(n) \) time. □
Dynamic table: insert only

- When inserting into an empty table, allocate a table of capacity 1.
- When inserting into a full table, allocate a new table of twice the capacity and copy all items.
- Insert item into table.

<table>
<thead>
<tr>
<th>insert</th>
<th>old capacity</th>
<th>new capacity</th>
<th>insert cost</th>
<th>copy cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>8</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>8</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>8</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>8</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>16</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
</tr>
</tbody>
</table>

Cost model. Number of items written (due to insertion or copy).
Dynamic table: insert only (aggregate method)

Theorem. [via aggregate method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.
Dynamic table: insert only (aggregate method)

**Theorem.** [via aggregate method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** Let $c_i$ denote the cost of the $i^{th}$ insertion.

$$c_i = \begin{cases} 
  i & \text{if } i - 1 \text{ is an exact power of 2} \\
  1 & \text{otherwise}
\end{cases}$$
Theorem. [via aggregate method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

Pf. Let $c_i$ denote the cost of the $i^{th}$ insertion.

$$c_i = \begin{cases} i & \text{if } i - 1 \text{ is an exact power of 2} \\ 1 & \text{otherwise} \end{cases}$$

Starting from empty table, the cost of a sequence of $n$ INSERT operations is:
Dynamic table: insert only (aggregate method)

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$$< n + 2n$$
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$$= 3n \quad \blacksquare$$
Dynamic table demo: insert only (accounting method)

Insert. Charge 3 credits (use 1 credit to insert; save 2 with new item).
Invariant. 2 credits with each item in right half of table; none in left half.

insert N

capacity = 16

<table>
<thead>
<tr>
<th>A</th>
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Pf. [induction]

\[ \text{slight cheat if table capacity } = 1 \]
Dynamic table: insert only (accounting method)

Insert. Charge 3 credits (use 1 credit to insert; save 2 with new item).

Invariant. 2 credits with each item in right half of table; none in left half.

Pf. [induction]
  • Each newly inserted item gets 2 credits.
Dynamic table: insert only (accounting method)

**Insert.** Charge 3 credits (use 1 credit to insert; save 2 with new item).

**Invariant.** 2 credits with each item in right half of table; none in left half.

**Pf.** [induction]
- Each newly inserted item gets 2 credits.
- When table doubles from $k$ to $2k$, $k/2$ items in the table have 2 credits.

↑ slight cheat if table capacity = 1
Dynamic table: insert only (accounting method)

**Insert.** Charge 3 credits (use 1 credit to insert; save 2 with new item).

**Invariant.** 2 credits with each item in right half of table; none in left half.

**Pf.** [induction]
- Each newly inserted item gets 2 credits.
- When table doubles from $k$ to $2k$, $k/2$ items in the table have 2 credits.
  - these $k$ credits pay for the work needed to copy the $k$ items

---

slight cheat if table capacity = 1
Dynamic table:  insert only (accounting method)

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**Invariant.** 2 credits with each item in right half of table; none in left half.

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  - these \( k \) credits pay for the work needed to copy the \( k \) items
  - now, all \( k \) items are in left half of table (and have 0 credits)

slight cheat if table capacity = 1
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**Theorem.** [via accounting method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.
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**Pf.**
- Invariant $\implies$ number of credits in data structure $\geq 0$. 
Dynamic table: insert only (accounting method)

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Pf.

- Invariant \( \Rightarrow \) number of credits in data structure \( \geq 0 \).
- Amortized cost per INSERT = 3.
Dynamic table: insert only (accounting method)

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- Invariant $\Rightarrow$ number of credits in data structure $\geq 0$.
- Amortized cost per INSERT = 3.
- Total actual cost of $n$ operations $\leq$ sum of amortized cost $\leq 3n$.  

slight cheat if table capacity = 1
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.
Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

Pf. Let \( \Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i) \).

number of elements

capacity of array
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

**Pf.** Let \( \Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i) \).

\[ \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \]

- number of elements
- capacity of array

\[
\text{size} = 6 \\
\text{capacity} = 8 \\
\Phi = 4
\]
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

**Pf.** Let \( \Phi(D_i) = 2 \text{ size}(D_i) - \text{capacity}(D_i) \).

- \( \Phi(D_0) = 0 \).

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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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size = 6  
capacity = 8  
\( \Phi = 4 \)
Dynamic table: insert only (potential method)

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**Case 0.** [first insertion]
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

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**Case 0.** [first insertion]

- Actual cost \( c_1 = 1 \).
Theorem. [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

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**Case 0.** [first insertion]

- Actual cost $c_1 = 1$.
- $\Phi(D_1) - \Phi(D_0) = (2 \text{size}(D_1) - \text{capacity}(D_1)) - (2 \text{size}(D_0) - \text{capacity}(D_0))$
  
  $= 1$.  

Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

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**Case 0.** [first insertion]

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- \( \Phi(D_1) - \Phi(D_0) = (2 \text{ size}(D_1) - \text{capacity}(D_1)) - (2 \text{ size}(D_0) - \text{capacity}(D_0)) \)
  \[ = 1. \]
- Amortized cost \( \hat{c}_i = c_i + (\Phi(D_1) - \Phi(D_0)) \)
  \[ = 1 + 1 \]
  \[ = 2. \]
Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

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Case 1. [no array expansion] $\text{capacity}(D_i) = \text{capacity}(D_{i-1})$. 

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2 \cdot \text{size}(D_i) - \text{capacity}(D_i)$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

Case 1. [no array expansion] $\text{capacity}(D_i) = \text{capacity}(D_{i-1})$.

- Actual cost $c_i = 1$. 
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**Pf.** Let \( \Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i) \).

\[ \begin{align*}
\Phi(D_0) &= 0, \\
\Phi(D_i) &\geq 0 \text{ for each } D_i.
\end{align*} \]

**Case 1.** [no array expansion] \( \text{capacity}(D_i) = \text{capacity}(D_{i-1}) \).

\[ \begin{align*}
\text{Actual cost } c_i &= 1, \\
\Phi(D_i) - \Phi(D_{i-1}) &= (2 \text{size}(D_i) - \text{capacity}(D_i)) - (2 \text{size}(D_{i-1}) - \text{capacity}(D_{i-1})) \\
&= 2.
\end{align*} \]
Dynamic table: insert only (potential method)

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  \[= 2.\]

- Amortized cost $\hat{c}_i = c_i + (\Phi(D_i) - \Phi(D_{i-1}))$
  
  \[= 1 + 2\]
  
  \[= 3.\]
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

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- \( \Phi(D_i) \geq 0 \) for each \( D_i \).

**Case 2.** [array expansion] \( \text{capacity}(D_i) = 2 \text{ capacity}(D_{i-1}) \).
Dynamic table: insert only (potential method)

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**Case 2.** [array expansion] $\text{capacity}(D_i) = 2 \text{capacity}(D_{i-1})$.

- Actual cost $c_i = 1 + \text{capacity}(D_{i-1})$. 
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

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**Case 2.** [array expansion] \( \text{capacity}(D_i) = 2 \text{ capacity}(D_{i-1}) \).

- Actual cost \( c_i = 1 + \text{ capacity}(D_{i-1}) \).
- \( \Phi(D_i) - \Phi(D_{i-1}) = (2 \text{ size}(D_i) - \text{ capacity}(D_i)) - (2 \text{ size}(D_{i-1}) - \text{ capacity}(D_{i-1})) \)
  
  \[ = 2 - \text{ capacity}(D_i) + \text{ capacity}(D_{i-1}) \]
  
  \[ = 2 - \text{ capacity}(D_{i-1}). \]
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

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- Amortized cost \( \hat{c}_i = c_i + (\Phi(D_i) - \Phi(D_{i-1})) \)
  \[ = 1 + \text{capacity}(D_{i-1}) + (2 - \text{capacity}(D_{i-1})) \]
  \[ = 3 \].
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

**Pf.** Let \( \Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i) \).

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[putting everything together]
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

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[putting everything together]
- Amortized cost per operation $\hat{c}_i \leq 3$. 

\[\begin{align*}
\text{number of elements} & \quad \uparrow \\
\text{capacity of array} & \quad \uparrow
\end{align*}\]
Dynamic table: insert only (potential method)

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[putting everything together]

- Amortized cost per operation $\hat{c}_i \leq 3$.
- Total actual cost of $n$ operations $\leq$ sum of amortized cost $\leq 3n$. □
Dynamic table: doubling and halving

Thrashing.

• **INSERT**: when inserting into a full table, double capacity.
• **DELETE**: when deleting from a table that is $\frac{1}{2}$-full, halve capacity.
Dynamic table: doubling and halving

Thrashing.

- **INSERT:** when inserting into a full table, double capacity.
- **DELETE:** when deleting from a table that is $\frac{1}{2}$-full, halve capacity.

Efficient solution.

- When inserting into an empty table, initialize table size to 1; when deleting from a table of size 1, free the table.
- **INSERT:** when inserting into a full table, double capacity.
- **DELETE:** when deleting from a table that is $\frac{1}{4}$-full, halve capacity.
Dynamic table: doubling and halving

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Memory usage. A dynamic table uses $\Theta(n)$ memory to store $n$ items.
Dynamic table: doubling and halving

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Memory usage. A dynamic table uses \( \Theta(n) \) memory to store \( n \) items.

Pf. Table is always between 25% and 100% full. □
Dynamic table demo: insert and delete (accounting method)

**Insert.** Charge 3 credits (1 to insert; save 2 with item if in right half).

**Delete.** Charge 2 credits (1 to delete; save 1 in empty slot if in left half).

**Invariant 1.** 2 credits with each item in right half of table.

**Invariant 2.** 1 credit with each empty slot in left half of table.

delete M

capacity = 16

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GB
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![Table diagram]
Dynamic table: insert and delete (accounting method)

**Insert.** Charge 3 credits (1 to insert; save 2 with item if in right half).
Dynamic table: insert and delete (accounting method)

**Insert.** Charge 3 credits (1 to insert; save 2 with item if in right half).

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Dynamic table: insert and delete (accounting method)

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Invariant 2. 1 credit with each empty slot in left half of table.

Theorem. [via accounting method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.
Dynamic table: insert and delete (accounting method)

**Insert.** Charge 3 credits (1 to insert; save 2 with item if in right half).

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**Theorem.** [via accounting method] Starting from an empty dynamic table, any intermixed sequence of \( n \) INSERT and DELETE operations takes \( O(n) \) time.

**Pf.**
Dynamic table: insert and delete (accounting method)

Insert. Charge 3 credits (1 to insert; save 2 with item if in right half).
Delete. Charge 2 credits (1 to delete; save 1 in empty slot if in left half).

Invariant 1. 2 credits with each item in right half of table.
Invariant 2. 1 credit with each empty slot in left half of table.

discard any existing or extra credits
to pay for expansion
to pay for contraction

Theorem. [via accounting method] Starting from an empty dynamic table, any intermixed sequence of \( n \) INSERT and DELETE operations takes \( O(n) \) time.

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- Invariants $\Rightarrow$ number of credits in data structure $\geq 0$.
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**Theorem.** [via accounting method] Starting from an empty dynamic table, any intermixed sequence of $n$ \textsc{insert} and \textsc{delete} operations takes $O(n)$ time.

**Pf.**
\begin{itemize}
  \item Invariants $\Rightarrow$ number of credits in data structure $\geq 0$.
  \item Amortized cost per operation $\leq 3$.
  \item Total actual cost of $n$ operations $\leq$ sum of amortized cost $\leq 3n$.  
\end{itemize}

\[ \text{accounting method theorem} \]
Dynamic table: insert and delete (potential method)

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Pf sketch.
Dynamic table: insert and delete (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.

**Pf sketch.**

- Let $\alpha(D_i) = \text{size}(D_i) / \text{capacity}(D_i)$. 
Theorem. [via potential method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.

Pf sketch.

- Let $\alpha(D_i) = \frac{\text{size}(D_i)}{\text{capacity}(D_i)}$.

- Define $\Phi(D_i) = \begin{cases} 
  2 \text{size}(D_i) - \text{capacity}(D_i) & \text{if } \alpha(D_i) \geq 1/2 \\
  \frac{1}{2} \text{capacity}(D_i) - \text{size}(D_i) & \text{if } \alpha(D_i) < 1/2 
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- $\Phi(D_0) = 0$, $\Phi(D_i) \geq 0$. [a potential function]
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- When $\alpha(D_i) = 1$, $\Phi(D_i) = \text{size}(D_i)$. [can pay for expansion]
- When $\alpha(D_i) = 1/4$, $\Phi(D_i) = \text{size}(D_i)$. [can pay for contraction]

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