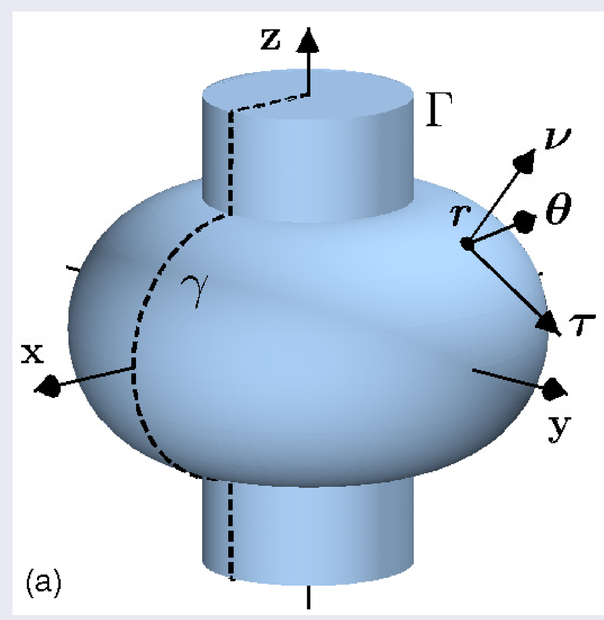
High-order convergent and accurate electromagnetic solvers on Lipschitz domains

Johan Helsing, Anders Karlsson, and Karl-Mikael Perfekt
Lund University

Emerging Topics in Optics, IMA, April 24-28, 2017

Problem, challenges, and methods

Problem: Solve the time harmonic Maxwell equations and find eigenwavenumbers $\boldsymbol{k}$ and eigenfields $\boldsymbol{E}$ of perfectly conducting axially symmetric cavities and for dielectric objects in vacuum.
Challenges: Large structures. High accuracy. Finding all eigenmodes. Resolution of singular fields at sharp edges. Normalization $\|\boldsymbol{E}\|^2 = 1$ of eigenfields.
Methods: ChIE extensions of MFIE and EFIE (no surface divergence for surface charge densities), 16th-order explicit kernel-split panel-based Fourier–Nyström discretization, recursively compressed inverse preconditioning (RCIP), volume⟶surface integral for $\|\boldsymbol{E}_n\|^2 = 1$, robust search for eigenwavenumbers.



Integral equation for eigenproblem

The ChIE extended MFIE system can be written on modal block operator form as

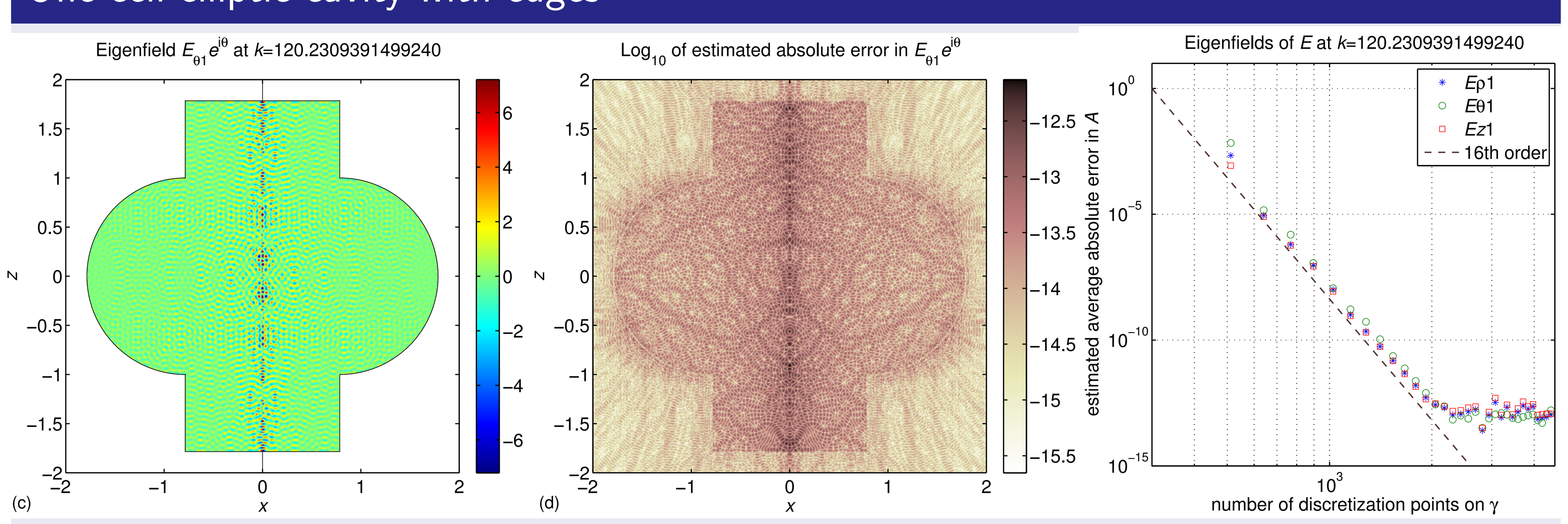
$$\begin{bmatrix} I - 2K_{\nu n} & 2ikS_{5n} & -2kS_{6n} \\ 0 & I + K_{1n} & iK_{2n} \\ 0 & iK_{3n} & I + K_{4n} \end{bmatrix} \begin{bmatrix} \varrho_{sn}(r) \\ J_{\tau n}(r) \\ J_{\theta n}(r) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where $K_{\nu n}$, K_{1n}, K_{2n}, K_{3n}, K_{4n} are modal double-layer type integral operators and S_{5n}, S_{6n} are single-layer type operators. All operators are weakly singular.

The RCIP idea

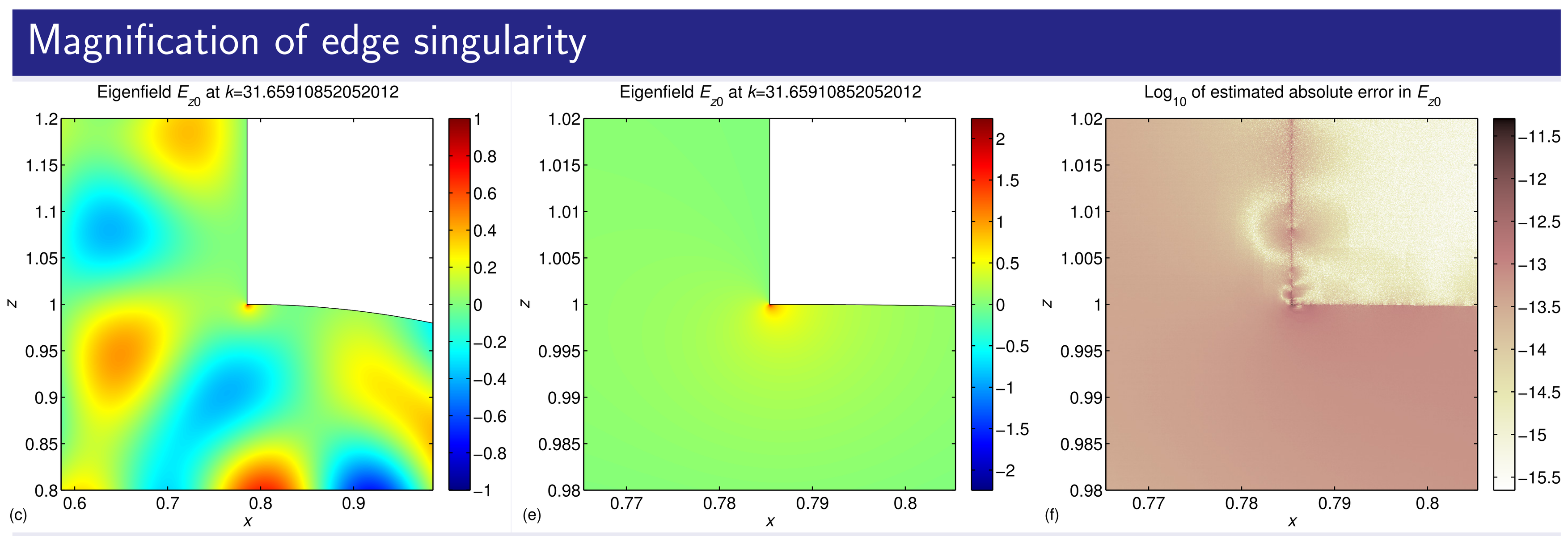
$(\mathrm{I}+\mathrm{K})\rho = g$ is difficult to solve on Lipschitz domains. The split $\mathrm{K} = \mathrm{K}^{\star} + \mathrm{K}^{\circ}$ and the change of variables $\rho = (\mathrm{I}+\mathrm{K}^{\star})^{-1}\tilde{\rho}$ give the simpler compressed preconditioned equation $(\mathrm{I} + \mathrm{K}^{\circ}\mathrm{P}_W^T(\mathrm{I}+\mathrm{K}^{\star})^{-1}\mathrm{P})\tilde{\rho} = g$, where P is a prolongation operator. Recursion on nested grids is used for the lossless compression of $\mathrm{P}_W^T(\mathrm{I}+\mathrm{K}^{\star})^{-1}\mathrm{P}$.

One cell elliptic cavity with edges



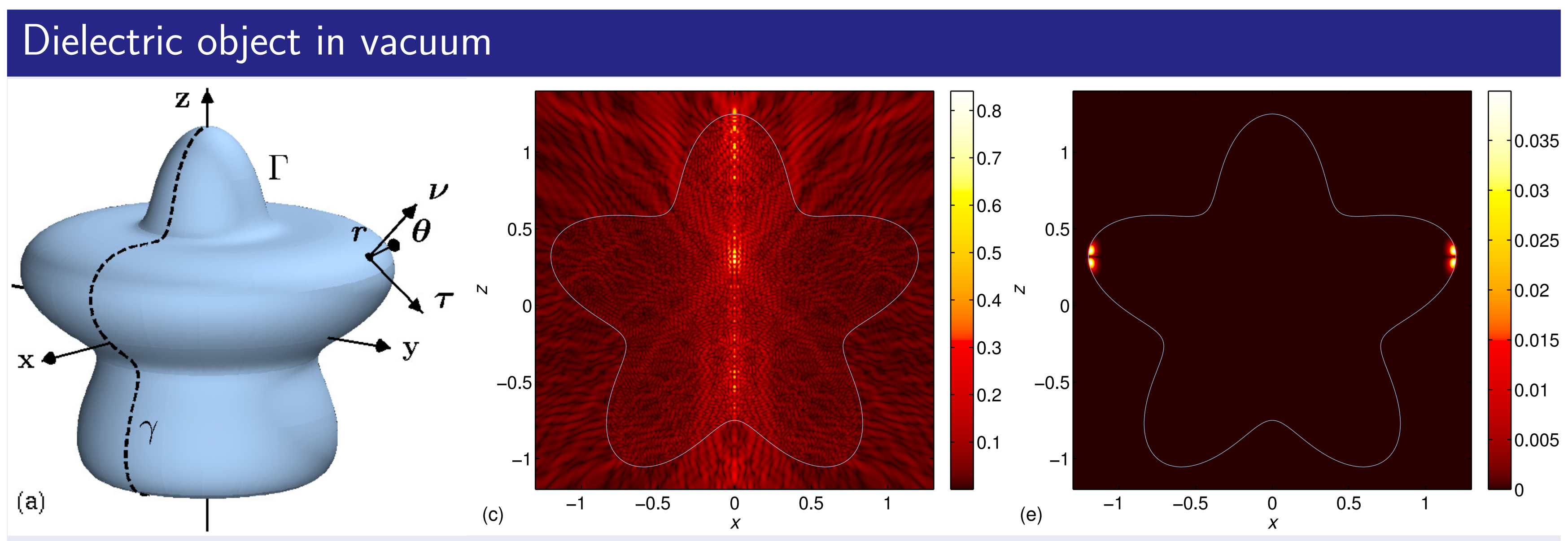
Left: The azimuthal component of normalized electric field for the 9928th $n = 1$ eigenmode with eigenwavenumber $k = 120.2309391499240$ evaluated at $4.9 \cdot 10^5$ points on a cartesian grid. Object diameter $\approx 75\lambda$. Center: $\log_{10}$ of estimated absolute pointwise error. Right: stable 16th-order convergence with mesh refinement.

Magnification of edge singularity



Left and center: zoom of z-component of normalized electric field for the 662nd $n = 0$ eigenmode with eigenwavenumber $k = 31.65910852052012$. The field diverges in the reentrant corners. Right: $\log_{10}$ of estimated absolute pointwise error.

Dielectric object in vacuum

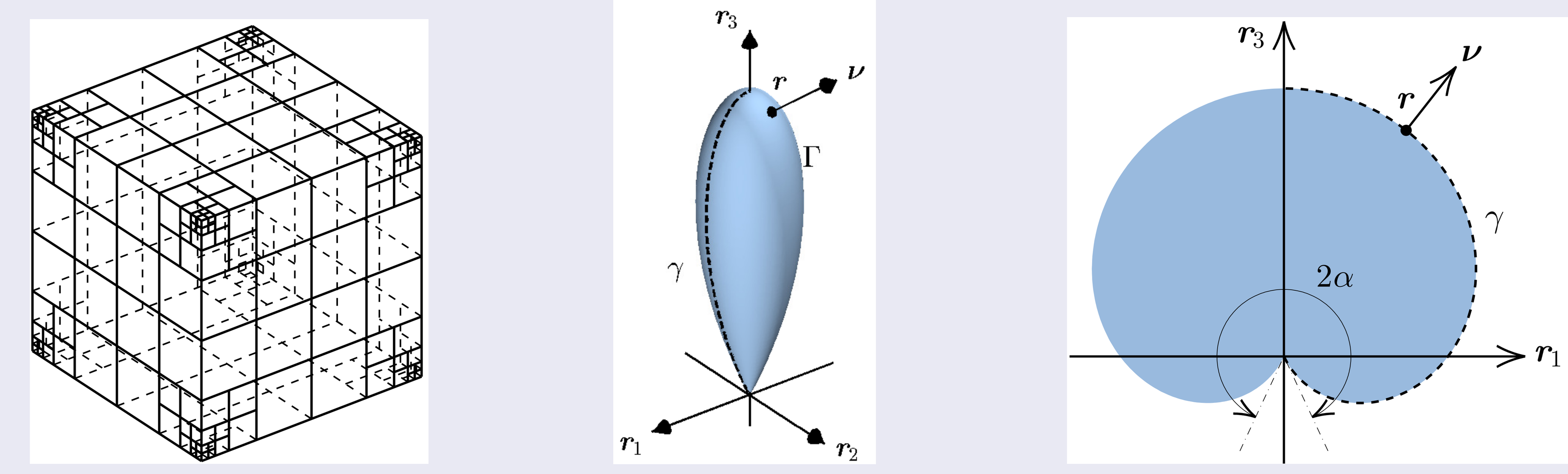


Left: Object. Center: Azimuthal component of electric field for the $n = 1$ eigenmode with eigenwavenumber $k = 110.041232211051 - 0.404177078290\mathrm{i}$. The refractive index is $m = 1.5$. The object diameter is around 46 vacuum wavelengths. Right: Whispering gallery $n = 450$ mode with eigenwavenumber $k = 258.059066513439$, $m = 1.5$, and object diameter around 108 vacuum wavelengths. The absolute pointwise error is less than 10^{-11} in both field plots.

Problem, challenges and methods

Problem: Solve the electrostatic transmission problem for an inclusion with Lipschitz surface $\boldsymbol{\Gamma}$ and permittivity $\boldsymbol{\epsilon}$ embedded in a background medium with unit permittivity. Then compute the polarizability tensor $\omega_{ij}(\epsilon)$ and its spectral measure.
Challenges: High accuracy. Resolution of singular fields at sharp edges and in corners. Integral operator spectra depend on function spaces considered: $L^2(\Gamma)$ or energy space.
Methods: Classic integral equation, 16th- and 32nd-order explicit kernel-split panel-based (Fourier–)Nyström discretization, recursively compressed inverse preconditioning (RCIP), fixed-point iteration, Newton's method, homotopy.

Geometries and meshes



Left: a three times dyadically refined mesh on the surface of a cube. Middle: An axially symmetric surface Γ with a conical point of opening angle $\alpha = 5\pi/36$, denoted a *snow cone*. Right: A cross section of the snow cone interior for $\alpha = 31\pi/36$.

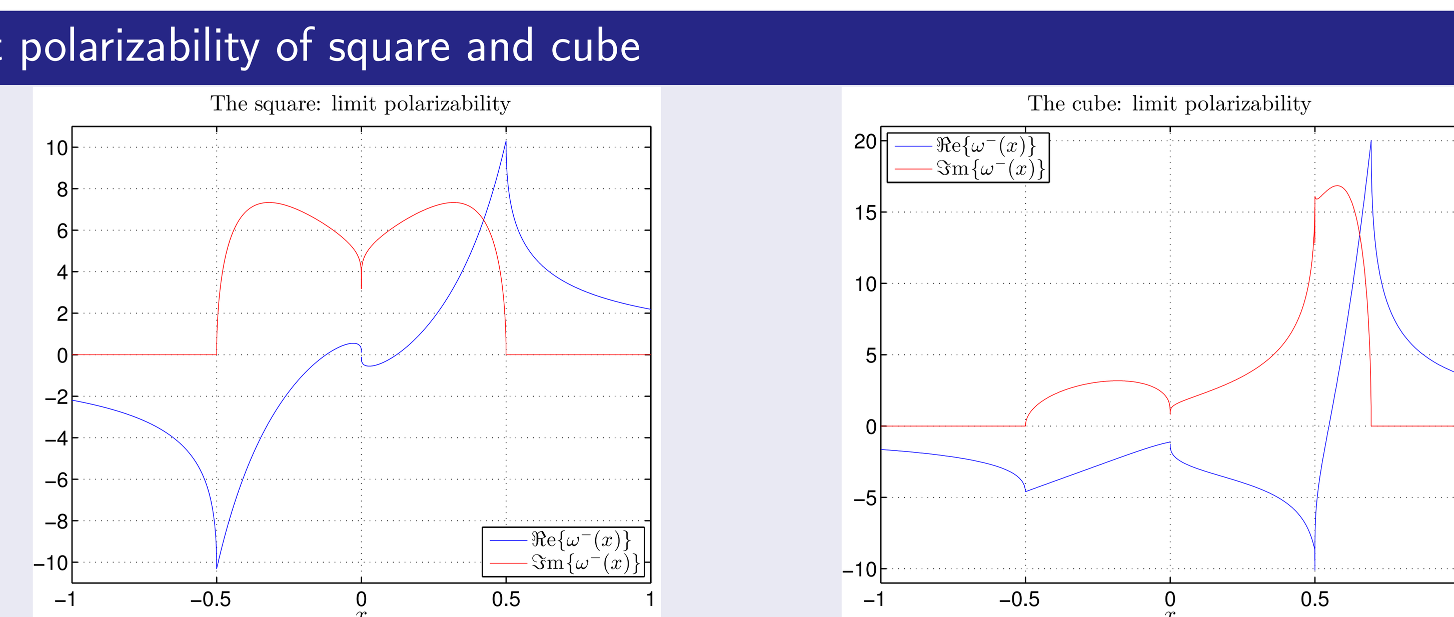
Integral equation

With standard basis e_i, the element ω_{ij} of the polarizability tensor of an inclusion is

$$\omega_{ij}(z) = \frac{2}{|V|}\int_{\Gamma}(e_i \cdot r)\left[(K_\nu - z)^{-1}(e_j \cdot \nu)\right](r)\,\mathrm{d}\sigma(r),$$

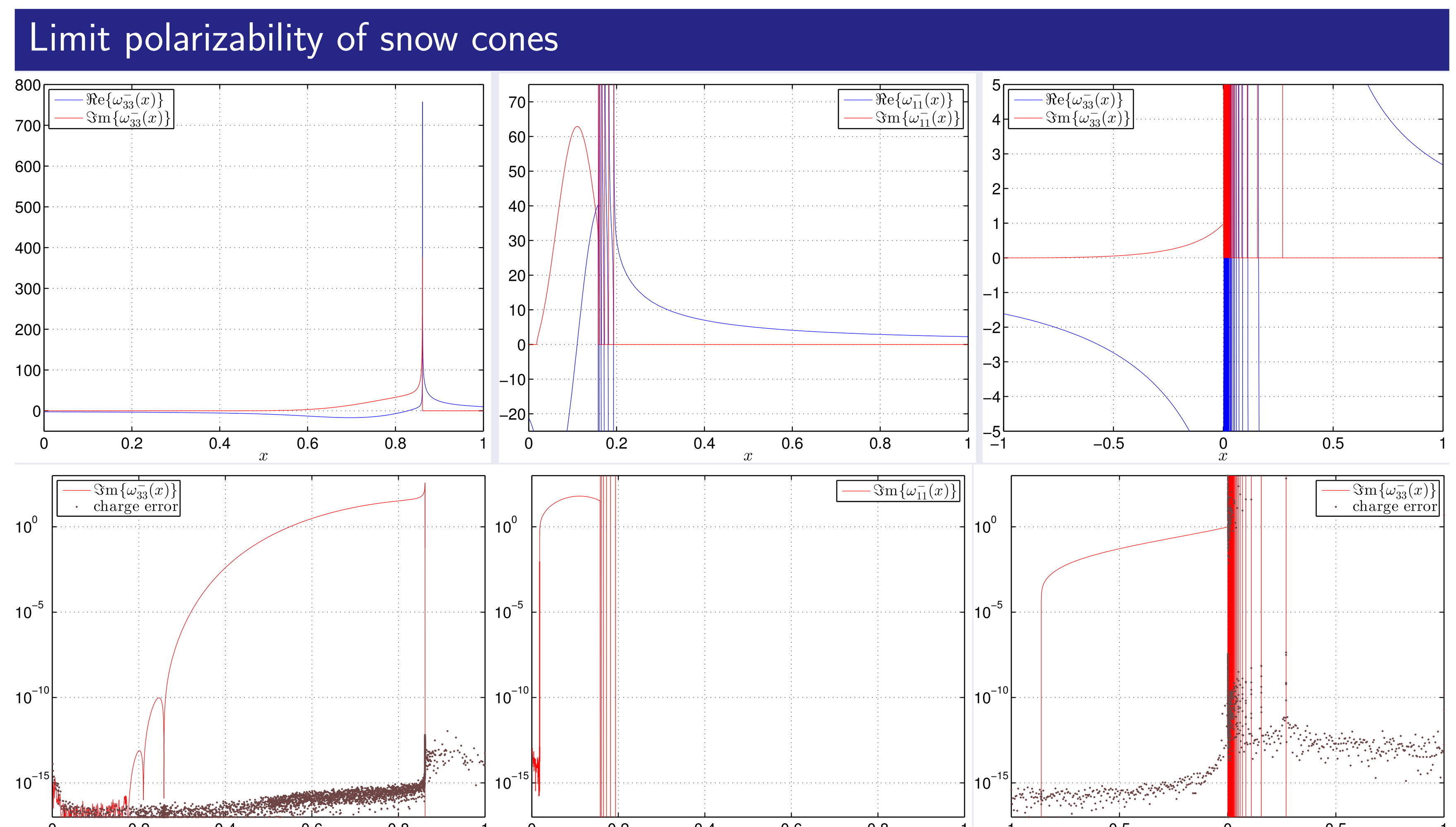
where K_ν is two times the adjoint of the Neumann–Poincaré operator (double layer potential), $z \equiv x + \mathrm{i}y = (\epsilon+1)/(\epsilon-1)$, and $|V|$ is the volume of the inclusion.

Limit polarizability of square and cube



Limit values of $\omega(z)$ as $y \to 0^-$. The accuracy varies from full machine precision to about five digits. $\Im\mathrm{m}\{\omega^-(x)\}$ of the cube has support for $x \in (-0.5, 0.694526)$.

Limit polarizability of snow cones



Top row: (a,b) limit values of $\omega_{33}(z)$ and $\omega_{11}(z)$ as $y \to 0^-$ for $\alpha = 5\pi/36$; (c) $\omega_{33}(z)$ for $\alpha = 31\pi/36$. Bottom row: imaginary parts with logarithmic scales on the vertical axes. The spectral measure $\mu_i(x)$, associated with $\omega_{ii}(z)$, is determined by $\mu_i'(x) = -\Im\mathrm{m}\{\omega_{ii}^-(x)\}/\pi$. The numerical accuracy in (a,b) is such that $\int_{-1}^{1}\mathrm{d}\mu_i(s) = -2$ holds to almost machine precision. Note the infinite number of poles for $\omega_{33}^-(x)$, $x > 0$, and $\alpha = 31\pi/36$, of which 275 are located and drawn.

References

'Solving integral equations on piecewise smooth boundaries using the RCIP method ...', arXiv:1207.6737.
'On the polarizability and capacitance of the cube', Appl. Comput. Harmon. Anal., **34**, 445–468, 2013.
'Determination of normalized electric eigenfields in microwave cavities with ...', JCP, **304**, 465–486, 2016.
'Resonances in axially symmetric dielectric objects', IEEE Trans. Microw. Theory Tech., (in press 2017).
'The spectra of harmonic layer potential operators on domains with rotationally symmetric conical points', arXiv:1703.01628.