ASYMPTOTIC EIGENVALUE BEHAVIOR FOR (NON-NORMAL) MATRIX SEQUENCES AND APPLICATIONS

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Abstract. When approximating infinite dimensional linear (and nonlinear) equations, sequences of matrices $\{A_n\}$ of increasing dimension d_n (i.e., $d_n < d_{n+1}$, $\forall n \in \mathbf{N}^d$, $d \geq 1$) arise in a natural way. There exist many reasons for studying the spectral properties of these sequences: approximation quality provided by the finite dimensional operators, convergence features of Krylov type methods, design of preconditioners and projection operators for accelerating iterative methods etc.

Here we concentrate our attention on global distribution results in the more intriguing and tricky non normal setting (for several results connected with this topic see the reported list of references).

Definition
We say that $\{A_n\}$ is distributed as the measurable function θ defined over $K \subset \mathbf{C}^d$, $0 < \mu(K) < \infty$, if and only if

$$\lim_{n\to\infty} \sum_{j=1}^{d_n} \frac{F\big(\lambda_j(A_n)\big)}{d_n} = \frac{1}{\mu(K)} \int_K F\big(\theta(s)\big)\, ds$$

for any F continuous with bounded support. In short we write $\{A_n\} \sim_\lambda (\theta, K)$.
Analogous definition holds for the singular values. In that case $|\theta|$ replaces θ and $\{A_n\} \sim_\sigma (\theta, K)$ replaces $\{A_n\} \sim_\lambda (\theta, K)$.

Theorem
Assume

- $\{A_n\}$ is spectrally bounded,
- $\lim_{n\to\infty} \frac{\mathrm{trace}\left(A_n^q (A_n^*)^t\right)}{d_n} = \frac{1}{\mu(K)} \int_K \theta^q(s)\overline{\theta}^t(s)\, ds$, for some measurable θ defined over $K \subset \mathbf{C}^d$, $0 < \mu(K) < \infty$,
- the interior of range(θ) is empty and his complement is connected in $\mathbf{C}$

Then $\{A_n\} \sim_\lambda (\theta, K)$.

Some variations of this result, useful in specific settings, are illustrated together with some applications in PDE approximation and in the context of the algebra generated by Toeplitz sequences with bounded symbols.

Key words. Spectral distribution, symbol, Krylov methods, superlinear convergence.

AMS subject classifications. 65F10, 65F15, 15A18, 15A21, 15A51.

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