Priced Discrete-Timed Petri Nets

Parosh Aziz Abdulla

Uppsala University

February 8, 2010

(Joint work with Richard Mayr)
1 Petri Nets

2 Discrete-Timed Petri Nets

3 Priced Discrete-Timed Petri Nets (PTPNs)

4 From PTPNs to PNs

5 Algorithm

6 Further Results
Petri Nets

Markings

\[ M = [A, A, A, B, B, C] \]

\[ M = [A^3, B^2, C] \]

\[ M(A) = 3 \quad M(B) = 2 \quad M(C) = 3 \]
Petri Nets

Q-Markings

\[ M: \text{Q-Marking} \]

Tokens only in \( Q \). \( M(p) = 0 \) if \( p \notin Q \).

Example: \( \{ A, B \} \)-Marking

\[ M = [A^3, B^2] \]

\( M(A) = 3 \) \( M(B) = 2 \) \( M(C) = 0 \)
Petri Nets

ω-Markings

\[ M = [A^3, B^2, C^\omega] \]

\[ M = \{ [A^3, B^2], [A^3, B^2, C], [A^3, B^2, C^2], [A^3, B^2, C^3], [A^3, B^2, C^4], \ldots \} \]
Example: \(\{A, B\}-\omega\)-Marking

\[M = [A^3, B^\omega]\]

\[M = \{ [A^3], [A^3, B], [A^3, B^2], [A^3, B^3], [A^3, B^4], \ldots \}\]
[A^3, B^2, C] \rightarrow t_2 [A^2, B^3, C^2]
Petri Nets

Computations

\[
\begin{align*}
[A^3, B^2, C] & \xrightarrow{*} [A^4, B^2, C]
\end{align*}
\]

Ordering

\[
[A^2, B] \leq [A^3, B^2, C] \leq [A^3, B^\omega, C]
\]

Monotonicity

\[
M_1 \xrightarrow{\land} M_2 \\
M_3
\]

By Monotonicity:

- if \( F \) is upward closed
- then \( \left( \xrightarrow{*} F \right) \) is upward closed
**Petri Nets**

**Computations**

\[
[A^3, B^2, C] \xrightarrow{t_2} [A^2, B^3, C^2] \xrightarrow{t_1} [A^3, B^3, C^2] \xrightarrow{t_3} [A^4, B^2, C]
\]

**Ordering**

\[
[A^2, B] \leq [A^3, B^2, C] \leq [A^3, B^\omega, C]
\]

**Monotonicity**

\[
M_1 \xrightarrow{\wedge} M_2
\]

\[
M_3 \xrightarrow{\wedge} M_4
\]

**By Monotonicity:**

- if \( F \) is upward closed
- then \( \xrightarrow{*} F \) is upward closed
**Computations**

\[
\begin{align*}
& & & & [A^3, B^2, C] & \xrightarrow{\star} [A^4, B^2, C]
\end{align*}
\]

**Ordering**

\[
[A^2, B] \leq [A^3, B^2, C] \leq [A^3, B^\omega, C]
\]

**Monotonicity**

\[
\begin{align*}
M_1 & \xrightarrow{\star} M_2 \\
\uparrow & \uparrow \\
M_3 & \xrightarrow{\star} M_4
\end{align*}
\]

**By Monotonicity:**

- if \( F \) is upward closed
- then \( \left( \xrightarrow{\star} F \right) \) is upward closed
Petri Nets

The Coverability Problem

Upward Closure

\[ M^\uparrow := \{ M' \mid M \leq M' \} \]

\[
[A^2, B]^\uparrow = \{ [A^3, B], [A^2, B^2, C], [A^3, B^2, C], [A^3, B^2], [A^2, B^2, C^2], [A^3, B^2, C^2], \ldots \}
\]

Downward Closure

\[ M^\downarrow := \{ M' \mid M' \leq M \} \]

\[
[A^2, B]^\downarrow = \{ [A^2], [A, B], [A], [B], [] \}
\]

Minimal Elements

- \( \text{min}(P) \): minimal elements of \( P \) (w.r.t. \( \leq \))
- \( P \) upward closed: \( (\text{min}(P))^\uparrow = P \)
Petri Nets

The Coverability Problem

The Coverability Problem (I)

- Given:
  - \textbf{Init}: ω-marking
  - \textbf{Final}: finite set of markings
  - \textbf{Init} \rightarrow^* \textbf{Final} \uparrow ?

The Coverability Problem (II)

- Given:
  - \textbf{Final}: finite set of markings
  - Compute \textbf{min} \left( \rightarrow^* \textbf{Final} \uparrow \right).
The Valk-Jantzen Theorem

- $V \subseteq \mathbb{N}^k$: upward closed set
- if we can check
  - $(v \downarrow \cap V) = \emptyset$, for all $v \in \mathbb{N}_\omega^k$
- then we can compute
  - we can compute $\min(V)$.

Coverability (II) reducible to Coverability (I)

- $V := (\rightarrow^* \text{ Final} \uparrow)$
- $v \downarrow \rightarrow^* \text{ Final} \uparrow$ can be checked for all $\omega$-markings $v$
Generalized Ordering

\[[A^2, B, C] \leq_{\{A, B\}} [A^3, B^2, C] \quad [A^2, B] \leq_{\{A, B\}} [A^3, B^3]\]

Generalized Upward Closure

\[M \uparrow Q := \{M' | M \leq_Q M'\}\]

\[\left[ A^2, B \right] \uparrow \{A, B\} = \left\{ [A^3, B], [A^2, B^3], [A^3, B^3], [A^3, B^2], [A^2, B^4], [A^3, B^4], \ldots \right\}\]

(Generalized Definition of) Minimal Elements

- \(\text{min}_Q(P)\): minimal elements of \(P\) (w.r.t. \(\leq_Q\))
- \(P\) upward closed w.r.t. \(\leq_Q\): \((\text{min}_Q(P)) \uparrow Q = P\)
The Generalized Coverability Problem (I)

Given:

- \((P_1, P_2)\): partitioning of set of places
- \(Init = (Init_1, Init_2)\)
- \(Final = (Final_1, Final_2)\)

\(Init \rightarrow^* Final \uparrow P_1\) ?

The Generalized Coverability Problem (I) is solvable:

using Petri Net reachability (even if \(Init_1\) is an \(\omega\)-marking)

\[\{ M_1, (M_1, M_2) \rightarrow^* Final \uparrow P_1 \}\] is upward closed w.r.t. \(\leq_{P_1}\)
The Generalized Coverability Problem (II)

- Given:
  - \((P_1, P_2)\): partitioning of set of places
  - \(Init_2\): \(P_2\)-marking
  - \(Final = (Final_1, Final_2)\)

- Compute \(\min\left(\left\{ M_1 | (M_1, Init_2) \xrightarrow{*} Final \uparrow P_1 \right\}\right)\)

Generalized Coverability (II) reducible to Generalized Coverability (I)

- Follows from the Valk-Jantzen Theorem
- Corollary: Generalized Coverability (II) solvable
The (even more) Generalized Coverability Problem

Given:
- \((P_1, P_2)\): partitioning of set of places
- \(Init_2\): \(P_3\)-marking \((P_2 \subseteq P_3)\)
- \(Final = (Final_1, Final_2)\)

Compute \(\min \left\{ M_1 \mid (M_1, Init_2) \xrightarrow{*} Final^{\uparrow P_1} \right\}\)
Discrete-Timed Petri Nets
Markings

[A(2), A(3), A(6), B^2(0), C(4)]
Timed Transitions

\[ [A(2), A(3), A(6), B^2(0), C(4)] \xrightarrow{\text{Time}} [A(4), A(5), A(8), B^2(2), C(6)] \]
Discrete Transitions

\[ [A(4), A(5), A(8), B^2(2), C(6)] \rightarrow t_2 [A(5), A(8), B(4), B^2(2), C(3), C(6)] \]
Discrete-Timed Petri Nets

Computation

\[
\begin{align*}
[A(2), A(3), A(6), B^2(0), C(4)] & \longrightarrow \text{Time} \\
[A(4), A(5), A(8), B^2(2), C(6)] & \longrightarrow t_2 \\
[A(5), A(8), B(4), B^2(2), C(3), C(6)] & \longrightarrow \text{Time} \\
[A(7), A(10), B(6), B^2(4), C(5), C(8)] & \longrightarrow t_1 \\
[A(1), A(7), A(10), B(6), B(4), C(5), C(8)] & \\
\end{align*}
\]

\[
\begin{align*}
[A(2), A(3), A(6), B^2(0), C(4)] & \longrightarrow^* [A(1), A(7), A(10), B(6), B(4), C(5), C(8)]
\end{align*}
\]
Discrete-Timed Petri Nets

Ordering
\[ [A(2), A(6), B(0)] \leq [A(2), A(3), A(6), B(0), B(0), C(4)] \]

Upward Closure
\[ M \uparrow := \{ M' | M \leq M' \} \]
\[ [A(2), A(6), B(0)] \uparrow = \{ [A(2), A(3), A(6), B(0)] , [A(2), A(6), B(0), B(0), C(4)] , [A(2), A(3), A(6), B(0), B(0), C(4)] , \ldots \} \]

The Coverability Problem

- Given:
  - \textit{Init}: set of markings
  - \textit{Final}: finite set of markings
- \textit{Init} \xrightarrow{*} \textit{Final} \uparrow ?
Priced Discrete-Timed Petri Nets (PTPNs)

A

2

3

6

[1..4]

[3..6]

[2..\infty]

B

0

0

(1,2)

C

4

(0,3)

Parosh Aziz Abdulla (Uppsala University)
Priced Discrete-Timed Petri Nets

Timed Transitions

\[
\begin{align*}
\text{Cost} & = 2 \cdot (3 \cdot (2,3) + 2 \cdot (1,2) + 1 \cdot (0,3)) = (16,32) \\
\end{align*}
\]

\[
[A(2), A(3), A(6), B(0), A(0), C(4)] \xrightarrow{(16,32)} Time [A(4), A(5), A(8), B(2), B(2), C(6)]
\]
Cost = (0,2)

\[ [A(4), A(5), A(8), B^2(2), C(6)] \xrightarrow{(0,2)} \text{Disc} \ [A(5), A(8), B(4), B^2(2), C(3), C(6)] \]
Priced Discrete-Timed Petri Nets

Computation

\[
\begin{align*}
[A(2), A(3), A(6), B^2(0), C(4)] & \xrightarrow{(16,32)} Time \\
[A(4), A(5), A(8), B^2(2), C(6)] & \xrightarrow{(0,2)} t_2 \\
[A(5), A(8), B(4), B^2(2), C(3), C(6)] & \xrightarrow{(7,18)} Time \\
[A(7), A(10), B(6), B^2(4), C(5), C(8)] & \xrightarrow{(1,1)} t_1 \\
[A(1), A(7), A(10), B(6), B(4), C(5), C(8)] &
\end{align*}
\]

Total Cost = 

\[(16,32) + (0,2) + (7,18) + (1,1) = (24,53)\]

\[
A(2), A(3), A(6), B^2(0), C(4)] \xrightarrow{(24,53)} [A(1), A(7), A(10), B(6), B(4), C(5), C(8)]
\]
The **Threshold** Priced Coverability Problem

- **Given:**
  - \( \text{Init} \): set of markings
  - \( \text{Final} \): finite set of markings
  - \( v \): price vector.

\[ \exists u \leq v. \text{Init} \xrightarrow{u} \text{Final} \uparrow ? \]

The **Optimal** Priced Coverability Problem

- **Given:**
  - \( \text{Init} \): set of markings
  - \( \text{Final} \): finite set of markings

Find minimal \( v \) such that \( \text{Init} \xrightarrow{v} \text{Final} \uparrow \)
The Optimal Priced Coverability Problem is reducible to
The Threshold Priced Coverability Problem

\[ \{ v \mid \exists u \leq v. \text{ Init} \xrightarrow{u} \text{ Final} \uparrow \} \text{ is upward closed} \]

- The Valk-Jantzen Theorem.
From PTPNs to PNs

Translation Scheme

Solving The Cost Threshold Problem

- Reduce The Cost Threshold Problem to The Generalized Coverability Problem for PNs

Translation Scheme - Encoding of Places

- marking $M$ in PTPN $\rightarrow$ marking $\text{Encoding}(M)$ in PN.
- Differentiate between free and priced places.
- Number of tokens in priced places bounded during timed transitions.

Translation Scheme - Encoding of Transitions

- Simulate discrete and timed transitions.

- Keep track of remaining allowed costs
From PTPNs to PNs

Free Places

\[ \max = 2 \quad \nu = (5, 5) \]

PTPN

PN
From PTPNs to PNs

Free Places

\[ \text{max} = 2 \quad \nu = (5,5) \]

PTPN \hspace{1cm} PN

(0,0)
From PTPNs to PNs

Free Places

\[ \text{max} = 2 \quad \nu = (5, 5) \]

PTPN

\((0,0)\)

\[ \text{P} \]

PN

\#A(0) \quad \#A(1) \quad \#A(2) \quad \#A(>2) \]
From PTPNs to PNs

Free Places

\[ \text{max} = 2 \quad v = (5,5) \]

PTPN

\[
\begin{array}{c}
\text{A} \\
(0,0) \\
0 \\
2 \\
7 \\
5
\end{array}
\]

\[
\begin{array}{c}
\text{A} \\
(0) \\
0 \\
0 \\
2 \\
7 \\
5
\end{array}
\]

PN

\[
\begin{array}{c}
#A(0) \\
#A(1) \\
#A(2) \\
#A(>2)
\end{array}
\]

\[
\begin{array}{c}
\text{A} \\
0 \\
0 \\
2 \\
7 \\
5
\end{array}
\]
From PTPNs to PNs

Free Places

\[ \text{max} = 2 \quad v = (5,5) \]

PTPN

\begin{align*}
(0,0) & \\
0 & 2 \\
7 & 5 \\
\end{align*}

PN

\begin{align*}
\#A(0) & \\
\#A(1) & \\
\#A(2) & \\
\#A(>2) & \\
\end{align*}
From PTPNs to PNs

Free Places

max = 2 \quad v = (5,5)

PTPN

<table>
<thead>
<tr>
<th>(0,0)</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
<td>5</td>
</tr>
</tbody>
</table>

PN

#A(0) #A(1) #A(2) #A(>2)

\[ \bullet \quad \bullet \quad \quad \quad \quad \quad \]

\[ \quad \quad \quad \quad \quad \quad \quad \]
From PTPNs to PNs

Free Places

\[ \text{max} = 2 \quad \nu = (5,5) \]

PTPN

A

(0,0)

0 2

0 7 5

PN

#A(0) #A(1) #A(2) #A(>2)

\[
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\]
From PTPNs to PNs

Free Places

\[ \text{max} = 2 \quad \nu = (5,5) \]

PTPN

\[(0,0)\]  
\[0 \quad 2 \quad 7 \quad 5\]

PN

\[\#A(0) \quad \#A(1) \quad \#A(2) \quad \#A(>2)\]

\[\bullet \quad \bullet \quad \bullet \quad \bullet \]
From PTPNs to PNs

Free Places

\[ \max = 2 \quad \nu = (5,5) \]

\[ [A^2(0), A(2), A(5), A(7)] \quad \rightarrow \quad [\#A(0), \#A(1), \#A(2), \#A(>2)] \]

Parosh Aziz Abdulla (Uppsala University)
Free Places

\[ \max = 2 \quad \nu = (5,5) \]

PTPN

\[ [A^4(0), A^2(2), A(5), A(7)] \]

PN

\#A(0) \quad \#A(1) \quad \#A(2) \quad \#A(>2)
From PTPNs to PNs

Priced Places

\[ \text{PTPN} \quad \text{max} = 2 \quad v = (5,5) \]

\[ \text{PN} \]
From PTPNs to PNs

Priced Places

PTPN \[\text{max} = 2 \quad \nu = (5,5)\]  

PN

\[(1,4)\]  

A
From PTPNs to PNs

Priced Places

PTPN  \[ \text{max} = 2 \quad v = (5,5) \]

\[ \begin{align*}
\#A(0) &= 0 \\
\#A(1) &= 0 \\
\#A(2) &= 0 \\
\#A(>2) &= 0
\end{align*} \]

\[ \begin{align*}
\#A(0) &= 1 \\
\#A(1) &= 1 \\
\#A(2) &= 1 \\
\#A(>2) &= 1
\end{align*} \]

\[ \begin{align*}
\#A(0) &= 2 \\
\#A(1) &= 2 \\
\#A(2) &= 2 \\
\#A(>2) &= 2
\end{align*} \]

\[ \begin{align*}
\#A(0) &= 3 \\
\#A(1) &= 3 \\
\#A(2) &= 3 \\
\#A(>2) &= 3
\end{align*} \]

\[ \begin{align*}
\#A(0) &= 4 \\
\#A(1) &= 4 \\
\#A(2) &= 4 \\
\#A(>2) &= 4
\end{align*} \]

\[ \begin{align*}
\#A(0) &\geq 5 \\
\#A(1) &\geq 5 \\
\#A(2) &\geq 5 \\
\#A(>2) &\geq 5
\end{align*} \]

\[ \begin{align*}
\#A(0) - 5 \\
\#A(1) - 5 \\
\#A(2) - 5 \\
\#A(>2) - 5
\end{align*} \]
From PTPNs to PNs

Priced Places

PTPN  max = 2  ν = (5,5)

PN

\[
\begin{align*}
\#A(0) &= 0 & #A(1) &= 0 & #A(2) &= 0 & #A(>2) &= 0 \\
\#A(0) &= 1 & #A(1) &= 1 & #A(2) &= 1 & #A(>2) &= 1 \\
\#A(0) &= 2 & #A(1) &= 2 & #A(2) &= 2 & #A(>2) &= 2 \\
\#A(0) &= 3 & #A(1) &= 3 & #A(2) &= 3 & #A(>2) &= 3 \\
\#A(0) &= 4 & #A(1) &= 4 & #A(2) &= 4 & #A(>2) &= 4 \\
\#A(0) &\geq 5 & #A(1) &\geq 5 & #A(2) &\geq 5 & #A(>2) &\geq 5 \\
\#A(0) - 5 &\geq 5 & #A(1) - 5 &\geq 5 & #A(2) - 5 &\geq 5 & #A(>2) - 5 &\geq 5 \\
\end{align*}
\]
From PTPNs to PNs

Priced Places

PTPN  $\max = 2$  $\nu = (5,5)$

$A$

(PN) $A(0) = 0$  $A(1) = 0$  $A(2) = 0$  $A(> 2) = 0$

$A(0) = 0$  $A(1) = 0$  $A(2) = 0$  $A(> 2) = 0$

$A(0) = 1$  $A(1) = 1$  $A(2) = 1$  $A(> 2) = 1$

$A(0) = 2$  $A(1) = 2$  $A(2) = 2$  $A(> 2) = 2$

$A(0) = 3$  $A(1) = 3$  $A(2) = 3$  $A(> 2) = 3$

$A(0) = 4$  $A(1) = 4$  $A(2) = 4$  $A(> 2) = 4$

$A(0) \geq 5$  $A(1) \geq 5$  $A(2) \geq 5$  $A(> 2) \geq 5$

$A(0) - 5$  $A(1) - 5$  $A(2) - 5$  $A(> 2) - 5$
From PTPNs to PNs

Priced Places

PTPN  \( \max = 2 \)  \( \nu = (5,5) \)

\[ \begin{align*}
#A(0) &= 0 \\
#A(1) &= 0 \\
#A(2) &= 0 \\
#A(>2) &= 0 \\
#A(0) &= 1 \\
#A(1) &= 1 \\
#A(2) &= 1 \\
#A(>2) &= 1 \\
#A(0) &= 2 \\
#A(1) &= 2 \\
#A(2) &= 2 \\
#A(>2) &= 2 \\
#A(0) &= 3 \\
#A(1) &= 3 \\
#A(2) &= 3 \\
#A(>2) &= 3 \\
#A(0) &= 4 \\
#A(1) &= 4 \\
#A(2) &= 4 \\
#A(>2) &= 4 \\
#A(\geq 5) &= 5 \\
#A(>2) &= 5 \\
#A(0) &= 5 \\
#A(1) &= 5 \\
#A(2) &= 5 \\
#A(>2) &= 5 \\
#A(0) &= 0 \\
#A(1) &= 0 \\
#A(2) &= 0 \\
#A(>2) &= 0 \\
\end{align*} \]
From PTPNs to PNs

Priced Places

PTPN max = 2 \( v = (5,5) \)

PN

A

\[ \begin{align*}
#A(0) &= 0 \\
#A(0) &= 1 \\
#A(0) &= 2 \\
#A(0) &= 3 \\
#A(0) &= 4 \\
#A(0) &\geq 5 \\
#A(0) - 5 \\
\end{align*} \]

\[ \begin{align*}
#A(1) &= 0 \\
#A(1) &= 1 \\
#A(1) &= 2 \\
#A(1) &= 3 \\
#A(1) &= 4 \\
#A(1) &\geq 5 \\
#A(1) - 5 \\
\end{align*} \]

\[ \begin{align*}
#A(2) &= 0 \\
#A(2) &= 1 \\
#A(2) &= 2 \\
#A(2) &= 3 \\
#A(2) &= 4 \\
#A(2) &\geq 5 \\
#A(2) - 5 \\
\end{align*} \]

\[ \begin{align*}
#A(>2) &= 0 \\
#A(>2) &= 1 \\
#A(>2) &= 2 \\
#A(>2) &= 3 \\
#A(>2) &= 4 \\
#A(>2) &\geq 5 \\
#A(>2) - 5 \\
\end{align*} \]
From PTPNs to PNs

Priced Places

PTPN: max = 2  \( v = (5,5) \)

PN:

\[
\begin{align*}
\#A(0) &= 0 \\
\#A(1) &= 0 \\
\#A(2) &= 0 \\
\#A(>2) &= 0 \\
\#A(0) &= 1 \\
\#A(1) &= 1 \\
\#A(2) &= 1 \\
\#A(>2) &= 1 \\
\#A(0) &= 2 \\
\#A(1) &= 2 \\
\#A(2) &= 2 \\
\#A(>2) &= 2 \\
\#A(0) &= 3 \\
\#A(1) &= 3 \\
\#A(2) &= 3 \\
\#A(>2) &= 3 \\
\#A(0) &= 4 \\
\#A(1) &= 4 \\
\#A(2) &= 4 \\
\#A(>2) &= 4 \\
\#A(0) &\geq 5 \\
\#A(1) &\geq 5 \\
\#A(2) &\geq 5 \\
\#A(>2) &\geq 5 \\
\#A(0) - 5 \\
\#A(1) - 5 \\
\#A(2) - 5 \\
\#A(>2) - 5
\end{align*}
\]
From PTPNs to PNs

Priced Places

**PTPN**  \[ \text{max} = 2 \quad v = (5,5) \]

**PN**

\[
\begin{align*}
\#A(0) &= 0 & \#A(1) &= 0 & \#A(2) &= 0 & \#A(\geq 2) &= 0 \\
\#A(0) &= 1 & \#A(1) &= 1 & \#A(2) &= 1 & \#A(\geq 2) &= 1 \\
\#A(0) &= 2 & \#A(1) &= 2 & \#A(2) &= 2 & \#A(\geq 2) &= 2 \\
\#A(0) &= 3 & \#A(1) &= 3 & \#A(2) &= 3 & \#A(\geq 2) &= 3 \\
\#A(0) &= 4 & \#A(1) &= 4 & \#A(2) &= 4 & \#A(\geq 2) &= 4 \\
\#A(0) &\geq 5 & \#A(1) &\geq 5 & \#A(2) &\geq 5 & \#A(\geq 2) &\geq 5 \\
\#A(0) &= 5 & \#A(1) &= 5 & \#A(2) &= 5 & \#A(\geq 2) &= 5 \\
\end{align*}
\]

\[ [A^2(0), A^7(2), A^2(8), A(9)] \rightarrow \]

Parosh Aziz Abdulla (Uppsala University)
From PTPNs to PNs

Priced Places

PTPN  \( \text{max} = 2 \)  \( v = (5,5) \)

PN

\[ [A^3(0), A^7(2), A^2(8), A(9)] \]

\[ \rightarrow \]

\[ A^3(0) = 0 \]
\[ A(0) = 0 \]
\[ A(1) = 0 \]
\[ A(2) = 0 \]
\[ A(>2) = 0 \]

\[ A^3(0) = 1 \]
\[ A(0) = 1 \]
\[ A(1) = 1 \]
\[ A(2) = 1 \]
\[ A(>2) = 1 \]

\[ A^3(0) = 2 \]
\[ A(0) = 2 \]
\[ A(1) = 2 \]
\[ A(2) = 2 \]
\[ A(>2) = 2 \]

\[ A^3(0) = 3 \]
\[ A(0) = 3 \]
\[ A(1) = 3 \]
\[ A(2) = 3 \]
\[ A(>2) = 3 \]

\[ A^3(0) \geq 5 \]
\[ A(0) \geq 5 \]
\[ A(1) \geq 5 \]
\[ A(2) \geq 5 \]
\[ A(>2) \geq 5 \]

\[ A^3(0) - 5 \]
\[ A(0) - 5 \]
\[ A(1) - 5 \]
\[ A(2) - 5 \]
\[ A(>2) - 5 \]
From PTPNs to PNs

Priced Places

PTPN  $\text{max} = 2$  $\nu = (5,5)$

PN

$[A^4(0), A^7(2), A^2(8), A(9)]$  $\rightarrow$

Parosh Aziz Abdulla (Uppsala University)  Priced Discrete-Timed Petri Nets  February 8, 2010  31 / 39
From PTPNs to PNs

Priced Places

\[
\text{PTPN} \quad \text{max} = 2 \quad \nu = (5,5)
\]

\[
[A^5(0), A^7(2), A^2(8), A(9)] \rightarrow
\]

\[
\begin{align*}
\#A(0) &= 0 & \#A(1) &= 0 & \#A(2) &= 0 & \#A(> 2) &= 0 \\
\#A(0) &= 1 & \#A(1) &= 1 & \#A(2) &= 1 & \#A(> 2) &= 1 \\
\#A(0) &= 2 & \#A(1) &= 2 & \#A(2) &= 2 & \#A(> 2) &= 2 \\
\#A(0) &= 3 & \#A(1) &= 3 & \#A(2) &= 3 & \#A(> 2) &= 3 \\
\#A(0) &= 4 & \#A(1) &= 4 & \#A(2) &= 4 & \#A(> 2) &= 4 \\
\#A(0) &\geq 5 & \#A(1) &\geq 5 & \#A(2) &\geq 5 & \#A(> 2) &\geq 5 \\
\#A(0) - 5 &\quad \#A(1) - 5 & \#A(2) - 5 & \#A(> 2) - 5
\end{align*}
\]
From PTPNs to PNs

Priced Places

**PTPN**  \( \max = 2 \)  \( v = (5,5) \)

\[
[\#A^6(0), \#A^7(2), \#A^2(8), \#A(9)]
\]

**PN**

\[
\#A(0) = 0 \quad \#A(1) = 0 \quad \#A(2) = 0 \quad \#A(>2) = 0
\]

\[
\#A(0) = 1 \quad \#A(1) = 1 \quad \#A(2) = 1 \quad \#A(>2) = 1
\]

\[
\#A(0) = 2 \quad \#A(1) = 2 \quad \#A(2) = 2 \quad \#A(>2) = 2
\]

\[
\#A(0) = 3 \quad \#A(1) = 3 \quad \#A(2) = 3 \quad \#A(>2) = 3
\]

\[
\#A(0) = 4 \quad \#A(1) = 4 \quad \#A(2) = 4 \quad \#A(>2) = 4
\]

\[
\#A(0) \geq 5 \quad \#A(1) \geq 5 \quad \#A(2) \geq 5 \quad \#A(>2) \geq 5
\]

\[
\#A(0) - 5 \quad \#A(1) - 5 \quad \#A(2) - 5 \quad \#A(>2) - 5
\]
From PTPNs to PNs

Priced Places

\[ \text{PTPN} \quad \text{max} = 2 \quad v = (5,5) \]

\[ \text{PN} \]
From PTPNs to PNs

Priced Places

PTPN \( \text{max} = 2 \quad \nu = (5,5) \)

\[ \begin{align*}
\#A(0) &= 0 \\
\#A(1) &= 0 \\
\#A(2) &= 0 \\
\#A(>2) &= 0
\end{align*} \]

\[ \begin{align*}
\#A(0) &= 1 \\
\#A(1) &= 1 \\
\#A(2) &= 1 \\
\#A(>2) &= 1
\end{align*} \]

\[ \begin{align*}
\#A(0) &= 2 \\
\#A(1) &= 2 \\
\#A(2) &= 2 \\
\#A(>2) &= 2
\end{align*} \]

\[ \begin{align*}
\#A(0) &= 3 \\
\#A(1) &= 3 \\
\#A(2) &= 3 \\
\#A(>2) &= 3
\end{align*} \]

\[ \begin{align*}
\#A(0) &= 4 \\
\#A(1) &= 4 \\
\#A(2) &= 4 \\
\#A(>2) &= 4
\end{align*} \]

\[ \begin{align*}
\#A(0) &\geq 5 \\
\#A(1) &\geq 5 \\
\#A(2) &\geq 5 \\
\#A(>2) &\geq 5
\end{align*} \]

\[ \begin{align*}
\#A(0) - 5 \\
\#A(1) - 5 \\
\#A(2) - 5 \\
\#A(>2) - 5
\end{align*} \]
From PTPNs to PNs

“Remaining Cost” Places

- keep track of remaining allowed costs.
- if \( v = (5, 5) \) then, a place \( \#R(i, j) \) for \( 0 \leq i, j \leq 5 \).
- token in \( \#R(i, j) \) indicates remaining cost = \((i, j)\).
From PTPNs to PNs

Discrete Transitions

PTPN

A

[1..2]

(1,2)

[2..\infty)

B

PN

A

B
From PTPNs to PNs

Discrete Transitions

**PTPN**

- \( [1..2] \)
- \( (1,2) \)
- \( [2..\infty) \)

**PN**

- \( \#A(1) \)
- \( \#B(>2) \)
- \( R = (4,3) \)
- \( R = (3,1) \)
From PTPNs to PNs

Discrete Transitions

PTPN

PN

A

[1..2]

(1,2)

[2..∞)

B
From PTPNs to PN

Discrete Transitions

PTPN

PN

A

[1..2]

B

(1,2)

[2..∞)

#A(1) = 3

#A(1) = 2

#B(> 2) = 1

#B(> 2) = 2

R = (4,3)

R = (3,1)
From PTPNs to PNs

Discrete Transitions

**PTPN**

\[(1, 2)\]

\[2 \cdots \infty\]

**PN**

\[\#A(1) - 5\]

\[\#B(> 2) = 1\]

\[R = (4, 3)\]

\[\#B(> 2) = 2\]

\[R = (3, 1)\]
From PTPNs to PN

Discrete Transitions

PTPN

\[
\begin{align*}
&A(1) = 3 \\
&R = (4, 3)
\end{align*}
\]

PN

\[
\begin{align*}
#A(1) = 2 \\
#B(> 2) = 5 \\
R = (3, 1)
\end{align*}
\]
From PTPNs to PNs

Timed Transitions

PTPN
From PTPNs to PNs

Timed Transitions

PTPN

\[(0,0)\]

\[0\]

\[0\]

\[2\]

\[2\]

\[2\]

\[2\]

\[2\]

\[7\]

\[7\]

\[8\]

A

\[(1,2)\]

\[1\]

B

Parosh Aziz Abdulla (Uppsala University)
From PTPNs to PNs
Timed Transitions

PTPN

A

B

(0,0)

0

2

2

7

8

(1,2)

1

2

3

3

3

8

B

Time
From PTPNs to PNs

Timed Transitions

PTPN

PN

A

B

A

B

(0,0) (1,2)

(0,0) (1,2)

Time
From PTPNs to PNs

Timed Transitions

From PTPNs to PNs

Timed Transitions

PTPN

PN

A
(0,0)
0
0
2
2
2
2
2
7
7
8

B
(1,2)
1

#A(0)

A
(0,0)
1
3
3
3
3
8
8
9

B
(1,2)
2

Time
From PTPNs to PNs

Timed Transitions

PTPN → PN

Time

Parosh Aziz Abdulla (Uppsala University)
From PTPNs to PNs

Timed Transitions

PTPN

\[
\begin{align*}
(0,0) & : 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 7 & 7 & 8 \\
A & \begin{array}{l}
0 \\
0 \\
2 \\
2 \\
2 \\
2 \\
7 \\
7 \\
8 \\
\end{array} \\
(1,2) & : 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 9 \\
B & \begin{array}{l}
1 \\
3 \\
3 \\
3 \\
3 \\
3 \\
3 \\
3 \\
9 \\
\end{array}
\end{align*}
\]

PN

\[
\begin{align*}
\#A(0) & : \bullet & \bullet \\
\#A(2) & : \bullet & \bullet & \bullet & \bullet \\
\#A(>2) & : \bullet & \bullet \\
\#B(1) & : \bullet \\
\end{align*}
\]
From PTPNs to PNs

Timed Transitions

PTPN

\[(0,0), 0, 0222222778, 322222, 1133333889, 2222222778\]

\[A(0), A(0), A(2), A(>2), B(1) = 1, R = (4,3)\]

PN

\[A(0), A(2), A(>2), B(1) = 1, R = (4,3)\]
From PTPNs to PNs

Timed Transitions

PTPN

(0,0)

0

0 2 2

2 2 2 2

8

A

(0,0)

1

1 3 3

3 3 3 3

3 8 8

9

A

Time

B

(1,2)

1

B

B

#A(0)

#A(2)

#A(> 2)

#B(1) = 1

#R = (4,3)

PN

Parosh Aziz Abdulla (Uppsala University)
From PTPNs to PNs

Timed Transitions

PTPN

\[
\begin{array}{c}
(0,0) \\
0 \\
0 \\
2 \\
2 \\
2 \\
2 \\
2 \\
8 \\
A
\end{array}
\]

\[
\begin{array}{c}
(0,0) \\
1 \\
1 \\
3 \\
3 \\
3 \\
3 \\
3 \\
9 \\
B
\end{array}
\]

\[
\begin{array}{c}
(1,2) \\
1 \\
B
\end{array}
\]

\[
\begin{array}{c}
(1,2) \\
2 \\
B
\end{array}
\]

Time

PN

\[
\begin{array}{c}
\#A(0) \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
\#A(2) \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
\#A(>2) \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
\#B(1)=1 \\
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
\#R=(4,3) \\
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
\#A(1) \\
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
g \\
\end{array}
\]

Parosh Aziz Abdulla (Uppsala University)
From PTPNs to PNs

Timed Transitions

**PTPN**

- **A**
  - \((0,0)\)
  - \((0,0)\)
  - \(0\)
  - \(0\)
  - \(2\)
  - \(2\)
  - \(2\)
  - \(7\)
  - \(7\)
  - \(8\)
  - \(B\)

- **B**
  - \((1,2)\)
  - \(1\)

**PN**

- **A**
  - \((0,0)\)
  - \(1\)
  - \(3\)
  - \(3\)
  - \(3\)
  - \(3\)
  - \(8\)
  - \(8\)
  - \(9\)

- **B**
  - \((1,2)\)
  - \(2\)

**Time**

- \#A\((0)\)
- \#A\((2)\)
- \#A\((>2)\)
- \#B\((1)\) = 1
- \#R = (4,3)
From PTPNs to PNs

Timed Transitions

**PTPN**

A

(0,0)

0

0 2 2

2 2 2 2 2

2 7 7 8

B

(1,2)

1

**PN**

A

(0,0)

1

1 3 3

3 3 3 3

3 8 8 9

B

(1,2)

1

2

#A(0)

#A(2)

#A(>2)

#B(1) = 1

#B(2) = 1

#R = (4,3)

Parosh Aziz Abdulla (Uppsala University)
From PTPNs to PN

Timed Transitions

PTPN

<table>
<thead>
<tr>
<th>(0,0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>

A

<table>
<thead>
<tr>
<th>(0,0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>

A

<table>
<thead>
<tr>
<th>(1,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

B

<table>
<thead>
<tr>
<th>(1,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

B

Time

PN

#A(0)

#A(2)

#A(> 2)

#B(1) = 1

#B(2) = 1

#R = (4,3)

#R = (3,1)

Parosh Aziz Abdulla (Uppsala University)
From PTPNs to PNs

\[
\begin{align*}
\mathcal{M}_1 & \xrightarrow{g} \mathcal{M}_2 \\
\textbullet \quad \text{Free Places:} & \\
& \mathcal{M}_2(\#A(0)) = 0 \\
& \mathcal{M}_2(\#A(j + 1)) = \mathcal{M}_1(\#A(j)), \quad 0 \leq j < \max \\
& \mathcal{M}_2(\#A(> \max)) = \mathcal{M}_1(\#A(> \! \max)) + \mathcal{M}_1(\#A(\max))
\end{align*}
\]

\[
\begin{align*}
\text{Priced Places:} & \\
& \mathcal{M}_2(\#A(0)=0) = 1 \\
& \mathcal{M}_2(\#A(0)=i) = 0, \quad 0 < i \leq \max. \\
& \mathcal{M}_2(\#A(i + 1)=j) = \mathcal{M}_2(\#A(i)=j), \quad 0 \leq i < \max, \quad j \leq 0 \leq R - 1 \\
& \mathcal{M}_2(\#A(i + 1) \geq R) = \mathcal{M}_2(\#A(i) \geq R), \quad 0 \leq i < \max \\
& \mathcal{M}_2(\#A(i + 1) \geq R) = \mathcal{M}_2(\#A(i) \geq R), \quad 0 \leq i < \max 
\end{align*}
\]
Algorithm

The **Threshold** Priced Coverability Problem:

\[ \exists u \leq v. \text{Init} \xrightarrow{u} \text{Final} \uparrow ? \]

Algorithm:

- \( F_0, G_0, F_1, G_1, F_2, G_2, \ldots \)

- \( F'_0 := \text{encoding}(\text{Final}) \).
- \( F_0 \) is upward closure of \( F'_0 \) w.r.t. the ordering on PN.
- \( F_i \): set of PN-markings \( M \) that
  - can reach \( F_0 \) via a computation starting with a \( g \)-transition.
  - \( M \) is zero on all cost places.
- \( G_i \): defined analogously, except that we start with a normal transition.
Algorithm:

\[ F_0, G_0, F_1, G_1, F_2, G_2, \ldots \]

Since \( F_0 \) is upward closed, all sets \( F_i \) and \( G_i \) are upward closed on the free places.

- follows from monotonicity of Petri nets and monotonicity of \( g \)-transitions.

Define \( H_\ell := \bigcup_i F_i \).

Markings in \( H_\ell \) are

- upward closed on free places.
- have empty cost places

The sequence converges at some \( \ell \) by Dickson's lemma.

Check whether \( Encoding(Init) \) can reach \( H_\ell \) via PN-transitions.
Further Results

- Coverability for Petri Nets with one-inhibitor arc.
- Priced Reachability for Priced (untimed) Petri nets.
- The Dense-Time case.