Well-Quasi-Orderings

Well-Quasi-Orderings

- Quasi-Orderings
- Well-Quasi-Orderings (WQOs)
- Very-Well-Quasi-Orderings
- Building WQOs

Quasi-Ordering

$$\langle A, \sqsubseteq \rangle$$

set

quasi-order

binary relation

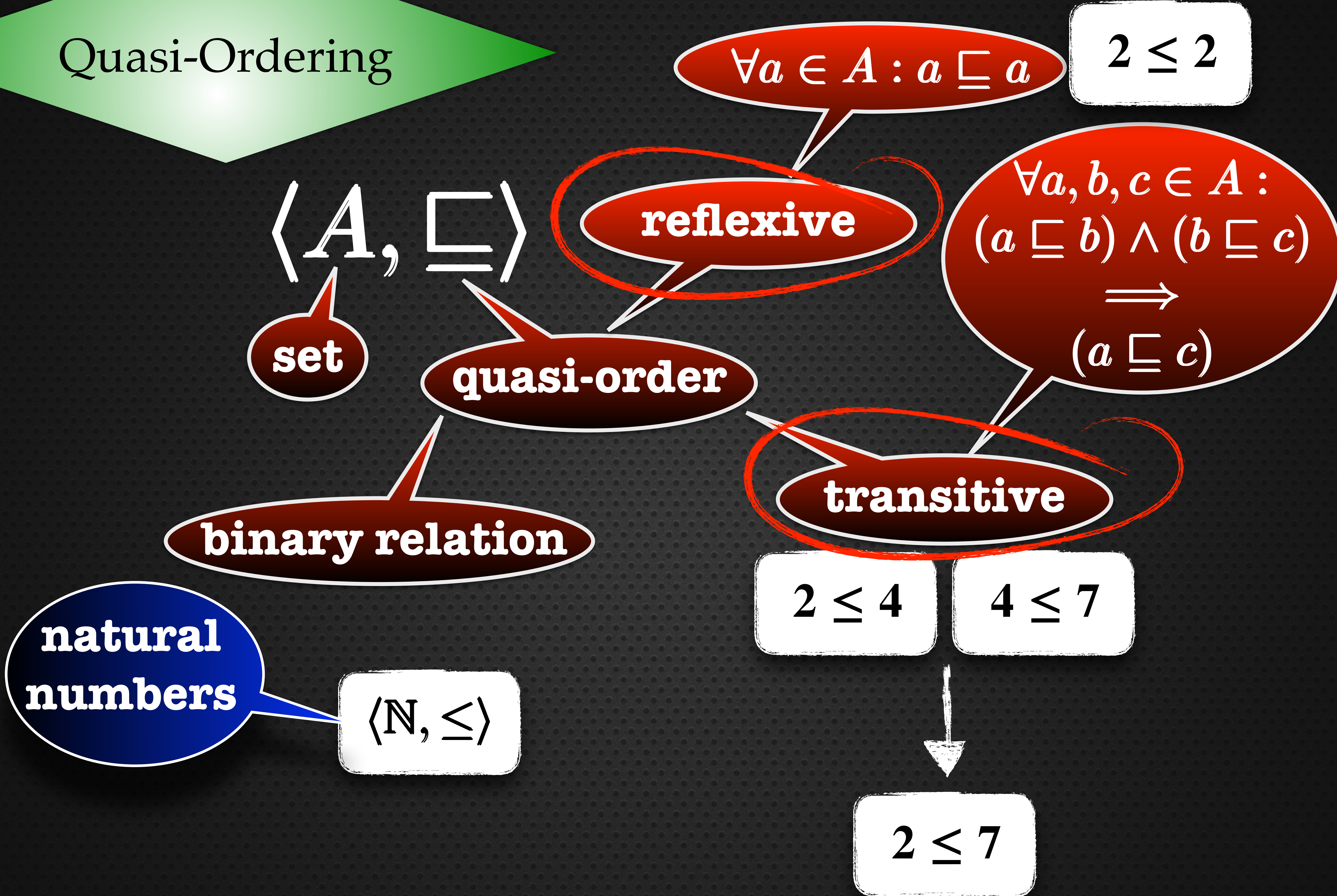
reflexive

$$\forall a \in A : a \sqsubseteq a$$

transitive

$$\forall a, b, c \in A : (a \sqsubseteq b) \wedge (b \sqsubseteq c) \Longrightarrow (a \sqsubseteq c)$$

Quasi-Ordering



Quasi-Ordering

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set

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$\forall a \in A : a \sqsubseteq a$

transitive

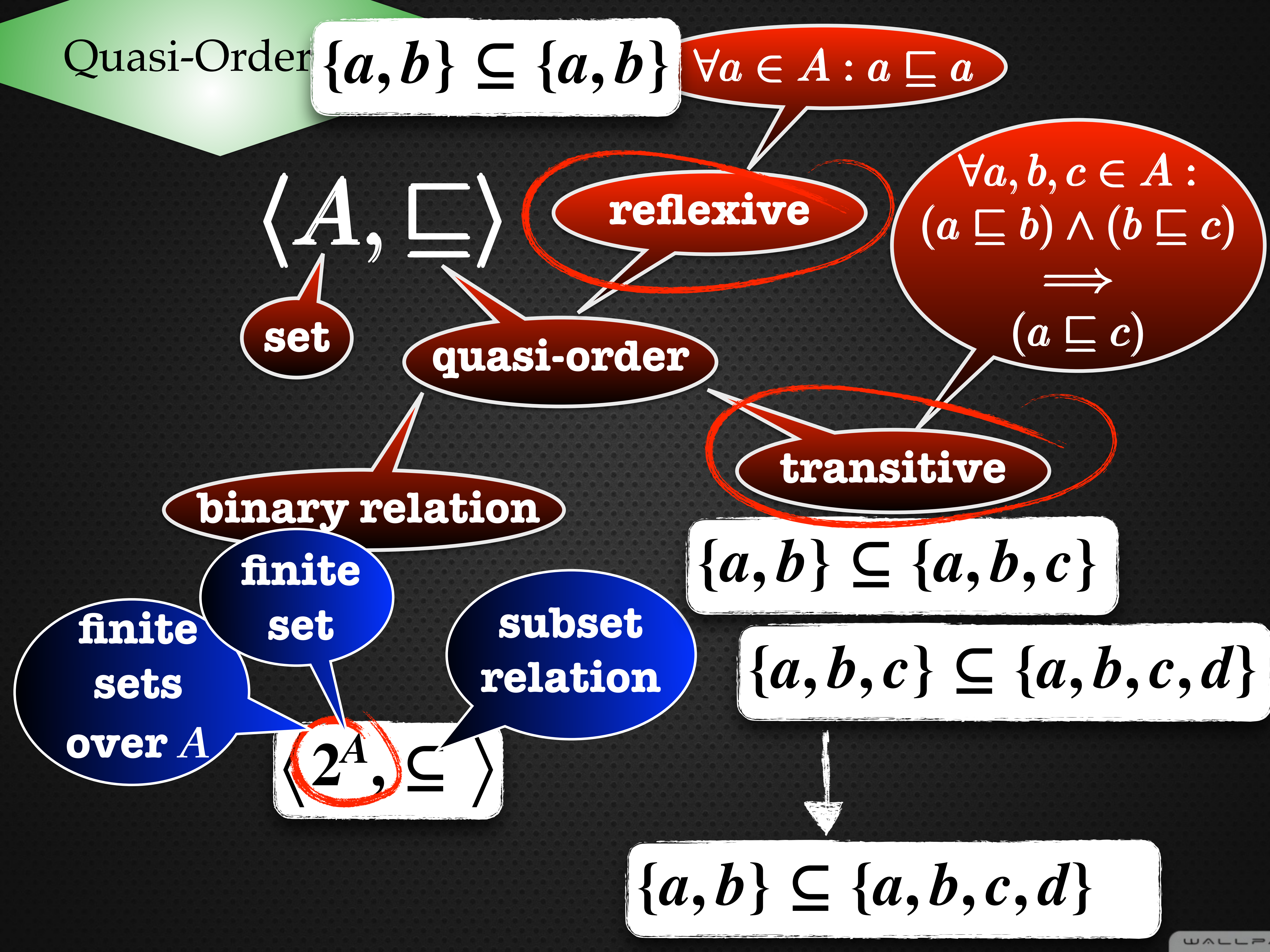
$\forall a, b, c \in A : (a \sqsubseteq b) \wedge (b \sqsubseteq c) \Longrightarrow (a \sqsubseteq c)$

natural numbers $\langle \mathbb{N}, \leq \rangle$

natural numbers $\langle \mathbb{N}, = \rangle$

integers $\langle \mathbb{I}, \leq \rangle$

finite sets $\langle A, = \rangle$



Quasi-Order
$\{a,b\} \subseteq \{a,b\}$
$\forall a \in A : a \sqsubseteq a$
$\langle A, \sqsubseteq \rangle$
reflexive
$\forall a,b,c \in A : (a \sqsubseteq b) \wedge (b \sqsubseteq c) \Longrightarrow (a \sqsubseteq c)$
set
quasi-order
transitive
binary relation
$\{a,b\} \subseteq \{a,b,c\}$
finite set
finite sets over A
subset relation
$\{a,b,c\} \subseteq \{a,b,c,d\}$
$\langle 2^A, \subseteq \rangle$
$\{a,b\} \subseteq \{a,b,c,d\}$

Quasi-Ordering

$$\langle A, \sqsubseteq \rangle$$

set

quasi-order

reflexive

$\forall a \in A : a \sqsubseteq a$

transitive

$\forall a, b, c \in A :$
$(a \sqsubseteq b) \wedge (b \sqsubseteq c)$
$\Longrightarrow$
$(a \sqsubseteq c)$

binary relation

finite multisets over A

finite set

multiset ordering

$$\langle A^{\circledast}, \sqsubseteq \rangle$$

$$A = \{a, b, c, d\}$$

multiset over A

multiset over A

$$[a, a, b, c] \sqsubseteq [a, a, a, b, c, c, d]$$

Quasi-Ordering

$\langle A, \sqsubseteq \rangle$

set

quasi-order

binary relation

reflexive

$\forall a \in A : a \sqsubseteq a$

transitive

$\forall a, b, c \in A :$
$(a \sqsubseteq b) \wedge (b \sqsubseteq c)$
$\Longrightarrow$
$(a \sqsubseteq c)$

finite multisets over A

finite set

multiset ordering

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$A = \{a, b, c, d\}$

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Quasi-Ordering

$\langle A, \sqsubseteq \rangle$

set

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binary relation

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$A = \{a, b, c, d\}$

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Quasi-Ordering

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transitive

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$(a \sqsubseteq b) \wedge (b \sqsubseteq c)$
$\Longrightarrow$
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finite multisets over A

finite set

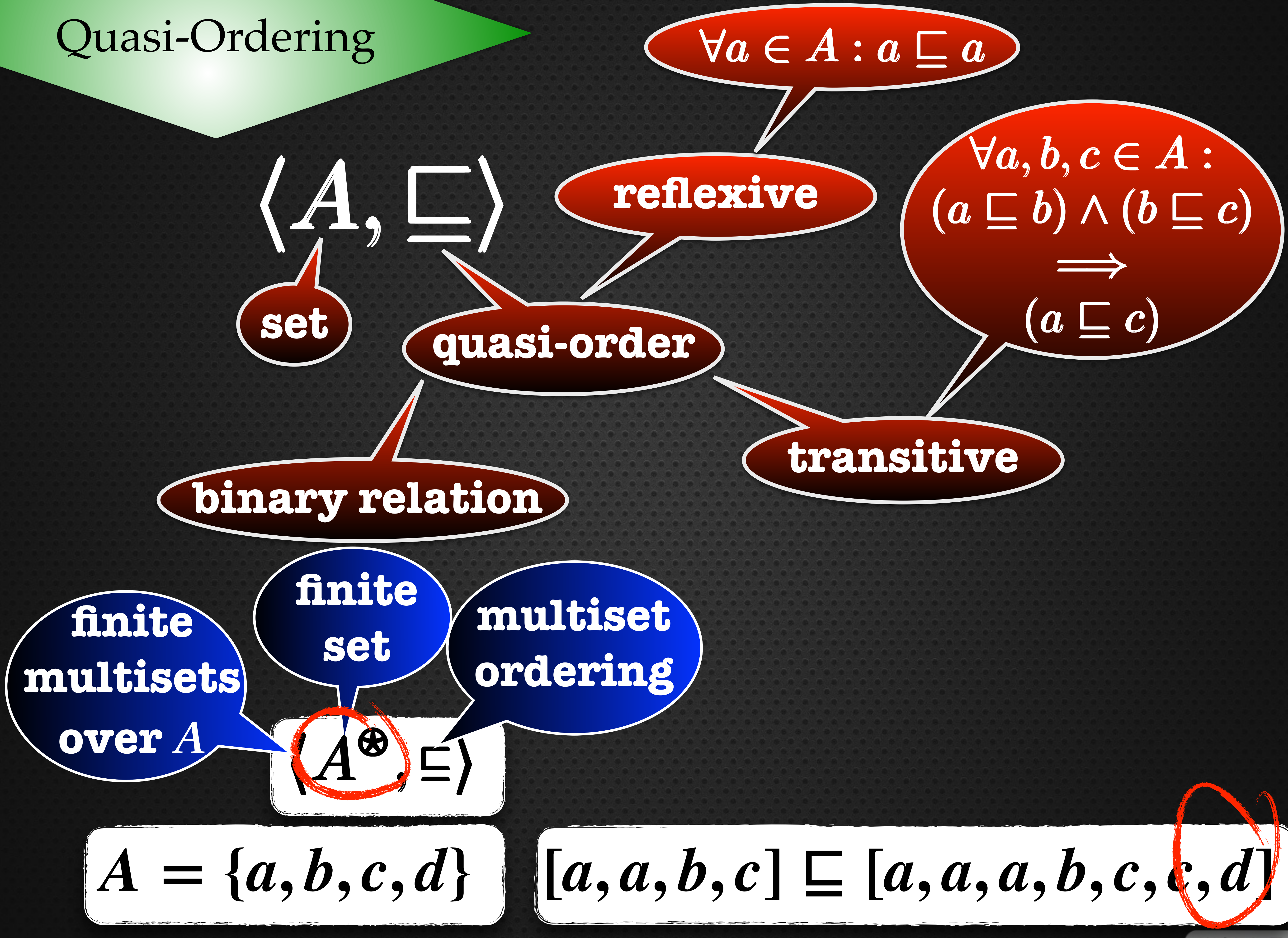
multiset ordering

$\langle A^{\circledast}, \sqsubseteq \rangle$

$A = \{a, b, c, d\}$

$[a, a, b, c] \sqsubseteq [a, a, a, b, c, c, d]$

Quasi-Ordering



Quasi-Ordering

alternative representations

$$[2,1,1,0] \sqsubseteq [3,1,2,1]$$

$$[a^2,b,c] \sqsubseteq [a^3,b,c^2,d]$$

$$A = \{a,b,c,d\}$$

$$[a,a,b,c] \sqsubseteq [a,a,a,b,c,c,d]$$

Quasi-Ordering

$\forall a \in A : a \sqsubseteq a$

$$\langle A, \sqsubseteq \rangle$$

set

quasi-order

reflexive

$\forall a, b, c \in A :$
$(a \sqsubseteq b) \wedge (b \sqsubseteq c)$
$\Longrightarrow$
$(a \sqsubseteq c)$

transitive

binary relation

finite multisets over A

finite set

multiset ordering

$\langle A^{\circledast}, \sqsubseteq \rangle$

finite multisets over A

finite set

$m_1 \sqsubseteq m_2 :$
$|m_1| \leq |m_2|$

$\langle A^{\circledast}, \sqsubseteq \rangle$

Well-Quasi-Orderings

- Quasi-Orderings
- Well-Quasi-Orderings (WQOs)
- Very-Well-Quasi-Orderings
- Building WQOs

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$

quasi-order

well-quasi-order

infinite sequence of elements from A

good sequence

$a_0, a_1, a_2, \ldots\ldots, a_i, \ldots\ldots, a_j, \ldots\ldots$

$\exists i, j : (i < j) \wedge (a_i \sqsubseteq a_j)$

$\sqsubseteq$

WQO = all sequences are good

"… for any infinite sequence of elements in A, there are two elements such that the later element is larger (wrt. $\sqsubseteq$) than the earlier element …"

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$

quasi-order

well-quasi-order

infinite sequence of elements from A

good sequence

$a_0, a_1, a_2, \ldots\ldots, a_i, \ldots\ldots, a_j, \ldots\ldots$

$\sqsubseteq$

$\exists i, j : (i < j) \wedge (a_i \sqsubseteq a_j)$

$\langle \mathbb{N}, \leq \rangle$ ✔

natural numbers

WQO = all sequences are good

9 7 5 4 3 0 8

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$ — quasi-order

infinite sequence of elements from A

$a_0, a_1, a_2, \ldots\ldots, a_i, \ldots\ldots, a_j, \ldots\ldots$

$a_i \sqsubseteq a_j$

good sequence

$\exists i, j : (i < j) \wedge (a_i \sqsubseteq a_j)$

$\langle \mathbb{N}, \leq \rangle$ ✔

$\langle \mathbb{I}, \leq \rangle$ ✗ — integers

bad sequence

9 7 0 -2 -5 -10 -15 ...

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$ — quasi-order

infinite sequence of elements from A

good sequence

$a_0, a_1, a_2, \ldots\ldots, a_i, \ldots\ldots, a_j, \ldots\ldots$

$\sqsubseteq$

$\exists i, j : (i < j) \wedge (a_i \sqsubseteq a_j)$

$\langle \mathbb{N}, \leq \rangle$ ✔

$\langle \mathbb{I}, \leq \rangle$ ✗

$\langle \mathbb{N}, = \rangle$ ✗ — natural numbers

bad sequence

9 7 0 6 5 10 15 …

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$ — quasi-order

infinite sequence of elements from A

good sequence

$a_0, a_1, a_2, \ldots\ldots, a_i, \ldots\ldots, a_j, \ldots\ldots$

$\exists i, j : (i < j) \wedge (a_i \sqsubseteq a_j)$

$\sqsubseteq$

$\langle \mathbb{N}, \leq \rangle$ ✔

$\langle A, = \rangle$ ✔ $A = \{a, b, c\}$

finite set

a b c b

$\langle \mathbb{I}, \leq \rangle$ ✗

$\langle \mathbb{N}, = \rangle$ ✗

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$ — quasi-order

infinite sequence of elements from A

good sequence

$a_0, a_1, a_2, \ldots\ldots, a_i, \ldots\ldots, a_j, \ldots\ldots$

$a_i \sqsubseteq a_j$

$\exists i, j : (i < j) \wedge (a_i \sqsubseteq a_j)$

$\langle A^{\circledast}, \sqsubseteq \rangle$

finite multisets over A

finite set

multiset ordering

Well-Quasi-Ordering

$$\langle A, \sqsubseteq \rangle$$

quasi-order

infinite sequence of elements from A

good sequence

$$a_0, a_1, a_2, \ldots\ldots, a_i, \ldots\ldots, a_j, \ldots\ldots$$

$$\exists i, j : (i < j) \wedge (a_i \sqsubseteq a_j)$$

$$\sqsubseteq$$

$A = \{a, b\}$

$[a, a, b, b, b, b] \in A^{\circledast}$

(2,4)

a = 2

b = 4

finite multisets over A

finite set

multiset ordering

$$\langle A^{\circledast}, \sqsubseteq \rangle$$

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$ — quasi-order

infinite sequence of elements from A

good sequence

$a_0, a_1, a_2, \ldots\ldots, a_i, \ldots\ldots, a_j, \ldots\ldots$

$\exists i, j : (i < j) \wedge (a_i \sqsubseteq a_j)$

$a_i \sqsubseteq a_j$

$A = \{a, b\}$

$[a, a, b, b, b, b] \in A^{\circledast}$

(2,4)

a = 2

b = 4

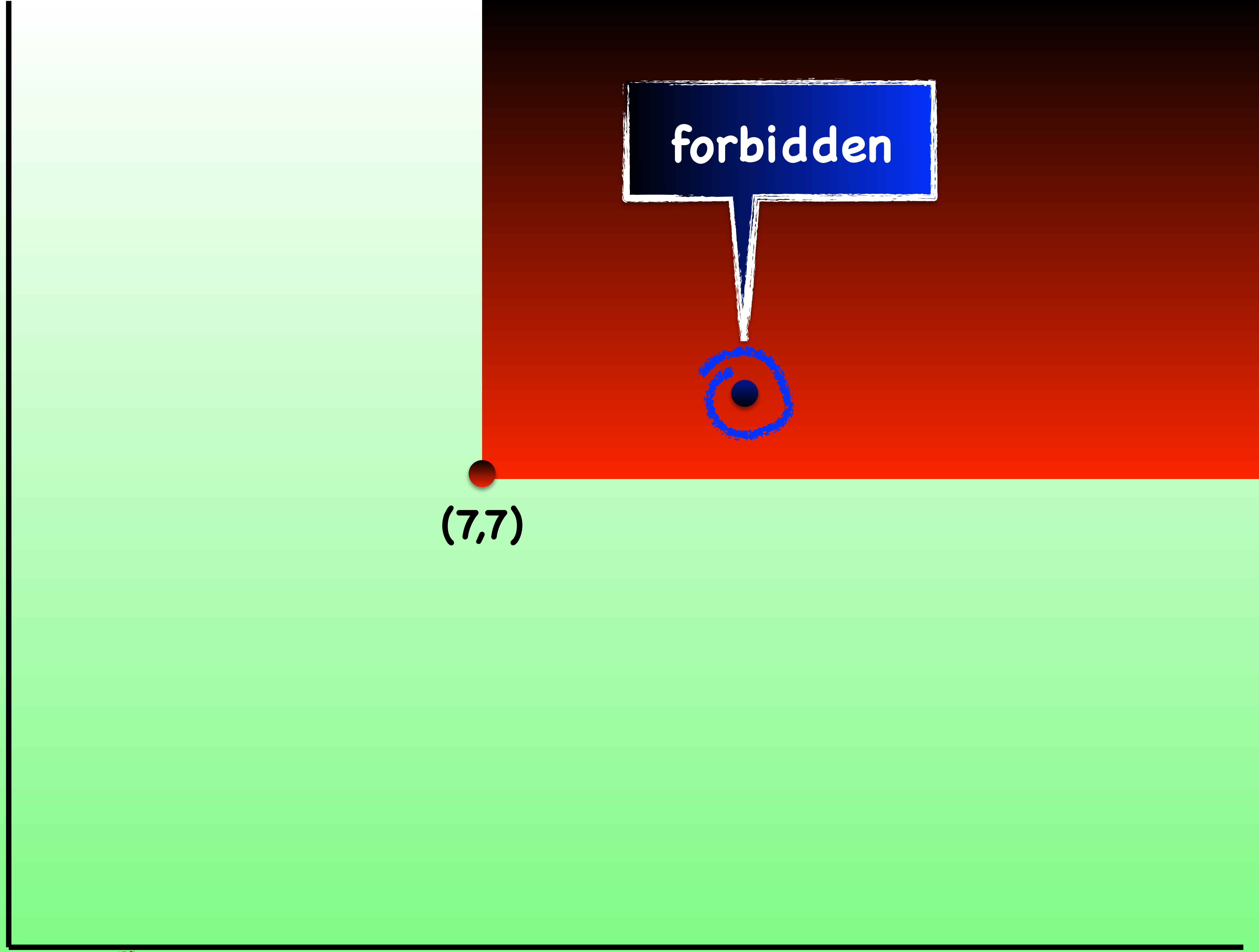
finite multisets over A

finite set

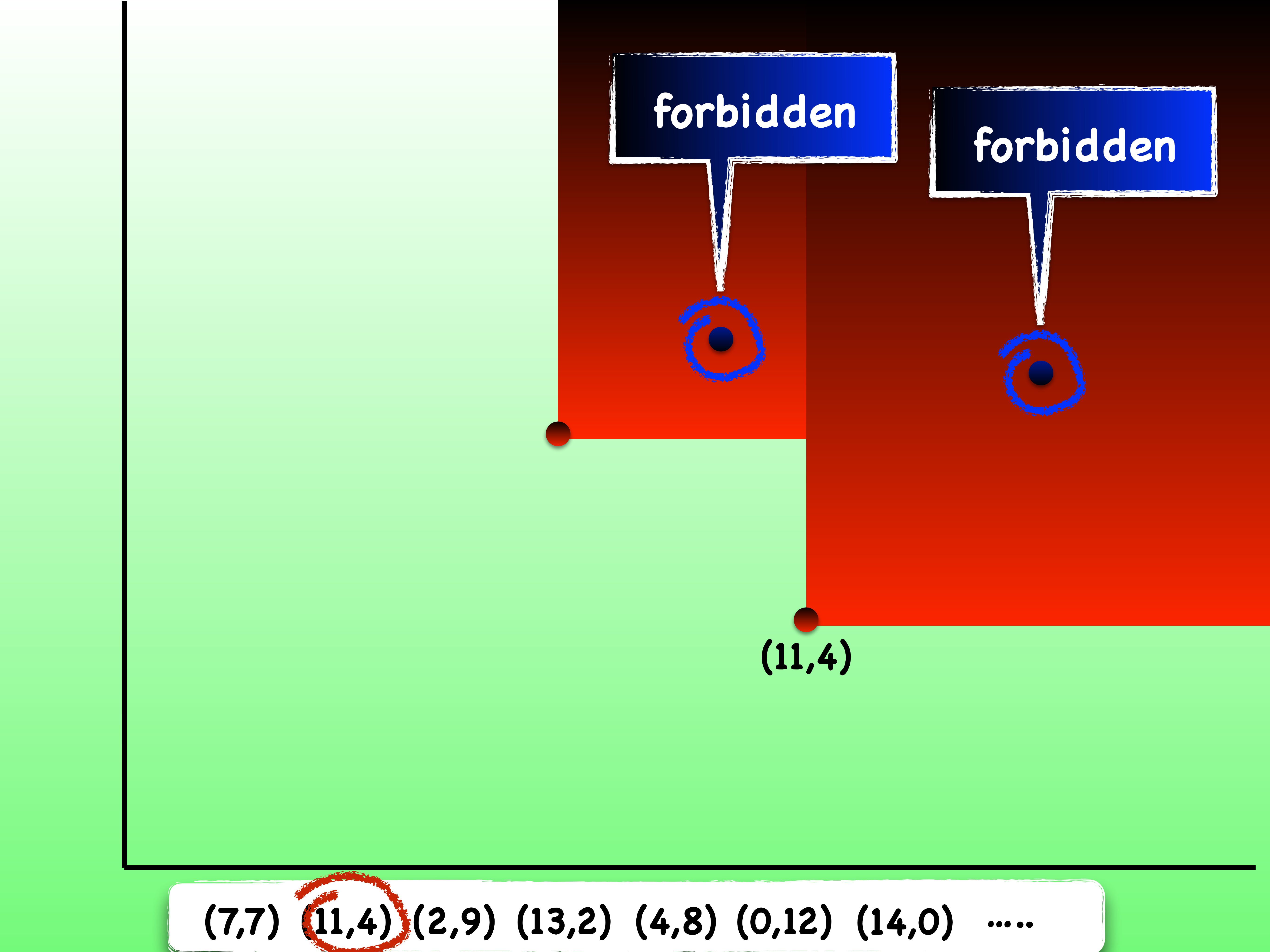
multiset ordering

$\langle A^{\circledast}, \sqsubseteq \rangle$

(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) .....



(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) .....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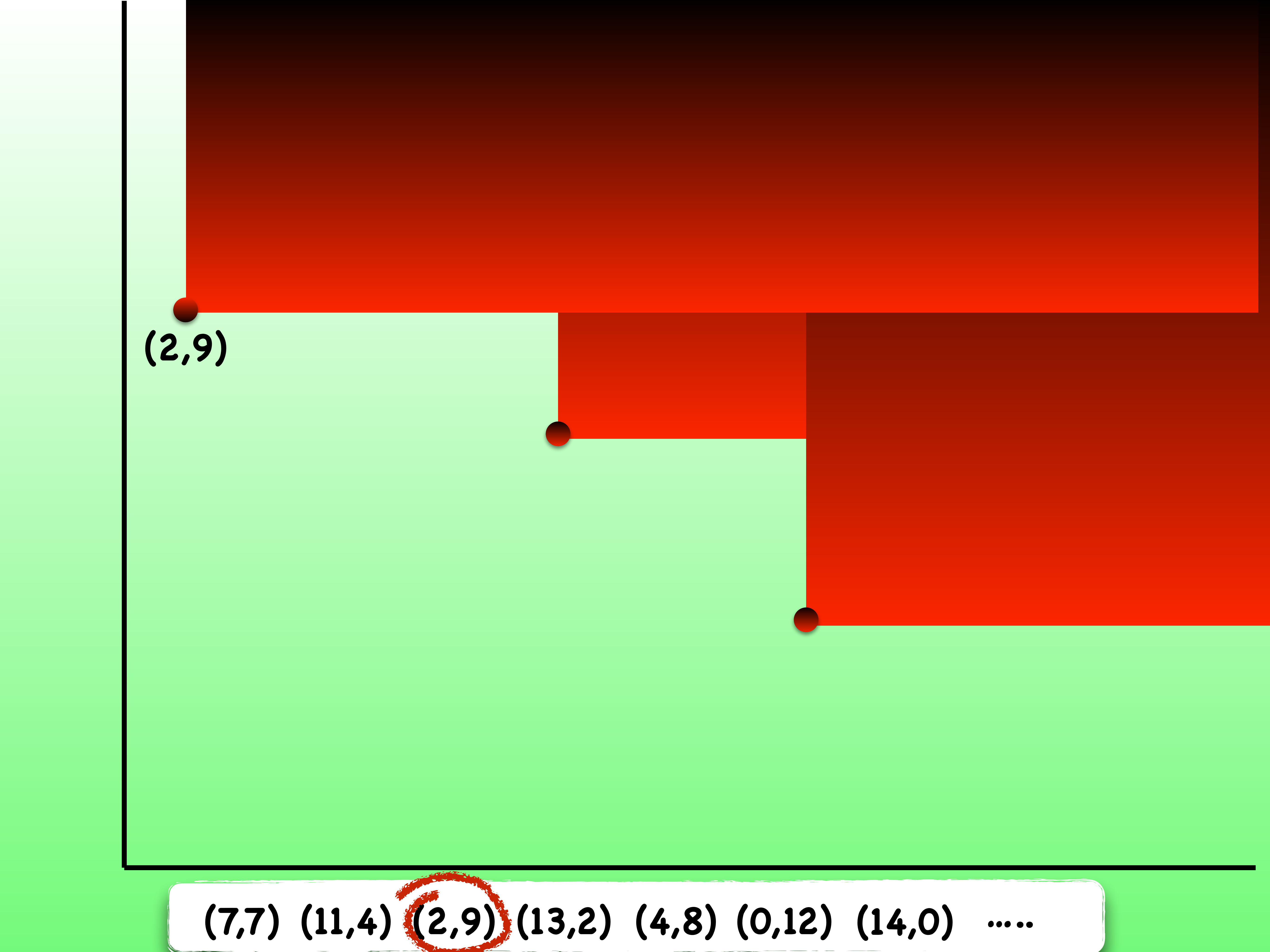
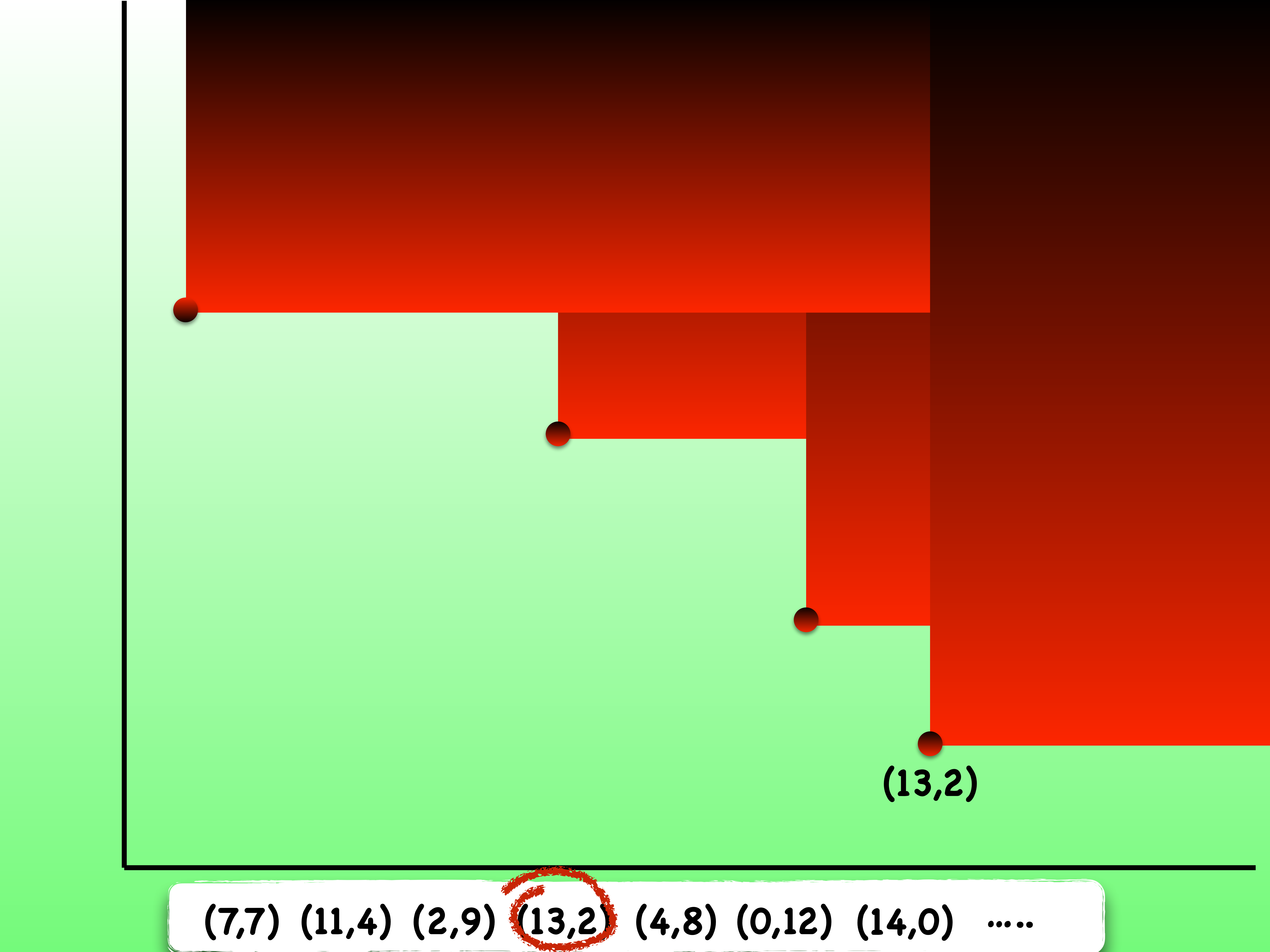
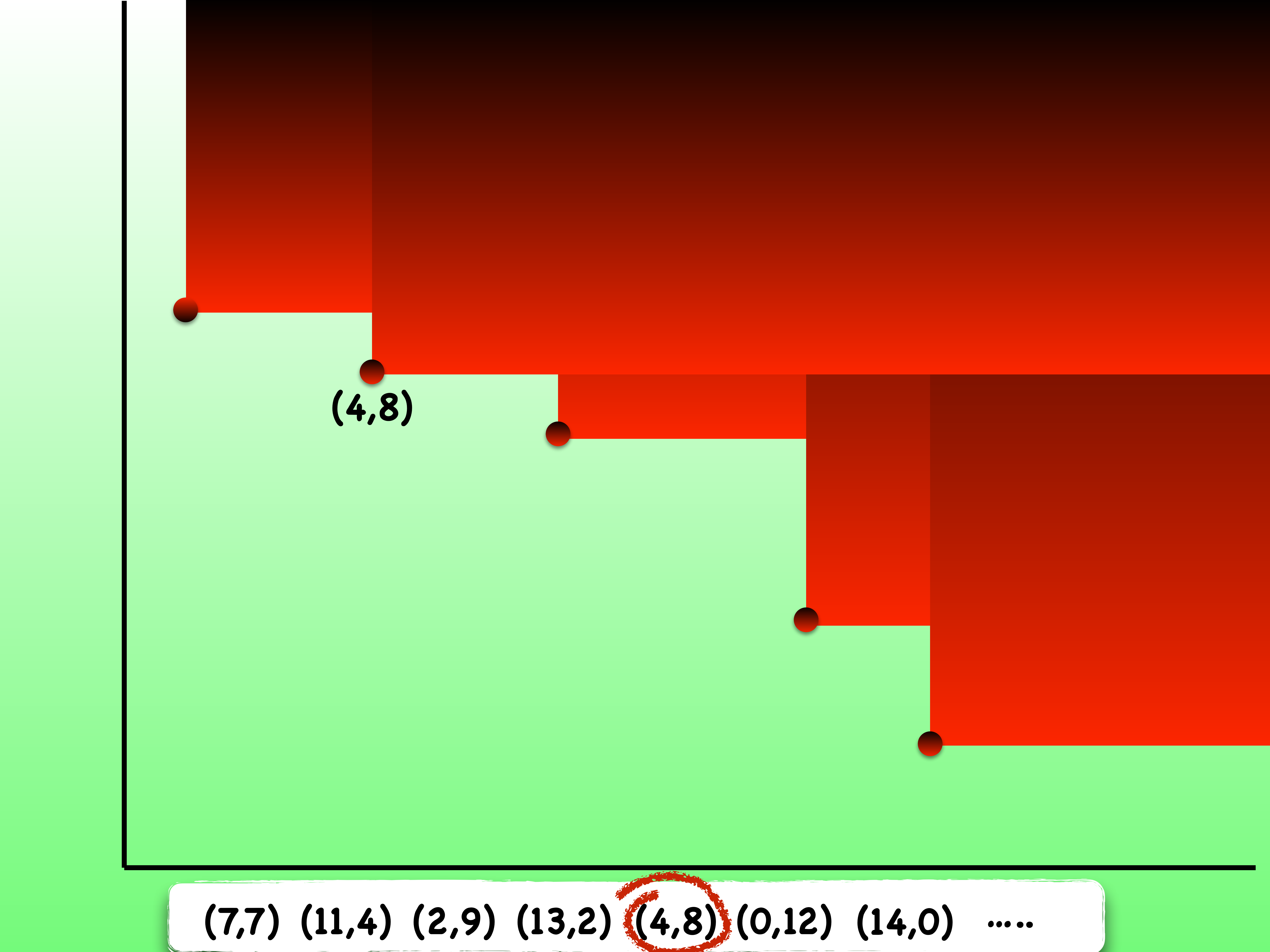
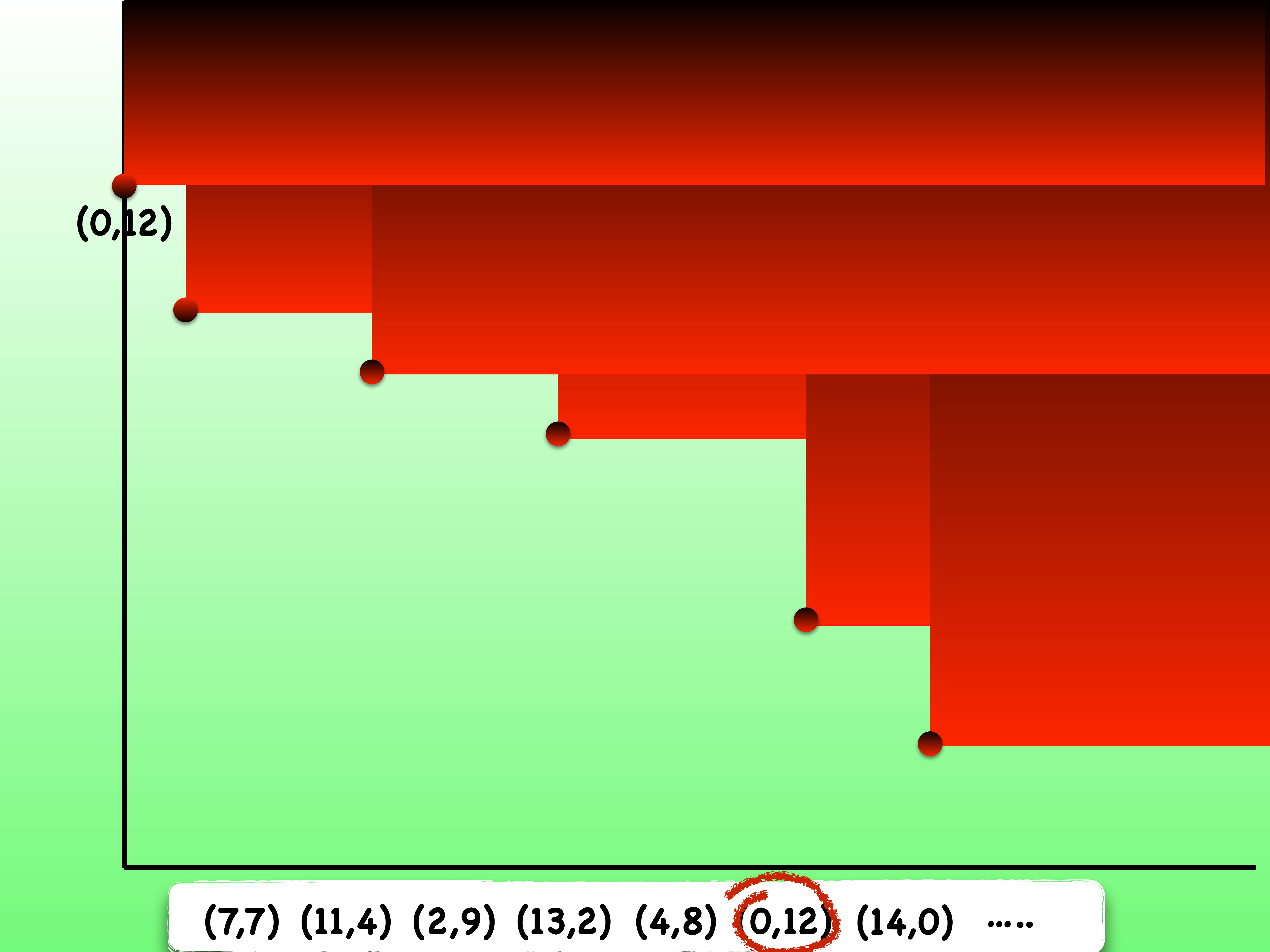
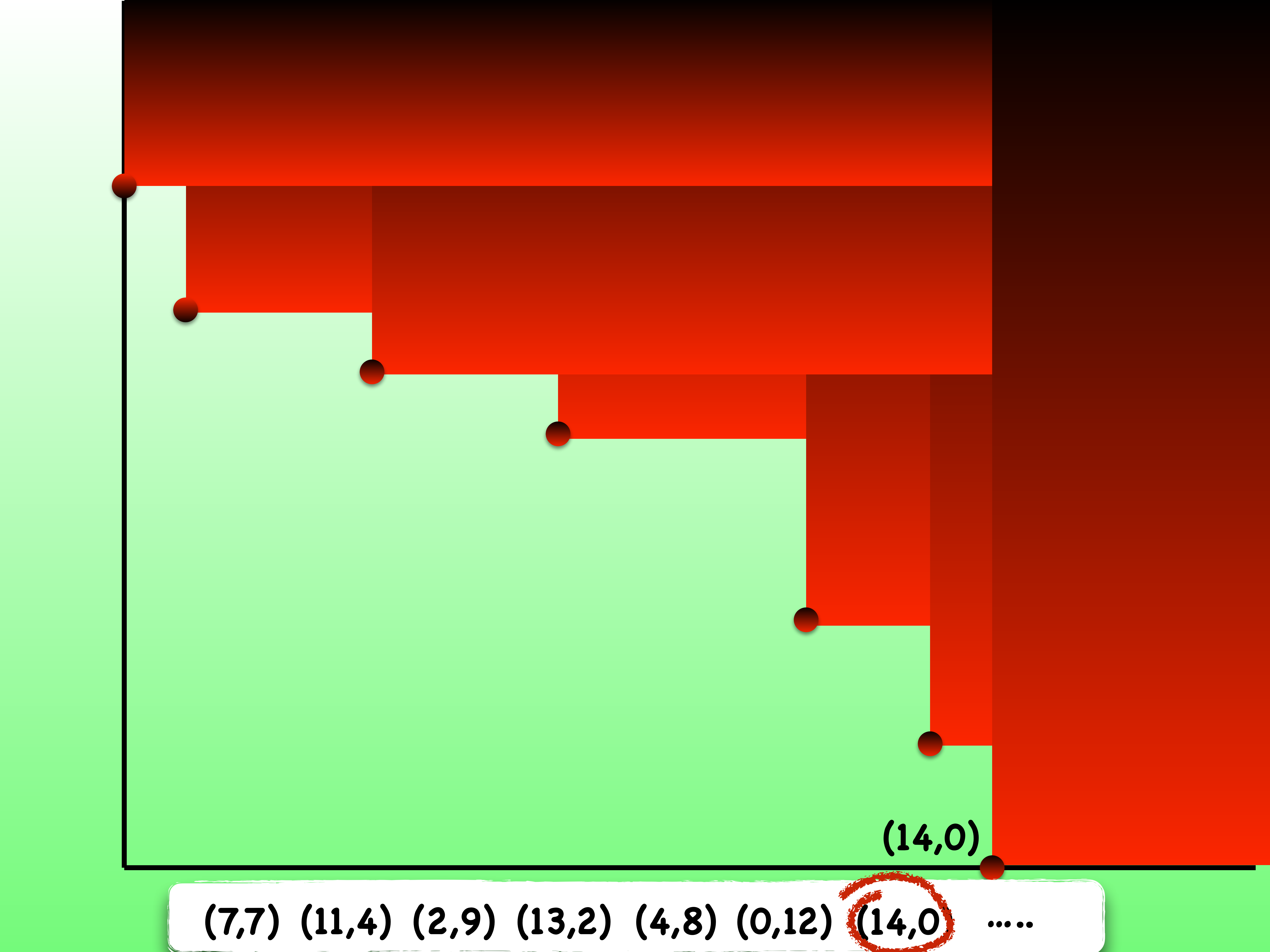
forbidden
(11,4)
(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) .....


(2,9)
(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) .....


(13,2)
(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) .....


(4,8)
(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) .....


(0,12)
(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) .....



(14,0)
(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) .....

Dickson's Lemma, 1910

$A = \{a, b\}$

$[a, a, b, b, b, b] \in A^{\circledast}$

(2,4)

a = 2

b = 4

finite multisets over A

finite set

multiset ordering

$\langle A^{\circledast}, \sqsubseteq \rangle$ ✔

(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) .....

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$ — quasi-order

infinite sequence of elements from A

good sequence

$a_0, a_1, a_2, \ldots\ldots, a_i, \ldots\ldots, a_j, \ldots\ldots$

$\exists i, j : (i < j) \wedge (a_i \sqsubseteq a_j)$

$\sqsubseteq$

$\langle \mathbb{N}, \leq \rangle$ ✔

$\langle \mathbb{I}, \leq \rangle$ ✗

$\langle \mathbb{N}, = \rangle$ ✗

$\langle A, = \rangle$ ✔

finite multisets over A

m ordering

$\langle A^{\circledast}, \sqsubseteq \rangle$ ✔

finite set

$m_1 \sqsubseteq m_2 : |m_1| \leq |m_2|$

$\langle A^{\circledast}, \sqsubseteq \rangle$ ✔

Well-Quasi-Orderings

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- Building WQOs

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$ — quasi-order

infinite sequence of elements from A

good sequence

$a_0, a_1, a_2, \ldots\ldots, a_i, \ldots\ldots, a_j, \ldots\ldots$

$a_i \sqsubseteq a_j$

$\exists i, j : (i < j) \land (a_i \sqsubseteq a_j)$

Very WQO = all sequences are very good

very good sequence

$a_0, a_1, a_2, \ldots, a_{i_1}, \ldots, a_{i_2}, \ldots, a_{i_3}, \ldots\ldots$

$a_{i_1} \sqsubseteq a_{i_2} \sqsubseteq a_{i_3}$

$\exists i_1, i_2, i_3, \ldots : (i_1 < i_2 < i_3 < \cdots) \land (a_{i_1} \sqsubseteq a_{i_2} \sqsubseteq a_{i_3} \sqsubseteq \cdots)$

Well-Quasi-Orderi

WQO = very WQO? yes

- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult .... yes why?**

Well-Quasi-Orderi

WQO = very WQO? yes

- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult .... yes** **why?**

$\forall j > i : a_i \not\sqsubseteq a_j$

terminal

$$a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_m \quad a_{m+1} \quad \ldots, \quad a_n \, , \quad \ldots$$

Well-Quasi-Orderi

WQO = very WQO? yes

- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult .... yes** **why?**

$\forall j > i : a_i \not\trianglelefteq a_j$

terminal

$a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_m \quad a_{m+1} \quad \ldots, \quad a_n \,, \quad \ldots$

$\not\trianglelefteq$

Well-Quasi-Orderi

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$\forall j > i : a_i \not\sqsubseteq a_j$

terminal

$$a_0 \quad a_1 \quad a_2 \quad \dots \quad a_m \quad a_{m+1} \quad \dots, \quad a_n, \quad \dots$$

$$a_m \not\sqsubseteq a_n$$

Well-Quasi-Orderi

WQO = very WQO? yes

- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult yes why?**
  - **finitely many terminals why?**

assume there are ininitely many terminals

Well-Quasi-Orderi

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- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult yes why?**
  - **finitely many terminals why?**

bad sequence

a_{i_0}

assume there are ininitely many terminals

$\forall j > i : a_i \not\sqsubseteq a_j$

terminal terminal terminal terminal

$a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_{i_0} \quad \ldots \quad a_{i_1} \quad \ldots \quad a_{i_2} \quad \ldots \quad a_{i_3} \quad \ldots$

Well-Quasi-Orderi

WQO = very WQO? yes

- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult** yes why?
  - **finitely many terminals** why?

bad sequence

a_{i_0} a_{i_1}

assume there are ininitely many terminals

$\forall j > i : a_i \not\sqsubseteq a_j$

terminal terminal terminal terminal

a_0 a_1 a_2 … a_{i_0} … a_{i_1} … a_{i_2} … a_{i_3} …

$\not\sqsubseteq$

Well-Quasi-Orderi

WQO = very WQO? yes

- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult yes why?**
  - **finitely many terminals why?**

bad sequence

a_{i_0} a_{i_1} a_{i_2}

assume there are ininitely many terminals

$\forall j > i : a_i \not\sqsubseteq a_j$

terminal **terminal** **terminal** **terminal**

a_0 a_1 a_2 $\ldots$ a_{i_0} $\ldots$ a_{i_1} $\ldots$ a_{i_2} $\ldots$ a_{i_3} $\ldots$

$\not\sqsubseteq$ $\not\sqsubseteq$

Well-Quasi-Orderi

WQO = very WQO? yes

- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult yes why?**
  - **finitely many terminals why? ✔**

bad sequence

$a_{i_0} \quad a_{i_1} \quad a_{i_2} \quad a_{i_3} \quad \ldots$

assume there are ininitely many terminals

$\forall j > i : a_i \not\sqsubseteq a_j$

terminal **terminal** **terminal** **terminal**

$a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_{i_0} \quad \ldots \quad a_{i_1} \quad \ldots \quad a_{i_2} \quad \ldots \quad a_{i_3} \quad \ldots$

$\not\sqsubseteq \quad \not\sqsubseteq \quad \not\sqsubseteq$

Well-Quasi-Orderi

WQO = very WQO? yes

- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult …. yes** **why?**
  - **finitely many terminals** ✔

very good sequence

$\forall j > i : a_i \not\sqsubseteq a_j$

last terminal

$a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_m \quad \ldots$

Well-Quasi-Orderi

WQO = very WQO? yes

- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult .... yes** **why?**
  - **finitely many terminals** ✔

very good sequence

b_0

$\forall j > i : a_i \not\sqsubseteq a_j$

last terminal

$a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_m \quad \ldots \quad b_0 \quad \ldots$

Well-Quasi-Orderi

WQO = very WQO? yes

- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult …. yes** **why?**
  - **finitely many terminals** ✔

very good sequence

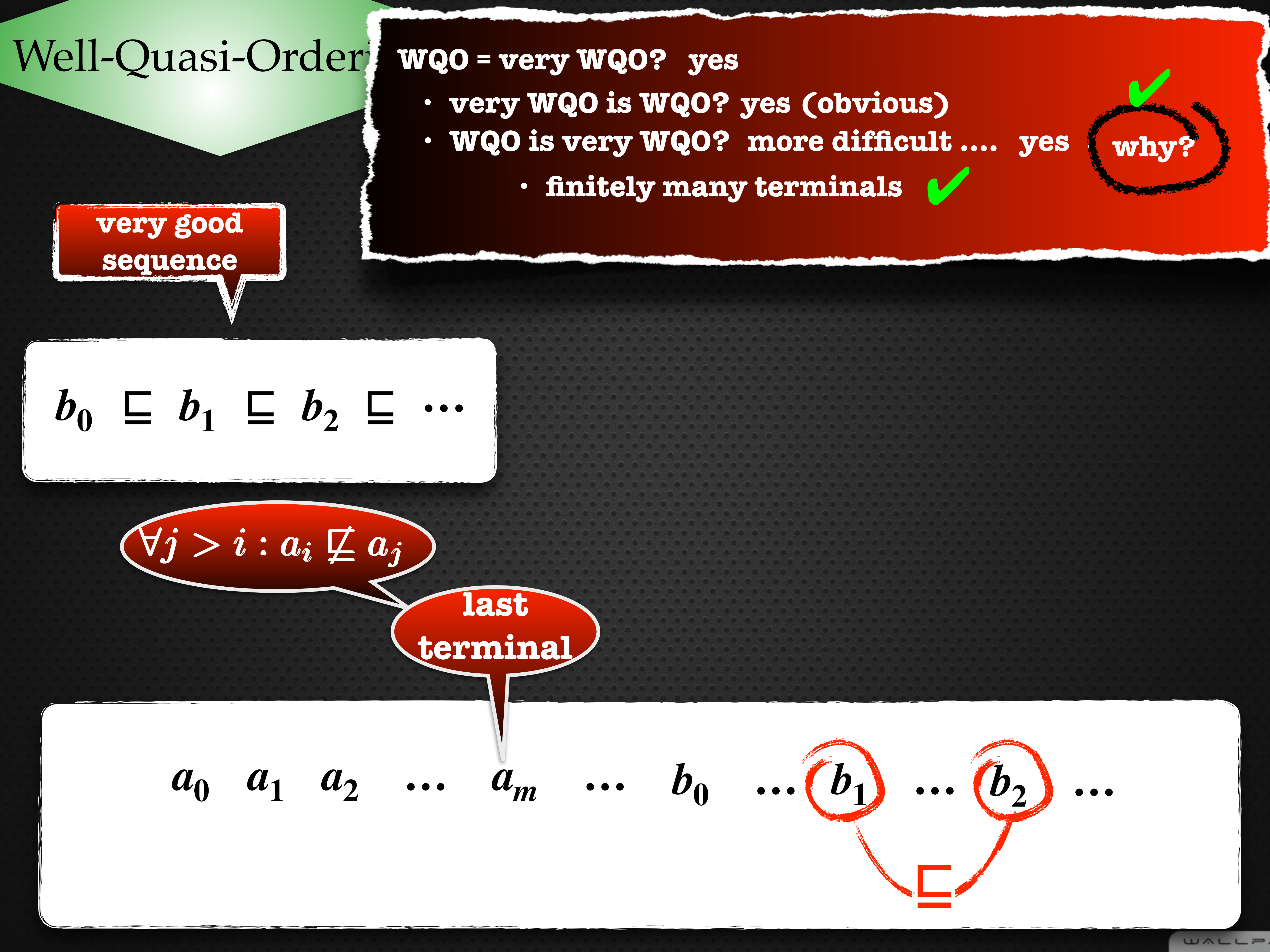
$b_0 \sqsubseteq b_1$

$\forall j > i : a_i \not\sqsubseteq a_j$

last terminal

$a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_m \quad \ldots \quad b_0 \quad \ldots \quad b_1 \quad \ldots$

$\sqsubseteq$


Well-Quasi-Orderi
WQO = very WQO? yes
• very WQO is WQO? yes (obvious)
• WQO is very WQO? more difficult .... yes
why?
• finitely many terminals
very good sequence
$b_0 \sqsubseteq b_1 \sqsubseteq b_2 \sqsubseteq \cdots$
$\forall j > i : a_i \not\sqsubseteq a_j$
last terminal
$a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_m \quad \ldots \quad b_0 \quad \ldots \quad b_1 \quad \ldots \quad b_2 \quad \ldots$
$\sqsubseteq$

Well-Quasi-Orderi

WQO = very WQO? yes

- **very WQO is WQO? yes (obvious)**
- **WQO is very WQO? more difficult .... yes** ✔

very good sequence

$$b_0 \sqsubseteq b_1 \sqsubseteq b_2 \sqsubseteq \cdots$$

$\forall j > i : a_i \not\sqsubseteq a_j$

last terminal

$$a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_m \quad \ldots \quad b_0 \quad \ldots \quad b_1 \quad \ldots \quad b_2 \quad \ldots$$

$b_1 \sqsubseteq b_2$

Well-Quasi-Orderings

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Well-Qua... Higman's Lemma

Subword Relation

ab ⊑ xaybz

WQO

Higman's Lemma

finite words over A

finite set

sub-word ordering

$\langle A^*, \sqsubseteq \rangle$ ✔

finite words

$w_0, w_1, w_2, \ldots\ldots, w_i, \ldots\ldots, w_j, \ldots\ldots$

$\exists i, j : (i < j) \wedge (w_i \sqsubseteq w_j)$

$\sqsubseteq$

Well-Qu... Higman's Lemma

Subword Relation

ab $\sqsubseteq$ xaybz

WQO

"Proof"

$(x \sqsubseteq y) \implies (a \cdot x \sqsubseteq a \cdot y)$ $(a \cdot x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$

$(x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y)$ $(x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$

Well-Qua… Higman's Lemma

Subword Relation

ab ⊑ xaybz

WQO

"Proof"

$cabc \sqsubseteq cacbaca$

$abc \sqsubseteq acbaca$

$(x \sqsubseteq y) \implies (a \cdot x \sqsubseteq a \cdot y)$ $(a \cdot x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$

$(x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y)$ $(x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$

Well-Qua... Higman's Lemma

Subword Relation

ab $\sqsubseteq$ xaybz

WQO

"Proof"

$$cabc \sqsubseteq cacbaca$$

$$abc \sqsubseteq acbaca$$

$$abc \sqsubseteq cacbaca$$

$$(x \sqsubseteq y) \implies (a \cdot x \sqsubseteq a \cdot y) \quad (a \cdot x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$$

$$(x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y) \quad (x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$$

Well-Qu... Higman's Lemma

Subword Relation

ab $\sqsubseteq$ xaybz

WQO

"Proof"

$cabc \sqsubseteq cacbaca$

$abc \sqsubseteq acbaca$

$abc \sqsubseteq cacbaca$

$bc \not\sqsubseteq cacb$

$abc \not\sqsubseteq acacb$

$(x \sqsubseteq y) \implies (a \cdot x \sqsubseteq a \cdot y)$ $(a \cdot x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$

$(x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y)$ $(x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$

Well-Qua

Higman's Lemma

Subword Relation

ab $\sqsubseteq$ xaybz

WQO

"Proof"

$cabc \sqsubseteq cacbaca$

$abc \sqsubseteq acbaca$

$abc \sqsubseteq cacbaca$

$bc \not\sqsubseteq cacb$

$abc \not\sqsubseteq acacb$

$abc \not\sqsubseteq cacb$

$(x \sqsubseteq y) \implies (a \cdot x \sqsubseteq a \cdot y)$ $\quad$ $(a \cdot x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$

$(x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y)$ $\quad$ $(x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$

ab $\sqsubseteq$ xaybz

WQO

"Proof"

"minimal" bad sequence:

w_1 : a shortest word starting a bad sequence

minimal bad sequence

$w_1 \cdots$

Well-Qua… Higman's Lemma

Subword Relation

$ab \sqsubseteq xaybz$

WQO

"minimal" bad sequence:

w_1 **: a shortest word starting a bad sequence**

w_2 **: a shortest word** v **such that** $w_1 v \cdots$ **is bad**

minimal bad sequence

$w_1 \quad w_2 \quad \cdots$

Subword Relation

ab $\sqsubseteq$ xaybz

WQO

"minimal" bad sequence:

w_1 **: a shortest word starting a bad sequence**

w_2 **: a shortest word** v **such that** $w_1 v \cdots$ **is bad**

w_3 **: a shortest word** v **such that** $w_1 w_2 v \cdots$ **is bad**

minimal bad sequence

$w_1 \quad w_2 \quad w_3 \quad \cdots$

Subword Relation

ab $\sqsubseteq$ xaybz

WQO

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w_1 : a shortest word starting a bad sequence

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w_n : a shortest word v such that $w_1 w_2 w_3 \cdots w_{n-1} v$ is bad

minimal bad sequence

infinite sequence

infinitely many start with some "a"

$w_1 \quad w_2 \quad w_3 \quad \cdots \quad w_n$

Well-Qua… Higman's Lemma

Subword Relation

ab $\sqsubseteq$ xaybz

WQO

"minimal" bad sequence:

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minimal bad sequence

infinite sequence

infinitely many start with some "a"

$w_1 \quad w_2 \quad w_3 \quad \cdots \quad w_n \quad a \cdot v_{i_1} \quad \cdots \quad a \cdot v_{i_2} \quad \cdots \quad a \cdot v_{i_3} \quad \cdots$

Well-Qua… | Higman's Lemma

Subword Relation

$ab \sqsubseteq xaybz$

WQO

"minimal" bad sequence:

- w_1 : a shortest word starting a bad sequence
- w_2 : a shortest word v such that $w_1 v \cdots$ is bad
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minimal bad sequence

infinite sequence

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$$w_1 \quad w_2 \quad w_3 \cdots w_n \quad a \cdot v_{i_1} \cdots a \cdot v_{i_2} \cdots a \cdot v_{i_3} \cdots$$

$\not\sqsubseteq$ $\not\sqsubseteq$ $\not\sqsubseteq$

Well-Qua… Higman's Lemma

Subword Relation

ab $\sqsubseteq$ xaybz

WQO

"minimal" bad sequence:

w_1 **: a shortest word starting a bad sequence**

$$(x \sqsubseteq y) \implies (a \cdot x \sqsubseteq a \cdot y) \quad (a \cdot x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$$

$$(x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y) \quad (x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$$

$w_1 \; w_2 \; w_3 \cdots w_n \; a \cdot v_{i_1} \cdots a \cdot v_{i_2} \cdots a \cdot v_{i_3} \cdots$

$\not\sqsubseteq$ $\not\sqsubseteq$ $\not\sqsubseteq$

$w_1 \; w_2 \; w_3$

$\not\sqsubseteq$

Well-Qua

Higman's Lemma

Subword Relation

ab $\sqsubseteq$ xaybz

WQO

"minimal" bad sequence:

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$$(x \sqsubseteq y) \implies (a \cdot x \sqsubseteq a \cdot y) \qquad (a \cdot x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$$

$$(x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y) \qquad (x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$$

$w_1 \quad w_2 \quad w_3 \cdots w_n \quad a \cdot v_{i_1} \cdots a \cdot v_{i_2} \cdots a \cdot v_{i_3} \cdots$

$\not\sqsubseteq \qquad \not\sqsubseteq \qquad \not\sqsubseteq$

$w_1 \quad w_2 \quad w_3 \qquad w_n \quad v_{i_1}$

$\not\sqsubseteq \qquad \not\sqsubseteq$

Well-Qua

Higman's Lemma

Subword Relation

ab $\sqsubseteq$ xaybz

WQO

"minimal" bad sequence:

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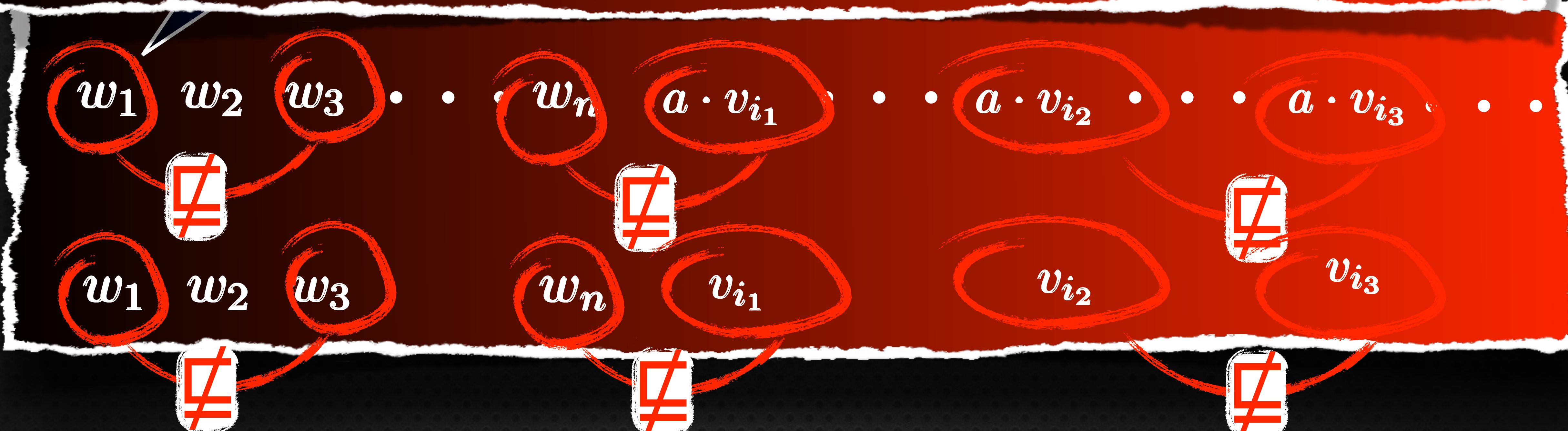
$$(x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y) \qquad (x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$$



Well-Qua…

Higman's Lemma

Subword Relation

$$ab \sqsubseteq xaybz$$

WQO

"minimal" bad sequence:

- w_1 **: a shortest word starting a bad sequence**
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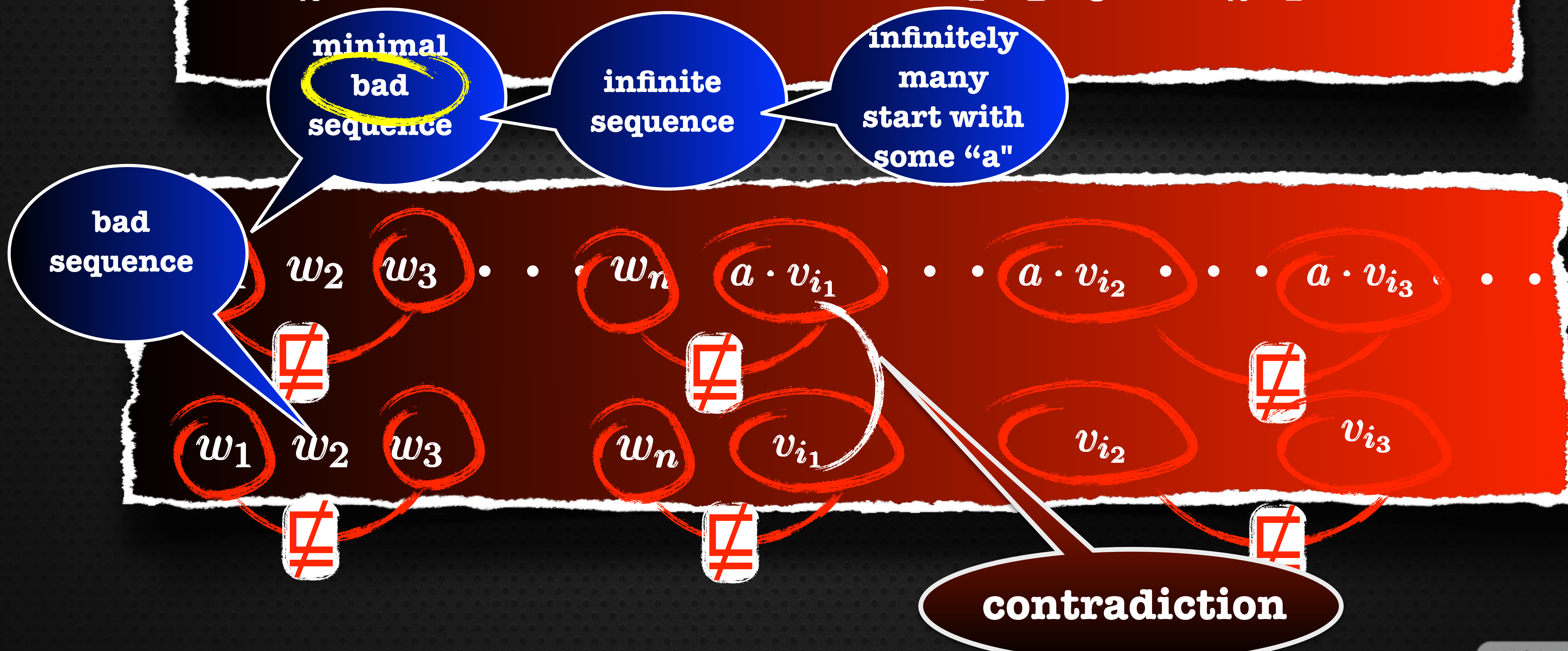
- w_n **: a shortest word** v **such that** $w_1 w_2 w_3 \cdots w_{n-1} v$ **is bad**



Well-Qua… Higman's Lemma

Subword Relation

$$ab \sqsubseteq xaybz$$

WQO

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minimal bad sequence

infinite sequence

infinitely many start with some "a"

$$w_1 \quad w_2 \quad w_3 \quad \bullet \bullet \bullet \quad w_n \quad a \cdot v_{i_1} \quad \bullet \bullet \bullet \quad a \cdot v_{i_2} \quad \bullet \bullet \bullet \quad a \cdot v_{i_3} \quad \bullet \bullet \bullet$$

bad sequence

$$w_1 \quad w_2 \quad w_3 \qquad w_n \quad v_{i_1} \qquad v_{i_2} \qquad v_{i_3}$$

contradiction

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$

WQO

set of finite words over A

$\langle A^*, \sqsubseteq^* \rangle$

WQO?

Yes

$$a_1 a_2 \cdots a_n \sqsubseteq^* x_0 \cdot b_1 \cdot x_1 \cdot b_2 \cdot x_2 \cdots x_{n-1} \cdot b_n \cdot x_n$$

if

$$a_1 \sqsubseteq b_1 \quad a_2 \sqsubseteq b_2 \quad \cdots \quad a_n \sqsubseteq b_n$$

$\langle \mathbb{N}, \leq \rangle$

$\langle \mathbb{N}^*, \leq^* \rangle$

$1\,7\,5 \leq^* 0\,3\,2\,9\,3\,6\,8$

Well-Qu... Higman's Lemma

"minimal" bad sequence:

w_1 **: a shortest word starting a bad sequence**

minimal bad sequence

$w_1 \cdots$

Higman's Lemma

"minimal" bad sequence:

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minimal bad sequence

$w_1 \quad w_2 \quad \cdots$

Well-Qua… Higman's Lemma

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minimal bad sequence

$w_1 \quad w_2 \quad w_3 \quad \cdots$

Well-Qua

Higman's Lemma

"minimal" bad sequence:

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w_n : **a shortest word** v **such that** $w_1 w_2 w_3 \cdots w_{n-1} v$ **is bad**

minimal bad sequence

infinite sequence

$$w_1 \quad w_2 \quad w_3 \quad \cdots \; w_n \; \cdots$$

Well-Qua…

Higman's Lemma

"minimal" bad sequence:

w_1 : a shortest word starting a bad sequence

w_2 : a shortest word v such that $w_1 v \cdots$ is bad

w_3 : a shortest word v such that $w_1 w_2 v \cdots$ is bad

w_n : a shortest word v such that $w_1 w_2 w_3 \cdots w_{n-1} v$ is bad

minimal bad sequence

infinite sequence

very WQO

$a_1 \sqsubseteq a_2 \sqsubseteq a_3 \sqsubseteq \cdots$

$w_1 \quad w_2 \quad w_3 \cdots w_n \quad a_1 \cdot v_1 \cdots a_2 \cdot v_2 \cdots a_3 \cdot v_3 \cdots$

$\not\sqsubseteq \qquad \not\sqsubseteq \qquad \not\sqsubseteq$

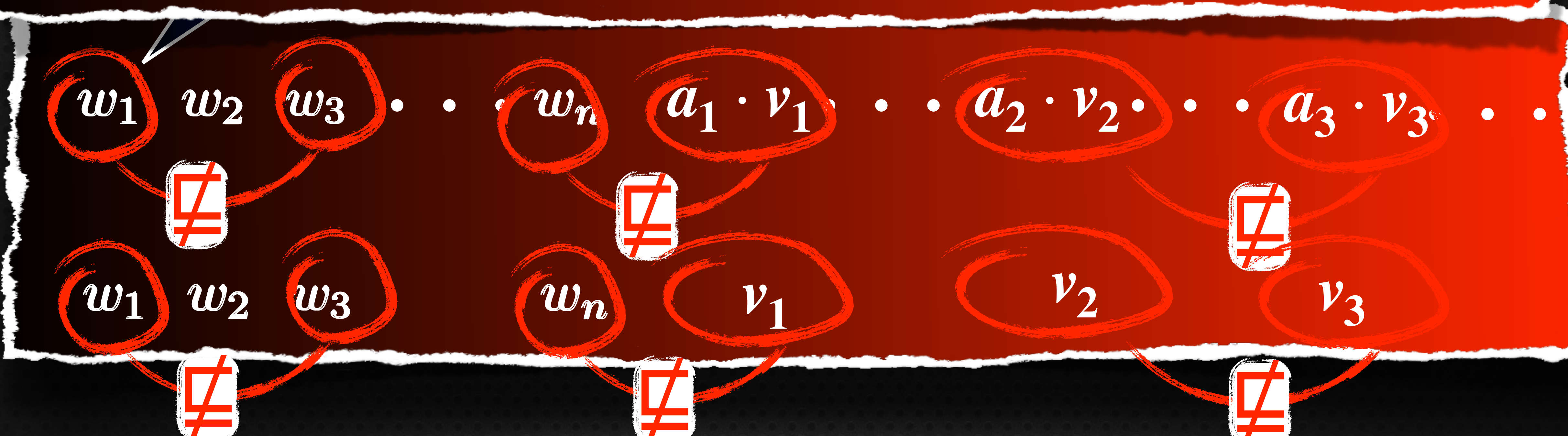
Well-Qua… Higman's Lemma

"minimal" bad sequence:

w_1 : a shortest word starting a bad sequence

$$(x \sqsubseteq y) \wedge (a \sqsubseteq b) \implies (a \cdot x \sqsubseteq b \cdot y)$$

$$(x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y)$$



Well-Qu… Higman's Lemma

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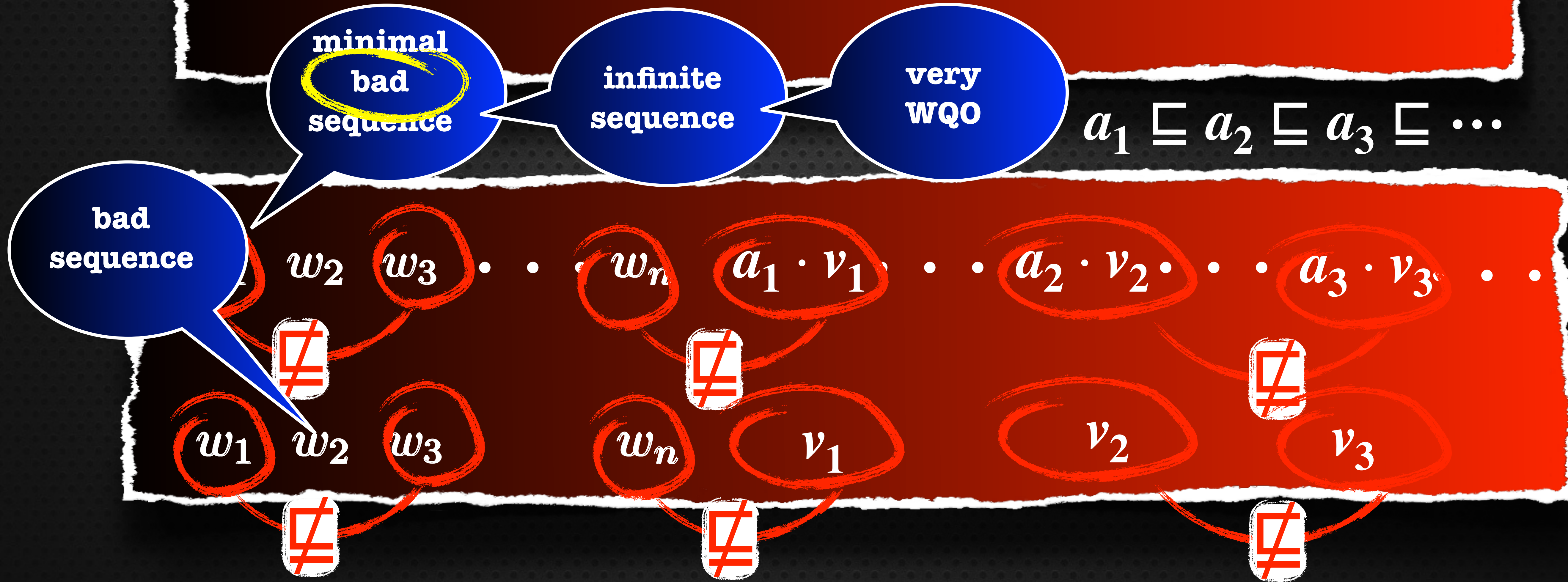

Well-Qu

Higman's Lemma

"minimal" bad sequence:

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w_2 **: a shortest word** v **such that** $w_1 v \cdots$ **is bad**

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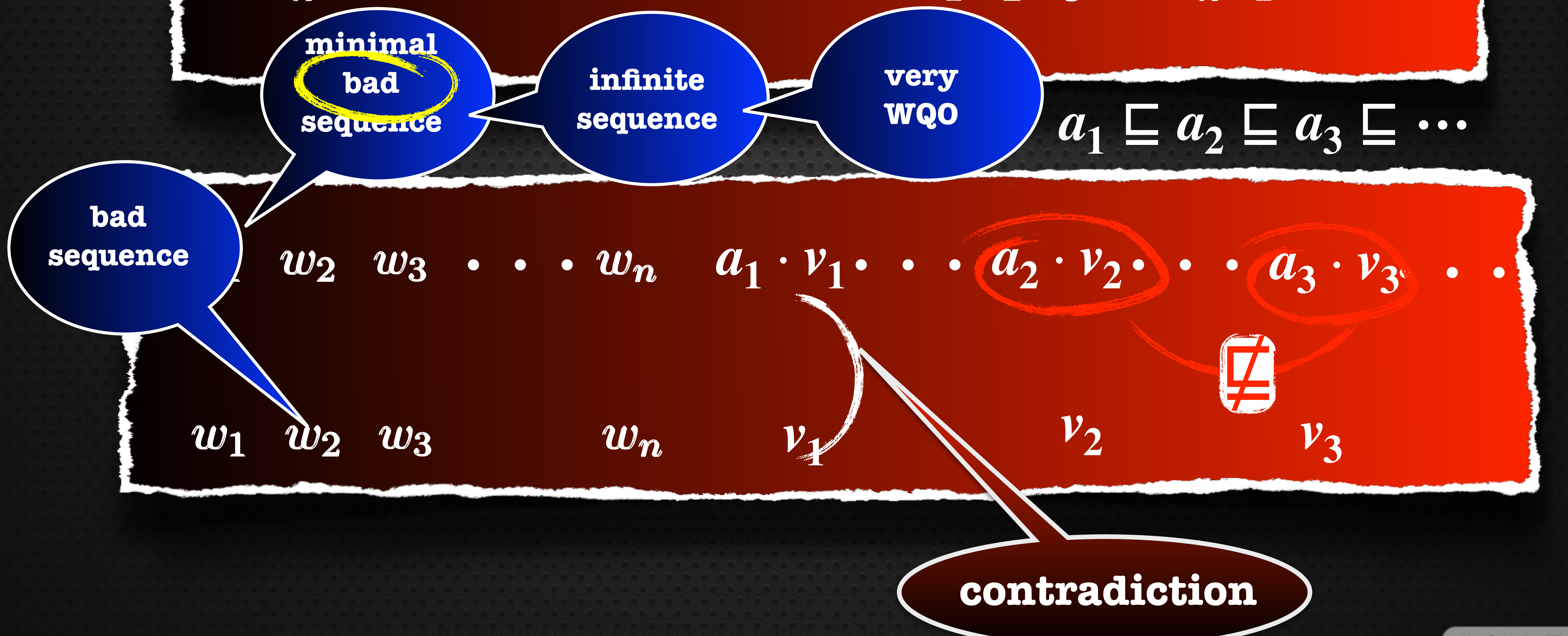


Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$

WQO

$\langle A^*, \sqsubseteq^* \rangle$ ✔

$\langle A^{\circledast}, \sqsubseteq^{\circledast} \rangle$ ✔

set of finite multisets over A

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$

WQO

$\langle A^*, \sqsubseteq^* \rangle$ ✔

$\langle A^{\circledast}, \sqsubseteq^{\circledast} \rangle$ ✔

$\langle A^k, \sqsubseteq^k \rangle$ ✔

set of vectors of length k over A

$\langle A_1, \sqsubseteq_1 \rangle$ $\langle A_2, \sqsubseteq_2 \rangle$ .... $\langle A_k, \sqsubseteq_k \rangle$

WQO WQO WQO

$\langle A_1 \times A_2 \times \cdots \times A_k, \sqsubseteq^k \rangle$ ✔

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$

WQO

$\langle A^*, \sqsubseteq^* \rangle$ ✔

$\langle A^\circledast, \sqsubseteq^\circledast \rangle$ ✔

$\langle A^k, \sqsubseteq^k \rangle$ ✔

$\langle 2^A \sqsubseteq^{2^A} \rangle$ ✔

set of finite sets over A

Well-Quasi-Ordering

$\langle A, \sqsubseteq \rangle$ — WQO

$\langle A^*, \sqsubseteq^* \rangle$ ✔ — **set of finite words over A**

$\langle A^{\circledast}, \sqsubseteq^{\circledast} \rangle$ ✔ — **set of finite multisets over A**

$\langle A^k, \sqsubseteq^k \rangle$ ✔ — **set of vectors of length k over A**

$\langle 2^A \sqsubseteq^{2^A} \rangle$ ✔ — **set of finite sets over A**

Well-Quasi-Ordering

natural numbers

$\langle \mathbb{N}, \leq \rangle$

standard ordering

finite set

$\langle A, = \rangle$

$\langle A^*, \sqsubseteq^* \rangle$

$\langle A^\circledast, \sqsubseteq^\circledast \rangle$

$\langle A^k, \sqsubseteq^k \rangle$

$\langle 2^A \sqsubseteq^{2^A} \rangle$

$\langle A^*, \preceq \rangle$

subword relation

Well-Quasi-Ordering

$\langle \mathbb{N}, \leq \rangle$

$\langle A, = \rangle$

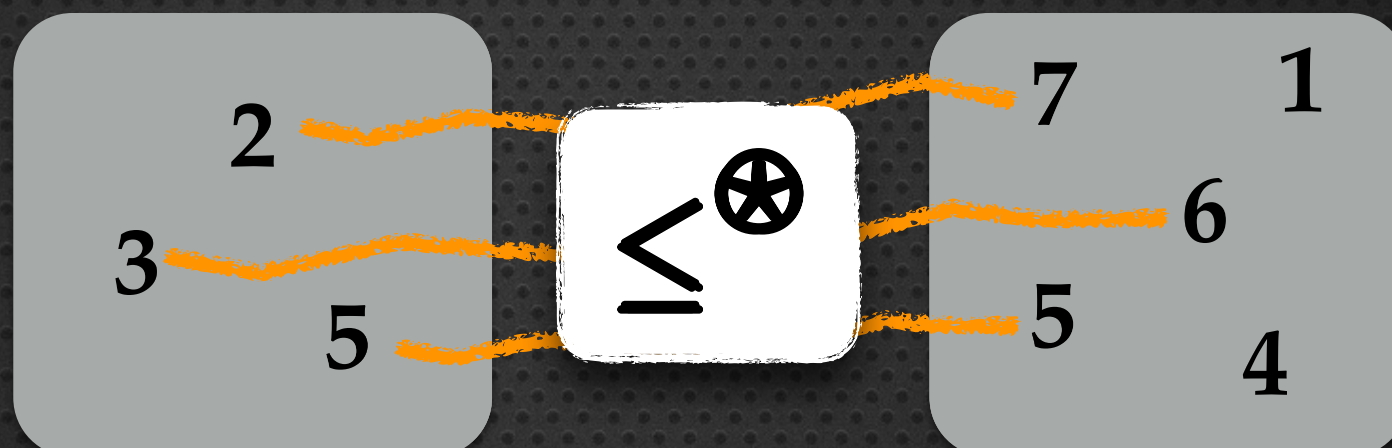
$\langle \mathbb{N}^{\circledast}, \leq^{\circledast} \rangle$

$\langle A^*, \sqsubseteq^* \rangle$

$\langle A^{\circledast}, \sqsubseteq^{\circledast} \rangle$

$\langle A^k, \sqsubseteq^k \rangle$

$\langle 2^A \sqsubseteq^{2^A} \rangle$



Well-Quasi-Ordering

$\left\langle (\mathbb{N}^{\circledast})^{*}, (\leq^{\circledast})^{*} \right\rangle$

$\langle \mathbb{N}, \leq \rangle$

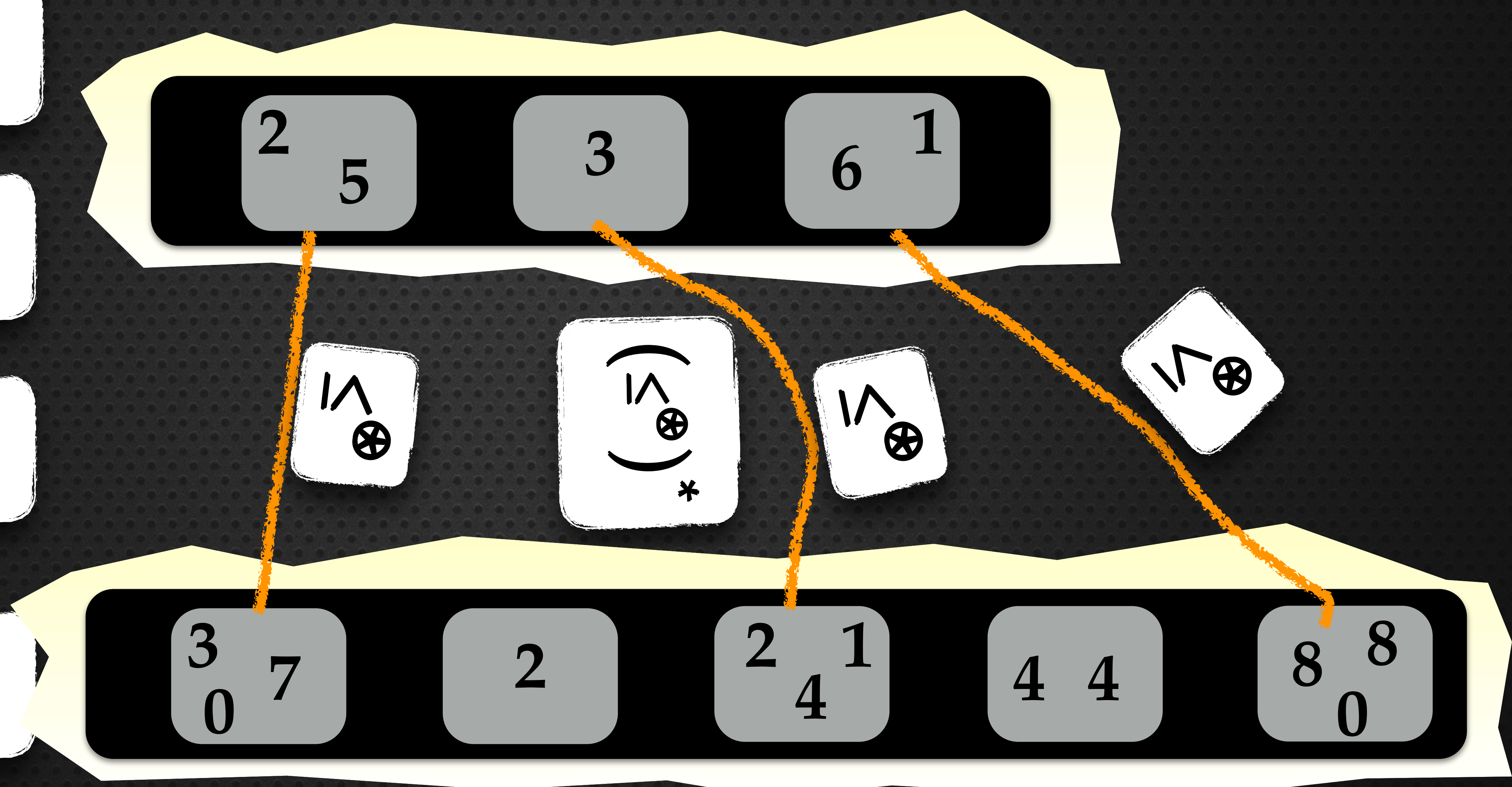
$\langle A, = \rangle$

$\langle A^{*}, \sqsubseteq^{*} \rangle$

$\langle A^{\circledast}, \sqsubseteq^{\circledast} \rangle$

$\langle A^{k}, \sqsubseteq^{k} \rangle$

$\langle 2^{A}, \sqsubseteq^{2^{A}} \rangle$



Well-Quasi-Orderings

- Quasi-Orderings
- Well-Quasi-Orderings (WQOs)
- Very-Well-Quasi-Orderings
- Building WQOs