On the Qualitative Analysis of Conformon P-Systems

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Abstract. We study computational properties of conformon P-systems, an extension of P-systems in which symbol objects are labelled by their current amount of energy. We focus here our attention to decision problems like reachability and coverability of a configuration and give positive and negative results for the full model and for some of its fragments. Furthermore, we investigate the relation between conformon P-systems and other concurrency models like *nested Petri nets* and *constrained multiset rewriting systems*.

1 Introduction

P-systems [10] are a basic model of the living cell defined by a set of hierarchically organized membranes and by rules that dynamically distribute elementary objects in the component membranes. Conformon P-Systems [5] are an extension of P-systems in which symbol objects (conformons) are labelled with their current amount of energy. In a conformon P-system membranes are organized into a directed graph. Furthermore, a symbol object is a pair name-value, where name ranges over a given alphabet, and value is a natural number. The value associated to a conformon denotes its current amount of energy. Conformon P-systems provide rules for the exchange of energy from a conformon to another and for passing through membranes. Passage rules are conditioned by predicates defined over the values of conformons. In [6] Frisco and Corne applied conformon P-systems to model the dynamics of HIV infection. Concerning the expressive power of conformon P-systems, in [5] Frisco has shown that the model is Turing equivalent even without the use of priority or maximal parallelism.

In this paper we investigate restricted fragments of conformon P-systems for which decision problems related to verification of qualitative properties are decidable. We focus our attention to verification of safety properties and decision problems like coverability of a configuration [1]. The fragment we consider put some restrictions on the form of predicates used as conditions of passage rules. Namely, we only admit passage rules with lower bound constraints as conditions (i.e. $p(x) = x \geq c$ for $c \in \mathbb{N}$). The resulting fragment, we will refer to as *restricted conformon P-systems*, is still interesting as a model of natural processes. Indeed, we can use it to specify systems in which conformons pass through a membrane when a given amount of energy is reached.

For restricted conformon P-systems, we apply the methodology of [1] to define an algorithm to decide the coverability problem. This algorithm performs a backward reachability analysis through the state space of a system. Since in our model the set of configurations is infinite, the analysis is made symbolic in order to finitely represent infinite sets of configurations. For this purpose, we use the theory of *well-quasi orderings* and its application to verification of concurrent systems [1].

In the paper we also investigate the relation between (restricted) conformon P-systems and other models used in concurrency like Petri nets [11], nested Petri nets [8], and constrained multiset rewriting systems (CMRS) [2]. Specifically, we show that conformon P-systems are a special class of nested Petri nets, and restricted P-systems are a special class of CMRS. This comparison gives us indirect proofs for decidability of coverability in restricted conformon P-systems that follows from the results obtained for nested nets and CMRS in [9, 2].

To our knowledge, this is the first work devoted to the analysis of problems like coverability for conformon P-systems, and to the comparison of the same models with other concurrency models like nested Petri nets and CMRS.

Plan of the paper In Section 2 we introduce the conformon P-systems model. In Section 3 we study decision problems like reachability and coverability In Section 4 we compare conformon P-systems with nested Petri nets and CMRS. Finally, in Section 5 we discuss related work and address some conclusions.

2 Conformon P-systems

Let V be a finite alphabet and $\mathbb{N}$ the set of natural numbers. A *conformon* is an element of $V \times \mathbb{N}_0$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, denoted by $[X, x]$. We will refer to X as the *name* of the conformon $[X, x]$ and to x as its *value*. In the rest of the paper we work with multisets of conformons. We use $\{\!\{a_1, \ldots, a_n\}\!\}$ to indicate a multiset with elements $a_1, \ldots, a_n$, and symbols $\oplus$ and $\ominus$ to indicate resp. multiset union and difference. We use $\mathcal{C}_V$ to denote the set of conformons defined over alphabet V.

Conformons are situated inside a finite set of membranes or regions. Let N be the set of membrane names. A *configuration* μ is a tuple (indexed on m) of multisets of conformons. For simplicity we often assume that membranes are numbered from 1 to n and that configurations are tupled $(\xi_1, \ldots, \xi_n)$ where ξ_i is a multiset of conformons in $\mathcal{C}_V$.

The dynamic behavior of conformons is described via a set of rules of the following form:

- A *creation* rule has the form $\xrightarrow[m]{e} A$, where $A \in V$, $e \in \mathbb{N}_0$, and $m \in N$ and defines the creation of a conformon $[A, e]$ inside membrane m. A creation rule for conformon $[A, e]$ in membrane m corresponds to a conformon $[A, e]$ with cardinality ω in [5]. The use of creation rules allows us to obtain a better comparison with other Petri net models as discussed later in the paper.
- An *internal* rule has the form $A \xrightarrow[m]{e} B$, where $A, B \in V$, $e \in \mathbb{N}$, $m \in N$ and defines the passage of a quantity e of energy from a conformon of type A to one of type B inside membrane m.
- A *passage* rule has the form $m \overset{p}{\hookrightarrow} n$ where $m, n \in N$ and $p(x)$ is a monadic predicate of one of the following forms $x = a$, $x > a$, $x < b$ for $a \in \mathbb{N}_0$ and $b \in \mathbb{N}$. With

this rule, a conformon $[X, x]$ inside m can move to membrane n if $p(x)$ is satisfied by the current value of X.

As in tissue P-systems, the underlying structure of membranes is here a finite graph whose nodes are the membranes in N and edges are defined by passage rules. We are ready now for a formal definition of conformon P-systems.

Definition 1 (Conformon P-system). *A* basic conformon P-system of degree $m \geq 1$ with unbounded values *(cP-system for short) is a tuple $\Pi = (V, N, R, \mu_0)$, V is a finite set of conformon names, N is a finite set of membranes names (we assume that each membrane has a distinct name), R is a set of rules, μ_0 is an initial configuration.*

Given a configuration μ, we say that an internal rule $r = A \xrightarrow[m]{e} B$ is enabled at μ if there exist a conformon $[A, x] \in \mu(m)$ and a conformon $[B, y] \in \mu(m)$ such that $x \geq e$; we say in this case that r operates on conformons $[A, x]$ and $[B, y]$ in μ. A passage rule $r = m \overset{p}{\hookrightarrow} n$ is enabled at μ if there exists a conformon $[A, x] \in \mu(m)$ such that $p(x)$ is satisfied; we say here that r operates on conformon $[A, x]$ in μ. Notice that creation rules are always enabled. The evolution of a conformon P-system Π is defined via a transition relation $\Rightarrow$ defined on configurations as follows. A configuration μ may evolve to μ', written $\mu \Rightarrow \mu'$, if one of the following conditions is satisfied:

– There exists a rule $r = A \xrightarrow[m]{e} B$ in R which is enabled in μ and operates on conformons $[A, x]$ and $[B, y]$, and the following conditions are satisfied:
  - $\mu'(m) = (\mu(m) \ominus \{\!\{[A, x], [B, y]\}\!\}) \oplus \{\!\{[A, x - e], [B, y + e]\}\!\}$;
  - $\mu'(n) = \mu(n)$ for any $n \neq m$.
– There exists a rule $r = m \overset{p}{\hookrightarrow} n$ in R which is enabled in μ and operates on conformon $[A, x]$ (i.e. $p(x)$ is true) and the following conditions are satisfied:
  - $\mu'(m) = \mu(m) \ominus \{\!\{[A, x]\}\!\}$;
  - $\mu'(n) = \mu(n) \oplus \{\!\{[A, x]\}\!\}$;
  - $\mu'(p) = \mu(p)$ for any $p \neq m, n$.
– There exists a rule $r = \xrightarrow[m]{e} A$ in R and the following conditions are satisfied:
  - $\mu'(m) = \mu(m) \oplus \{\!\{[A, e]\}\!\}$;
  - $\mu'(p) = \mu(p)$ for any $p \neq m$.

In the rest of the paper we use $\Rightarrow^*$ to indicate the reflexive and transitive closure of the transition relation $\Rightarrow$. Furthermore, we say that μ evolves into μ' if $\mu \Rightarrow^* \mu'$, i.e., there exists a finite sequence of configurations $\mu_1, \ldots, \mu_r$ such that $\mu = \mu_1 \Rightarrow \ldots \Rightarrow \mu_r = \mu'$. Furthermore, given a set of configurations S, the set of successor configurations is defined as

$$Post(S) = \{\mu' \mid \mu \Rightarrow \mu',\ \mu \in S\}$$

and the set of predecessor configurations is defined as

$$Pre(S) = \{\mu' \mid \mu' \Rightarrow \mu,\ \mu \in S\}$$

Notice that the transition relation $\Rightarrow$ defines an interleaving semantics for a *c*P-system Π, i.e., only a single rule among those enabled can be fired at each evolution step of Π. This semantics is slightly different from the original semantics in [5] where an arbitrary subset of all enable rules can be fired at each evolution step. It is important to remark however that the two semantics are equivalent with respect to the kind of qualitative properties (reachability problems) we consider in this paper.

As an example, consider the *c*P-system with two membranes m_1 and m_2 and $N = \{A, B, C\}$, and with the rules $\xrightarrow[m_1]{1} A$, $A \xrightarrow[m_1]{1} B$, and $m_1 \overset{p}{\hookrightarrow} m_2$ where $p(x)$ is defined by the equality $x = 3$. In this model the configuration $c = (\{\!\!\{[B, 0]\}\!\!\}, \emptyset)$ may evolve as follows:

$$
\begin{array}{l}
c \Rightarrow (\{\!\!\{[A,1],[B,0]\}\!\!\},\emptyset) \Rightarrow (\{\!\!\{[A,1],[A,1],[B,0]\}\!\!\},\emptyset) \Rightarrow \\
(\{\!\!\{[A,1],[A,1],[A,1],[B,0]\}\!\!\},\emptyset) \Rightarrow (\{\!\!\{[A,1],[A,1],[A,0],[B,1]\}\!\!\},\emptyset) \Rightarrow \\
(\{\!\!\{[A,1],[A,0],[A,0],[B,2]\}\!\!\},\emptyset) \Rightarrow (\{\!\!\{[A,0],[A,0],[A,0],[B,3]\}\!\!\},\emptyset) \Rightarrow \\
(\{\!\!\{[A,0],[A,0],[A,0]\}\!\!\},\{\!\!\{[B,3]\}\!\!\})
\end{array}
$$

Finally, notice that both our semantics and Frisco's semantics in [5] do not require all enabled rules to be fired simultaneously as in the semantics of P-systems (maximal parallelism). In general, maximal parallelism and interleaving semantics may lead to models with different computational power.

3 Qualitative analysis of *c*P-systems

In [5] Frisco introduced the class of *c*P-systems with *bounded values* in which the only type of admitted creation rules have the form $\xrightarrow[m]{0} A$, i.e., the only type of conformons for which there is no upper bound on the number of occurrences in reachable configurations (finite but unbounded multiplicity) are of the form $[A, 0]$. In *c*P-system with *bounded values* the total amount of energy in the system is always constant. Thus, with this restriction, the only dimension of infiniteness of the state-space is the number of occurrences of conformons. This kind of restricted systems, say *c*P-systems with bounded values, can be represented as Petri nets. Thus, several interesting qualitative properties like reachability and coverability of a configuration and can be decided for this fragment of *c*P-systems.

In the full model the set of configurations reachable from an initial one may be infinite in two dimensions, i.e., in the number of conformons occurring in the membrane system and in the amount of total energy exchanged in the system. In [5] Frisco has proved that full *c*P-systems are a Turing equivalent model. Despite of the power of the model, we prove next that a basic qualitative property called *reachability* can be decided for full *c*P-systems. Let us first define the reachability problem.

Definition 2 (Reachability problem).
The reachability problem is defined as follows: Given a c*P-system* $\Pi = (V, N, R, \mu_0)$ *and a configuration* μ_1*, does* $\mu_0 \Rightarrow^* \mu_1$ *hold?*

The following results then hold.

Theorem 1 (Decidability of reachability for full *c*P-systems).
The reachability problem (w.r.t. relation $\Rightarrow$*) is decidable for any* c*P-system.*

Proof. The proof is based on a reduction of reachability of configuration μ_1 in a *c*P-system Π to reachability in a finite-state system extracted from Π and μ_1. The reduction is based on the following key observation. For two configurations μ_0 and μ_1 the set Q of distinct configurations that may occur in all possible evolutions from μ_0 to μ_1 is finite. This property is a consequence of the fact that internal and passage rules maintain constant the total number of conformons and the total amount of energy of a system (sum of the values of all conformons) whereas creation rules may only increase both parameters. Thus, the total amount of conformons and of energy in configuration μ_1 gives us an upper bound U_C on the possible number of conformons and an upper bound U_V on their corresponding values in any evolution from μ_0 to μ_1. Based on this observation, it is simple to define a finite-state automata S with states in Q and transition relation δ defined by instantiating the rules in R over the elements in S. As an example, if $V = \{A, B\}$, $N = \{m, n\}$, $U_C = 10$ and $U_V = 4$ and R contains the rule $r = A \xrightarrow[m]{2} B$. Then, we have to consider a finite state automaton in which the states are all possible multisets of at most 10 elements taken from the alphabet $\Sigma = \{[X, n] \mid X \in V, 0 \leq n \leq 4\}$. The rule r generates a transition relation δ that put in relations two states q and q' iff q contains a pair of elements $[A, a], [B, b] \in \Sigma$ such that a and $a + 2$ satisfy the condition $2 \leq a, a + 2 \leq 4$ and $q' = (q \ominus \{\!\{[A, a], [B, b]\}\!\}) \oplus \{\!\{[A, a-2], [B, b+2]\}\!\}$. The finite automata S satisfy the property that μ_1 is reachable from μ_0 if and only if for the state $s \in Q$ that represents μ_0 and $s' \in Q$ that represents μ_1, (s, s') is in the transitive closure of δ. The thesis then follows from the decidability of configuration reachability in a finite-automata. □

In order to study verification of safety properties, we need to introduce an ordering between configurations similar to the coverability ordering used for models like Petri nets. We use here an ordering $\subseteq$ between configurations μ and μ' such that for each membrane m, each conformon in $\mu(m)$ is mapped to a distinguished conformon in $\mu'(m)$ that has the same name and greater or equal value. This ordering allows us to reason about the presence of a conformon with a given name and at least a given amount of energy inside a configuration.

Example 1. Consider the configurations

$$\begin{aligned} \mu_1 &= (\{\!\{[A, 2], [A, 4], [B, 3]\}\!\}, \{\!\{[A, 5]\}\!\}) \\ \mu_2 &= (\{\!\{[A, 4], [A, 5], [B, 6], [C, 8]\}\!\}, \{\!\{[A, 7], [B, 5]\}\!\}) \end{aligned}$$

Then $\mu_1 \sqsubseteq \mu_2$, since $[A, 2]$, $[A, 4]$ and $[B, 3]$ in membrane 1 of μ can be associated resp. to the conformons $[A, 4]$, $[A, 5]$ and $[B, 6]$ in membrane 1 of μ_2; furthermore, $[A, 5]$ in membrane 2 of μ can be associated to conformon $[A, 7]$ in membrane 2 of μ_2. Consider now the configurations

$$\begin{aligned} \mu_3 &= (\{\!\{[A, 4], [A, 5], [B, 1]\}\!\}, \{\!\{[A, 7], [B, 5]\}\!\}) \\ \mu_4 &= (\{\!\{[A, 5], [B, 6]\}\!\}, \{\!\{[A, 7], [B, 5]\}\!\}) \end{aligned}$$

Then, $\mu_1 \not\sqsubseteq \mu_4$ since there is no conformon in membrane 1 in μ_3 with name B and value greater or equal than 3. Furthermore, $\mu_1 \not\sqsubseteq \mu_4$ since we cannot associate two different conformons, namely $[A, 2]$ and $[A, 4]$ in μ_1, to the same conformon, namely $[A, 5]$, in μ_4. Finally, notice that the configuration $\mu = (\{\![A,0]\}\!\}, \emptyset)$ is such that $\mu \sqsubseteq \mu_i$ for $i : 1, \ldots, 4$. The configuration μ can be used to characterize the presence of a conformon with name A in membrane 1 no matter of how energy it has.

The ordering $\sqsubseteq$ is formally defined as follows.

Definition 3 (Ordering $\sqsubseteq$). *Given two configurations μ and μ', $\mu \sqsubseteq \mu'$ iff for each $m \in N$ there exists an* injective mapping *h_m from $\mu(m)$ to $\mu'(m)$*[4] *that satisfies the following condition: for each $[A, x] \in \mu(m)$, if $h_m([A, x]) = [B, y]$, then $A = B$ and $x \leq y$ ($[A, x]$ is associated to a conformon with the same name and larger amount of energy).*

A set S of configurations is said *upward closed* w.r.t. $\sqsubseteq$ if the following condition is satisfied: for any $\mu \in S$, if $\mu \sqsubseteq \mu'$ then $\mu' \in S$. In other words if a configuration μ belongs to an upward closed set S than all configurations greater than μ w.r.t. $\sqsubseteq$ belong to S either.

Consider now the following decision problem.

Definition 4 (Coverability problem). *The coverability problem is defined as follows: Given a* c*P-system $\Pi = (V, N, R, \mu_0)$ and a configuration μ_1, is there a configuration μ_2 such that $\mu_0 \Rightarrow^* \mu_2$ and $\mu_1 \sqsubseteq \mu_2$?*

Coverability can be viewed as a weak form of *configuration reachability* in which we check whether configurations with certain constraints can be reachable from the initial configuration. In concurrency theory, the coverability problem is strictly related to the verification of safety properties. This link can naturally be transferred to *qualitative properties* of natural systems. As an example, checking if a configuration in which two conformons with name A can occur in membrane m during the evolution of a system amounts to checking the coverability problem for the target configuration μ_2 defined as $\mu_2(m) = \{\![A, 0], [A, 0]\}\!\}$ and $\mu_2(m') = \emptyset$ for $m' \neq m$. The following negative result then holds.

Proposition 1. *Coverability is undecidable for full* c*P-systems.*

Proof. The encoding of a counter machine M in *c*P-systems can be adapted to our formulation with creation rules in a direct way: conformons with ω-cardinality are specified here by creation rules. In the encoding in [5] an execution of the counter machine M leading to location ℓ is simulated by the evolution of a *c*P-system Π_M that reaches a configuration containing a conformon $[\ell, 9]$ in a particular membrane, say m. Thus, coverability of the configuration with $[\ell, 9]$ inside m in Π_M corresponds to reachability of location ℓ in M. Since location reachability is undecidable for counter machines, coverability is undecidable for *c*P-systems.

3.1 A syntactic fragments of *c*P-systems

In this section we show that checking safety properties can be decided for a fragment of *c*P-systems with a restricted form of passage rules in which conditions are only defined by lower bound constraints.

[4] By injective, we mean that two distinct conformons in $\mu(m)$ cannot be mapped to the same conformon in $\mu'(m)$.

Definition 5 (Restricted *c*P-systems). *We call* restricted *the fragment of* c*P-systems in which we forbid the use of predicates of the form $x = c$ and $x < c$ as conditions of passage rules.*

Our main result is that, despite of the two dimension of infiniteness, the coverability problem is decidable for restricted *c*P-systems with an arbitrary number of conformons. To prove the result we adopt the methodology proposed in [1], i.e., we first show that restricted *c*P-systems are monotonic w.r.t. $\sqsubseteq$. We then show that $\sqsubseteq$ is a well-quasi ordering. This implies that any upward closed set is represented via a finite set of minimal (w.r.t. $\sqsubseteq$) configurations. Thus, minimal elements can be used to finitely represent infinite (upward closed) sets of configurations. Finally, we prove that, given an upward closed set S of configurations, it is possible to compute a finite representation of the set of predecessor configurations of S. Monotonicity ensures us that such a set is still upward closed. We compute it by operating on the minimal elements of S only.

Lemma 1 (Monotonicity). *Restricted* c*P-systems are monotonic w.r.t. $\sqsubseteq$, i.e., if $\mu_1 \Rightarrow \mu_2$ and $\mu_1 \sqsubseteq \mu'_1$, then there exists μ'_2 such that $\mu'_1 \Rightarrow \mu'_2$ and $\mu_2 \sqsubseteq \mu'_2$.*

Proof. Let μ_1 be a configuration evolving into μ_2, and let $\mu_1 \leq \mu'_1$. The proof is by case analysis on the type of rules applied in the execution step.
Internal rule. Let us consider a single application of an internal rule (A, e, B) operating on conformons $[A, x]$ and $[B, y]$ in membrane m. Since the rule is enabled we have that $x \geq e$. Furthermore, the application of the rule modifies the value of the two conformons as follows: $[A, x - e]$ and $[B, y + e]$.
Since $\mu_1 \sqsubseteq \mu'_1$ and by definition of $\sqsubseteq$, we have that there exist conformons $[A, x']$ and $[B, y']$ in membrane m of μ'_1 such that $x \leq x'$ and $y \leq y'$. Thus, the same rule can be applied to $[A, x']$ and $[B, y']$ leading to a configuration μ'_2 in which the two selected conformons are updated as follows: $[A, x' - e]$ and $[B, y' + e]$. Finally, we notice that, since $x' \geq x \geq e$, we have that $x - e \leq x' - e$ and $y + e \leq y' + e$. Thus, $\mu_2 \leq \mu'_2$
Passage rule. Let us consider a single application of a passage rule (e, p), $e = (m, n)$, operating on the conformons $[A, x]$ in membrane m such that $p(y) = y > e$. Since the rule is enabled we have that $x > e$. Furthermore, the application of the rule moves the conformon to membrane n in μ_2.
Since $\mu_1 \sqsubseteq \mu'_1$ and by definition of $\sqsubseteq$, we have that there exist conformons $[A, x']$ in membrane m of μ'_1 such that $x \leq x'$. Thus, the same passage rule is enabled in μ'_1 and can be applied to move $[A, x']$ in membrane n in μ'_2. Thus, we have that $\mu_2 \leq \mu'_2$. □

From the monotonicity property, we obtain the following corollary.

Corollary 1. *For any restricted* c*P-systems and any upward closed set (w.r.t. $\sqsubseteq$) S of configurations, the set of predecessor configurations of S, namely $Pre(S) = \{\mu \mid \mu \Rightarrow \mu',\ \mu' \in S\}$, is upward closed.*

It is important to notice that the last two properties do not hold for full *c*P-systems. As an example, a passage rule from membrane 1 to 2 with predicate $x = 0$ is not monotonic w.r.t. to the configurations $\mu_1 = (\{\!\{[A, 0]\}\!\}, \emptyset)$ and $\mu'_1 = (\{\!\{[A, 1]\}\!\}, \emptyset)$. Indeed, $\mu_1 \sqsubseteq \mu'_1$ and $\mu_1 \Rightarrow \mu_2 = (\emptyset, \{\!\{[A, 0]\}\!\})$ but μ'_1 has no successors. Furthermore, the set of predecessors of the upward closed set with minimal element μ_3 is the singleton containing μ_1 (clearly not an upward closed set).

Let us now go back to the properties of the ordering $\sqsubseteq$. We first have the following property.

Lemma 2. *Given a* c*P-system* Π *and two configurations* μ *and* μ'*, checking if* $\mu \sqsubseteq \mu'$ *holds (i.e. if* μ *is more general than* μ'*) is a decidable problem.*

Indeed, to decide it we have to select an appropriate injective mapping from a a finite set of mappings from μ to μ' and, then, to compute a finite set of multiset inclusions.

Let us now recall the notion of *well-quasi ordering* (see e.g. [7]).

Definition 6 ($\sqsubseteq$ is a wqo). *A quasi ordering* $\preceq$ *on a set* S *is a well-quasi ordering (wqo) if and only if for any infinite sequence* $a_1, a_2, \ldots$ *of elements in* S *(i.e.* $a_i \in S$ *for any* $i \geq 1$*) there exist indexes* $i < j$ *such that* $a_i \preceq a_j$*.*

The following important property then holds.

Lemma 3 ($\sqsubseteq$ is a wqo). *Given a* c*P-system* $\Pi = (V, N, R, \mu_0)$*, the ordering* $\sqsubseteq$ *defined on the set of all configuration of* Π *is a wqo.*

Proof. Assume $N = \{1, \ldots, m\}$ as the set of membrane names. Let us first notice that a configuration μ can be viewed as a multiset of multisets of objects over the alphabet $V^1 \cup \ldots \cup V^m$, where $V^i = \{v^i \mid v \in V\}$. Indeed, μ can be reformulated as the multiset union $\rho_1 \oplus \ldots \oplus \rho_m$ where for each $[A, x] \in \mu(m)$, ρ_i contains a multiset with x occurrences of A^m. E.g., $\mu_1 = (\{\!\{[A, 2], [B, 3]\}\!\}, \{\!\{[A, 5]\}\!\})$ can be reformulated as the multiset of multisets $\{\!\{\{\!\{A^1, A^1\}\!\}, \{\!\{B^1, B^1, B^1\}\!\}, \{\!\{A^2, A^2, A^2, A^2, A^2\}\!\}\}\!\}$.

When considering the aforementioned reformulation of configurations, the ordering $\sqsubseteq$ corresponds to the composition of multiset embedding (the existence of injective mapping $h_1, \ldots, h_m$) and multiset inclusion (the constraint on values). Since multiset inclusion is a well-quasi ordering, we can apply Higman's Lemma [7] to conclude that $\sqsubseteq$ is a well-quasi ordering. □

As a consequence of the latter property, we have that every upward closed set S of configurations is generated by a finite set of minimal elements, i.e., for any upward closed set S there exists a finite set F of configurations such that $S = \{\mu' \mid \mu \leq \mu', \mu \in F\}$. F is called the *finite basis* of S. As proved in the following lemma, given a finite basis of a set S, it is possible to effectively compute the finite basis of $Pre(S)$.

Lemma 4 (Computing Pre). *Given a finite basis* F *of a set* S*, there exists an algorithm that computes a finite basis* F' *of* $Pre(S)$*.*

Proof. The algorithm is defined by cases as follows.

Creation rules Assume $\xrightarrow[m]{e} A \in R$ and $\mu \in F$. Then, μ occurs in F'. Furthermore, suppose that $\mu(m)$ contains a conformon $[A, e]$. Then, F' also contains the configurations μ' that satisfies the following conditions:

- $\mu'(m) = \mu(m) \ominus \{\!\{[A, e]\}\!\}$;
- $\mu'(n) = \mu(n)$ for $m \neq n$.

Internal rules Assume a rule $r = A \xrightarrow[m]{e} B \in R$ and $\mu \in F$. We have several cases to consider.

- We first have to consider a possible application of r to two conformons that are not explicitly mentioned in μ. This leads to a predecessor configuration in which we require at least the presence of A with at least value e and the presence of B with any value. Thus, F' contains the configurations μ' that satisfies the following conditions:
  - $\mu'(m) = \mu(m) \oplus \{\!\{[A, e], [B, 0]\}\!\}$;
  - $\mu'(n) = \mu(n)$ for $m \neq n$.
- We now have to consider the application of r to a conformon A with value x in μ and to a conformon B not explicitly mentioned in μ. This leads to a predecessor configuration in which we require at least the presence of A with at least value $x + e$ and the presence of B with any value. Thus, if $[A, x] \in \mu(m)$, F' contains the configurations μ' that satisfies the following conditions:
  - $\mu'(m) = (\mu(m) \ominus \{\!\{[A, x]\}\!\}) \oplus \{\!\{[A, x + e], [B, 0]\}\!\}$;
  - $\mu'(n) = \mu(n)$ for $m \neq n$.
- Furthermore, we have to consider the application of r to a conformon B with value $y \geq e$ in μ and to a conformon A not explicitly mentioned in μ. This leads to a predecessor configuration in which we require at least the presence of A with at least value e and the presence of B with value $y - e$. Thus, if $[B, y] \in \mu(m)$ and $y \geq e$, F' contains the configurations μ' that satisfies the following conditions:
  - $\mu'(m) = (\mu(m) \ominus \{\!\{[B, y]\}\!\}) \oplus \{\!\{[A, e], [B, y - e]\}\!\}$;
  - $\mu'(n) = \mu(n)$ for $m \neq n$.
- Finally, we have to consider the application of r to a conformon B with value $y \geq e$ and to a conformon A with value x both in μ. This leads to a predecessor configuration in which we require at least the presence of A with at least value $x+e$ and the presence of B with value $y - e$. Thus, if $[A, x], [B, y] \in \mu(m)$ and $y \geq e$, F' contains the configurations μ' that satisfies the following conditions:
  - $\mu'(m) = (\mu(m) \ominus \{\!\{[A, x], [B, y]\}\!\}) \oplus \{\!\{[A, x + e], [B, y - e]\}\!\}$;
  - $\mu'(n) = \mu(n)$ for $m \neq n$.

Passage rules Assume $m \overset{p}{\hookrightarrow} n \in R$ with $p(x)$ defined by $x \geq c$ and $\mu \in F$. We first have to consider a possible application of r to a conformon that is not explicitly mentioned in μ. This leads to a predecessor configuration in which we require at least the presence of A with at least value e in membrane m. Thus, F' contains the configurations μ' that satisfies the following conditions:

- $\mu'(m) = \mu(m) \oplus \{\!\{[A, e]\}\!\}$;
- $\mu'(n) = \mu(n)$ for $m \neq n$.

Furthermore, suppose that $\mu(n)$ contains a conformon $[A, x]$ with $x \geq e$. Then, F' also contains the configurations μ' that satisfies the following conditions:

- $\mu'(n) = \mu(n) \ominus \{\!\{[A, x]\}\!\}$;
- $\mu'(m) = \mu(m) \oplus \{\!\{[A, x]\}\!\}$;
- $\mu'(p) = \mu(p)$ for $m, n \neq p$.

The correctness follows from a simple case analysis. □

Theorem 2 (Decidability of Coverability for Restricted *c*P-systems). *The coverability problem is decidable for restricted* c*P-systems.*

Proof. The thesis follows from Lemmas 1, 3, 4, and from Theorem 4.1 in [1].

4 Relation with other models

In this section we compare *c*P-systems with other models used in the concurrency field, namely the nested Petri nets of [8] and the constrained multiset rewriting systems (CMRS) of [2].

4.1 *c*P-systems vs nested Petri nets

Let us first recall that a Petri net (P/T system) [11] is a tuple (P, T, m_0) where P is a finite set of $places$, T is a finite set of $transitions$, and m_0 is the initial marking. Intuitively, places correspond to location or states of a given system. Places are populated with tokens, i.e., indistinguishable objects, that can be used e.g. to mark a given set of states of to model concurrent processes. Tokens have no internal structure. This means that we are only interested in the multiplicity of tokens inside a place. Transitions are used to control the flow of tokens in the net (they define links between different places and regulate the movement of tokens along the links). More formally, a $transition\ t$ has a pre-set $^\bullet t$ and a post-set $t^\bullet$ both defined by multisets of places in P. A marking is just a multiset with elements in P, a mapping from P to non-negative integers. Given a marking m and a place p, we say that the place p contains $m(p)$ *tokens*. A transition t is enabled at the marking m if $^\bullet t$ is contained as a sub-multiset in m. If it is the case, firing t produces a marking m', written $m \xrightarrow{t} m'$, defined as $(m \ominus^\bullet t) \oplus t^\bullet$, where $\oplus$ is multiset union and $\ominus$ is multiset difference. A firing sequence is a sequence of markings $m_0 m_1 \ldots$ such that m_i is obtained from m_{i-1} by firing a transition in T at m_i.

A Petri net with inhibitor arc is a Petri net in which transitions can be guarded by an emptyness test on a subset of the places. For instance, a transition with an inhibitor arc on place p is enabled only when p is empty.

Nested Petri nets Differently from P/T systems, in a *nested Petri net* tokens have an internal structure that can be arbitrarily complex (e.g. a token can be a P/T system, or a P/T system with tokens that are in turn P/T systems, and so on). For instance, a *2-level nested Petri net* is defined by a P/T system that describes the whole system, called *system net*, and by a P/T system that describes the internal structure of tokens, called *element net*.

The transitions of the system net can be used to manipulate tokens as black boxes, i.e. without changing their internal structure. These kind of transitions are called *transport rules* (they move complex objects around the places of the system net).

Transitions of the element nets can be used to change the internal structure of a token

without changing the marking of the system net. These kind of transitions are called *autonomous rules*.
Finally, we can use synchronization labels (i.e. labels in system/element net transitions) to enforce the simultaneous execution of a transition of the system net and of an element net (*vertical synchronization*), or the simultaneous execution of transitions of two distinct element nets residing in the same system place (*horizontal synchronization*).
Notice that vertical synchronization modifies both the marking of the system net and the internal structure of (some) tokens.

cP-systems as nested Petri nets In this section we show that *c*P-systems can be encoded as 2-level nested Petri nets in which the system net is a P/T system and the element net is a P/T system with inhibitor arcs.

Assume a *c*P-system $\Pi = (V, N, R, \mu_0)$. We build a 2-level nested Petri nets as follows. The system net is a P/T system with a places $CONF$ used to contain all conformons in a current configuration of Π, and a place $CREATE_r$ for each creation rule $r \in R$. The transitions of the system net are transport rules that model creation rules used to non-deterministically inject new conformons in place $CONF$. Namely, for each creation rule $r \in R$ we add a transport rule t_r with present $\{\!\{CREATE_r\}\!\}$ and postset $\{\!\{CREATE_r, CONF\}\!\}$. We assume here that $CREATE_r$ is initialized with a single element net that models the conformon created by rule r. Transition t_r makes a copy of such an element net and puts it in place $CONF$.

An element net N_c denotes a single conformon c. It is defined by a P/T system with places $P = V \cup N \cup \{E\}$. Only one place of those in N and only one place of those in V can be marked in the same instant. The marked places correspond to the name and current location of c. Furthermore, the number of tokens in place E denotes the current amount of energy of c.

To model an internal rule $r = A \xrightarrow[m]{e} B$ we use a horizontal step between two distinct element nets N_1 and N_2, i.e., a pair $(t_{r,1}, t_{r,2})$ of element net transitions with synchronized labels such that:

– ${}^\bullet t_{r,1}$ has one occurrence of A, one of m, and e of E, i.e., it is enabled if N_1 represents a conformon with name A in membrane m and at least e units of energy; those units are subtracted from place E in N_1.
– ${}^\bullet t_{r,2}$ has one occurrence of B and one of m, i.e., it is enabled if N_2 represents a conformon with name B in membrane m.
– $t_{r,1}^\bullet$ has one occurrence of A and one of m.
– $t_{r,2}^\bullet$ has one occurrence of B, one of m, and e of E, i.e., e units of energy are transferred to place E in N_2.

To model a passage rule $r = m \overset{p}{\hookrightarrow} n$ with condition $x \geq e$, we use an autonomous step. Specifically, we define an element net transition t_r such that:

– ${}^\bullet t_r$ has one occurence of m, and e occurrences of E, i.e., it is enabled if N_1 is in membrane m and at least e units of energy.
– $t_r^\bullet$ has one occurrences of n, and and e occurrences of E, i.e., N_1 represents now a conformon (with the same name) in membrane n. Its energy is not changed (we first subtract e tokens to check the condition $x \geq e$) and then add e tokens back to place E in N_1).

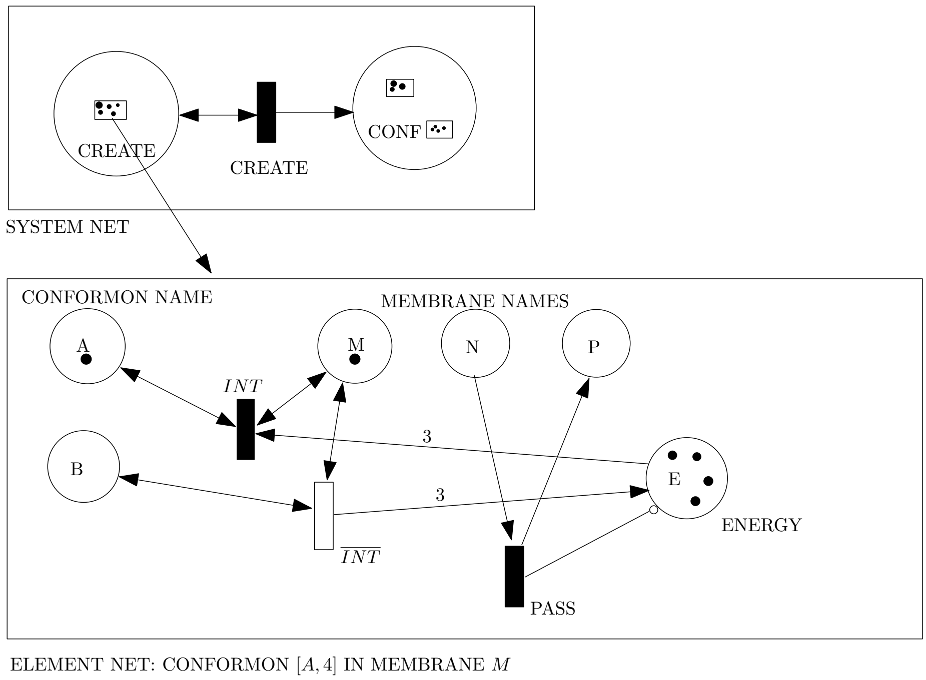

Fig. 1. Example of nested Petri net.

To model a passage rule r with condition $x = e$, we can add to each transition $t_{r,A}$ with $A \in V$ the test $= e$ on place E. It is easy to define this test by using P/T transitions with inhibitor arcs. Rules with conditions $x < e$ for $e > 1$ can be encoded by splitting the test into $x = 0, \ldots, x = e - 1$.

A marking of the resulting 2-level nested net specifies the number of element nets inside the system place $CONF$. Since each element net maintains information about name, value and location the content of place $CONF$ corresponds to the current configuration of Π.

Example 2. Assume a *c*P-system Π with $V = \{A, B\}$, $N = \{M, N, P\}$, creation rules $\xrightarrow[M]{4} A$ and $\xrightarrow[N]{2} B$, internal rule $A \xrightarrow[M]{3} B$, and passage rule $N \xhookrightarrow{p} P$ with condition $x = 0$. The 2-level nested Petri nets that encodes the *c*P-systems Π is shown in Fig. 1. We use here circles to denote places, rectangles to denote transitions, an arrow from a circle to a rectangle to denote places in the pre-set and an arrow from a rectangle to a circle to denote places in the post-sets of transitions; we label arrows with numbers to indicate a multiplicity > 1 of a place in the pre-/post-set. The system net place $CREATE$ is used to keep a copy resp. of the conformon $[A, 4]$ so as to non-deterministically inject new ones in the current configuration (place $CONF$). The element net has places to model names, membranes, and energy. The internal rule is modelled by the pair of transitions with labels INT and $\overline{INT}$. When executed simultaneously (within place $CONF$ of the system net) by two distinct element net (one executes INT and the other executes

$\overline{INT}$) their effect is to move 3 tokens from the E place of an element net marked A, M to the E place of an element net marked B, M. Notice that tokens of the element nets are objects with no structure. The passage rule is modelled by the element net transition with label $PASS$. It simply checks that E is empty with an inhibitor arc (arrow with circle) and then moves a token from the place N to the place P (it changes the location of the element net). Notice that the system net place $CONF$ may contain an arbitrary number of element nets (the corresponding P/T system is unbounded).

It is important to notice that 2-level nested Petri nets in which element nets have inhibitor arcs are Turing equivalent [9]. This result is consistent with the analysis of the expressive power of full *c*P-systems [5]. From the previous observations, restricted *c*P-systems are a subclass of nested Petri nets in which both the system and the element nets are defined by P/T systems. From the results obtained for well-structured subclasses of nested Petri nets in [9], we obtain an indirect proof for decidability of *coverability* of restricted *c*P-systems.

The connection between *c*P-systems and nested nets can be exploit to extend the model in several ways. As an example, for restricted passage rules, coverability remains decidable when extending *c*P-systems with: conformons defined by a list of pairs name-value instead of a single pair; rules that *transfer* all the energy from A to B; or conformons defined by a state machine (i.e. with an internal state instead of statically assigned type).

4.2 Restricted *c*P-systems vs CMRS

Restricted *c*P-systems can also be modelled in CMRS, an extension of Petri nets in which tokens carry natural numbers.

CMRS Constrained multiset rewriting systems (CMRS) [2] are inspired to formulations of colored Petri nets in term rewriting. A token with data d in place p is represented here as a term $p(d)$, a marking as a multiset of terms, and a transition as a (conditional) multiset rewriting rule. More precisely, let *term* be an element $p(x)$ where p belong to a finite set of predicate symbols $\mathbb{P}$ (places) and x is a variable ranging over natural numbers. We often call a term $p(t)$ with $p \in P$ a *p-term* or *P-term*. A element $p(v)$ with $p \in \mathbb{P}$ and $v \in Nat$ is called a *ground term*.
A configuration is a (finite) multiset of ground terms. A CMRS is a set of rewriting rules with constraints of the form $r = L \rightsquigarrow R : \Psi$ that allows to transform (rewrite) multisets into multisets. More precisely, L and R are multisets of terms (with variables) and Ψ is a (possibly empty) finite conjunction of *gap-order* constraints of the form: $x + c < y$, $x \leq y$, $x = y$, $x < c$, $x > c$, $x = c$ where x, y are variables appearing in L and/or R and $c \in Nat$ is a constant.
A rule r is enabled at a configuration c if there exists a valuation of the variables Val such that $Val(\Psi)$ is satisfied. Firing r at c leads to a new multiset $c' = c \ominus Val(L) \oplus Val(R)$ where $Val(L)$, resp. $Val(R)$, is the multiset of ground terms obtained from L, resp. R, by replacing each variable x by $Val(x)$.

As an example, consider the CMRS rule:

$$\rho = [p(x)\ ,\ q(y)]\ \rightsquigarrow\ [q(z)\ ,\ r(x)\ ,\ r(w)]\ \ :\ \ \{x+2<y\ ,\ x+4<z\ ,\ z<w\}$$

A valuation which satisfies the condition is $Val(x) = 1$, $Val(y) = 4$, $Val(z) = 8$, and $Val(w) = 10$. A CMRS configuration is a multisets of ground terms, e.g., $[p(1), p(3), q(4)]$. Therefore, we have that $[p(1), p(3), q(4)]t\rho[p(3), q(8), r(1), r(10)]$.

A CMRS is well-structured with respect to the well-quasi ordering $\preceq_c$ defined as follows. Given a configuration c, let $V(c) = \{i \in Nat \mid \exists p(i) \in c\}$, and $c_{=i} : \mathbb{P} \mapsto Nat$ with $i \in Nat$ be the multi set such that $c_{=i}(p) = c(p_i)$ for any $p \in \mathbb{P}$. Then, we have that $c \preceq_c c'$ iff there exists an injective function $h : V(c) \mapsto Nat$ such that (i) for any $i \in V(c) : c_{=i} \leq c'_{=h(i)}$; ($ii$) for any $i \in V(c)$ s.t. $i \leq cmax : i = h(i)$; (iii) for any $i, j \in V(c) \cup \{0\}$ s.t. $i < j$ and $j > cmax : j - i < h(j) - h(i)$. A symbolic algorithm to check coverability – w.r.t. $\preceq_c$ – is described in [2].

Restricted cP-systems as CMRS A cP-configuration μ is mapped to a CMRS configuration as follows. A conformon $c = [A, x]$ in membrane m is represented by means of a multiset of terms

$$\mathbb{M}^v_{c,m} = [conf_{A,m}(v)] \oplus \mathbb{O}^v_x$$

where $\mathbb{O}^x_v$ is the multiset with x occurrences of the term $u(v)$, i.e.,

$$\mathbb{O}^v_x = [\underbrace{u(v), \ldots, u(v)}_{x-times}]$$

where v is a natural number used as a unique identifier for the conformon c. The u-terms with parameter v are used to count the amount of energy of conformon with identifier v. E.g. if $c = [ATP, 4]$ then $\mathbb{M}^2_{c,m} = [conf_{ATP,m}(2), u(2), u(2), u(2), u(2)]$ – 4 occurrences of $u(2)$ – where 2 is the unique identifier of conformon c. Furthermore, if $c = [ATP, 0]$, then $\mathbb{M}^2_{c,m} = [conf_{ATP,m}(2)]$. Thus, we use $[conf_{A,m}(v)]$ to model a conformon with zero energy and identifier v.

A representation $Rep(\mu)$ of a cP-configuration μ is obtained by assigning a distinct indentifier to each conformon and by taking the (multiset) union of the representations of each conformons in μ. Formally, let μ contains r membranes such that $\mu(m_i)$ contains the conformons $c_{1,i}, \ldots, c_{i,n_i}$ for $i : 1, \ldots, r$ and $n_1 + \ldots + n_r = k$, then

$$Rep(\mu)^V = (\bigoplus_{j=1}^{n_1} \mathbb{M}^{v_{1,j}}_{c_{1,j},m_1}) \oplus \ldots \oplus (\bigoplus_{j=1}^{n_r} \mathbb{M}^{v_{r,j}}_{c_{r,j},m_r})$$

where $V = (v_{1,1}, \ldots, v_{1,n_1}, \ldots, v_{r,1}, \ldots, v_{r,n_r})$ are k distinct natural numbers working as identifiers of the k conformons in μ. Identifiers of conformons in the initial configuration μ_0 are non-deterministically chosen at the beginning of the simulation using the following rule:

$$[init] \rightsquigarrow [fresh(v)] \oplus Rep(\mu_0)^V \ : \ \{v_{1,1} < \ldots < v_{1,n_1} < v_{r,1} < v_{r,n_r} < v\}$$

where V is a vector of variables that denotes conformon indentifiers (as described in the def. of $Rep(\mu)^V$). Furthermore, we maintain a fresh identifier v in the $fresh$-term (used to dynamically create other conformons).

The rules of a restricted cP-system are simulated via the following CMRS rules working on CMRS representations of configurations.

– Creation of $c = [A, x]$ inside m:

$$[fresh(x)] \rightsquigarrow [fresh(y)] \oplus \mathbb{M}^x_{c,m} \ : \ \{x < y\}$$

We simply inject a new multiset of terms with parameter x stored in the $fresh$-term and reset the fresh value.

– A and B exchange e units of energy.
For each membrane m:

$$[conf_{A,m}(x), conf_{B,m}(y)] \oplus \mathbb{O}^x_e \rightsquigarrow [conf_{A,m}(x), conf_{B,m}(y)] \oplus \mathbb{O}^y_e \ : \ true$$

Notice that, by definition of the CMRS operational semantics, the rule is enabled only when there are at least e occurrences of u-terms with parameter x (identifier of A) and where there exists a conformon B with identifier y (x and y are variables ranging over natural numbers). The passage of energy from A (with identifier x) to B (with identifier y) is simply defined by changing the parameter x of e occurrences of u-terms into y.

– Passage rule form m to n conditioned by $x \geq c$:
For each membrane value A:

$$[conf_{A,m}(x)] \oplus \mathbb{O}^x_c \rightsquigarrow [conf_{A,n}(x)] \oplus \mathbb{O}^x_c$$

Notice that, by definition of the CMRS operational semantics, the rule is enabled only when there are at least c occurrences of u-terms with parameter x (identifier of A). The current location of A is stored in the term $conf_{A,m}(x)$. The passage to membrane n is defined by changing the term $conf_{A,m}(x)$ into $conf_{A,n}(x)$. The u-terms with the same parameter are not consumed (i.e. they occur both in the left-hand side and in the right-hand side of the rule).

From the results obtained for CMRS [2], we obtain another indirect proof for decidability of *coverability* of restricted *c*P-systems. The connection between *c*P-systems and CMRS can be used to devise extensions of the conformon model in which, e.g., conformon have different priorities or ordered with respect to some other parameter. This can be achieved by ordering the parameters of the multiset of terms used to encode each conformon. CMRS rules can deal with such an ordering by using conditions on parameters of terms in a rule of the form $x < y$.

5 Related Work and Conclusions

In the paper we have investigated the decidability of computational properties of conformon P-systems like reachability and coverability. More specifically, we have shown that, although undecidable for the full model, the coverability problem is decidable for a fragment with restricted types of predicates in passage rules.

To our knowledge, this is the first work devoted to the qualitative analysis of conformon P-systems, and to the comparison with other models like nested Petri nets and CMRS. The expressiveness of the conformon P-systems is studied in [5] by using a reduction to counter machines with zero test (Turing equivalent). We use such a result to show that coverability is undecidable for the full model. The decidability or reachability for the full model is not in contrast with its great expressive power. Indeed, in the reachability problem the target configuration contains precise information about the history of the computation, e.g., the total amount of energy exchanged during the computation. These information cannot be expressed in the coverability problem, where we can only fix part of the information of target configurations. In this sense, coverability seems a better measure for the expressiveness of this kind of computational models.

In the paper we have compared this result with similar results obtained for other models like nested Petri nets and constrained multiset rewriting systems. The direct proof presented in the paper and the corresponding algorithm can be viewed however as a first step towards the development of automated verification tools for biologically inspired models. The kind of qualitative analysis that can be performed using our algorithm is complementary to the simulation techniques used in quantitative analysis of natural and biological systems. Indeeed, in qualitative analysis we consider all possible executions with no probability distributions on transitions, whereas in quantitative analysis one often considers a single simulation by associating probabilities to each single transitions. Unfortunately, the rates of reactions are often unknown and, thus, extrapolated from known data to make the simulation feasible. Qualitative analysis requires instead only the knowledge of the dynamics of a given natural model. Automated verification methods can thus be useful to individuate structural properties of biological models.

References

1. P. A. Abdulla, K. Čerāns, B. Jonsson, and T. Yih-Kuen. General decidability theorems for infinite-state systems. LICS '96: 313–321.
2. P. A. Abdulla and G. Delzanno: On the Coverability Problem for Constrained Multiset Rewriting Systems. AVIS '06 (ETAPS-workshop). Available in the authors' home page.
3. Z. Dang, O. Ibarra, C. Li, G. Xie. On Model Checking of P Systems. UC 2005: 82-93.
4. A. Finkel, Ph. Schnoebelen. Well-structured transition systems everywhere! TCS 256(1-2), 2001: 63-92.
5. P. Frisco. The conformon-P system: a molecular and cell biology-inspired computability model. TCS 312(2-3), 2004: 295-319.
6. P.Frisco and D. W. Corne. Dynamics of HIV infection studied with Cellular Automata and conformon-P systems, BioSystems 91 (3), 2008: 531-544.
7. G. Higman. Ordering by divisibility in abstract algebras. London Math. Soc. (3), 2(7), 1952: 326336.
8. I. A. Lomazova. Nested petri nets. Fundamenta Informaticae 43(1-4), 2000: 195 - 214.
9. I. A. Lomazova and Ph. Schnoebelen. Some Decidability Results for Nested Petri Nets. Ershov Mem. Conf. 1999: 208-220.
10. G. Paun. Computing with membranes. JCSS 61(1), 2000: 108-143.
11. W. Reisig. Petri Nets: An Introduction. Springer, 1985.