Solving Parity Games on Integer Vectors

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Abstract. We consider parity games on infinite graphs where configurations are represented by control-states and integer vectors. This framework subsumes two classic game problems: parity games on vector addition systems with states (VASS) and multidimensional energy parity games. We show that the multidimensional energy parity game problem is inter-reducible with a subclass of single-sided parity games on VASS where just one player can modify the integer counters and the opponent can only change control-states. Our main result is that the minimal elements of the upward-closed winning set of these single-sided parity games on VASS are computable. This implies that the Pareto frontier of the minimal initial credit needed to win multidimensional energy parity games is also computable, solving an open question from the literature. Moreover, our main result implies the decidability of weak simulation preorder/equivalence between finite-state systems and VASS, and the decidability of model checking VASS with a large fragment of the modal μ-calculus.

1 Introduction

In this paper, we consider *integer games*: two-player turn-based games where a color (natural number) is associated to each state, and where the transitions allow incrementing and decrementing the values of a finite set of integer-valued counters by constants. We refer to the players as Player 0 and Player 1.

We consider the classical parity condition, together with two different semantics for integer games: the *energy semantics* and the VASS *semantics*. The former corresponds to *multidimensional energy parity games* [7], and the latter to parity games on VASS (a model essentially equivalent to Petri nets [8]). In energy parity games, the winning objective for Player 0 combines a qualitative property, the classical *parity condition*, with a quantitative property, namely the *energy condition*. The latter means that the values of all counters stay above a finite threshold during the entire run of the game. In VASS parity games, the counter values are restricted to *natural numbers*, and in particular any transition that may decrease the value of a counter below zero is disabled (unlike in energy games where such a transition would be immediately winning for Player 1). So for VASS games, the objective consists only of a parity condition, since the energy condition is trivially satisfied.

* Supported by Uppsala Programming for Multicore Architectures Research Center (UpMarc).
** Supported by Royal Society grant IE110996.
*** Supported by the project AMALFI (University of Turin/Compagnia di San Paolo) and the MIUR-PRIN project CINA.

We formulate and solve our problems using a generalized notion of game configurations, namely *partial configurations*, in which only a subset C of the counters may be defined. A partial configuration γ denotes a (possibly infinite) set of concrete configurations that are called *instantiations* of γ. A configuration γ' is an instantiation of γ if γ' agrees with γ on the values of the counters in C while the values of counters outside C can be chosen freely in γ'. We declare a partial configuration to be *winning* (for Player 0) if it has an instantiation that is winning. For each decision problem and each set of counters C, we will consider the *C-version* of the problem where we reason about configurations in which the counters in C are defined.

Previous Work. Two special cases of the general C-version are the *abstract* version in which no counters are defined, and the *concrete* version in which all counters are defined. In the energy semantics, the abstract version corresponds to the *unknown initial credit problem* for multidimensional energy parity games, which is coNP-complete [6, 7]. The concrete version corresponds to the *fixed initial credit problem*. For energy games without the parity condition, the fixed initial credit problem was solved in [4] (although it does not explicitly mention energy games but instead formulates the problem as a *zero-reachability* objective for Player 1). It follows from [4] that the fixed initial credit problem for d-dimensional energy games can be solved in d-EXPTIME (resp. $(d-1)$-EXPTIME for offsets encoded in unary) and even the upward-closed winning sets can be computed. An EXPSPACE lower bound is derived by a reduction from Petri net coverability. The subcase of one-dimensional energy parity games was considered in [5], where both the unknown and fixed initial credit problems are decidable, and the winning sets (i.e., the minimal required initial energy) can be computed. The assumption of having just one dimension is an important restriction that significantly simplifies the problem. This case is solved using an algorithm which is a generalization of the classical algorithms of McNaughton [13] and Zielonka [16].

However, for general multidimensional energy parity games, computing the winning sets was an open problem, mentioned, e.g., in [6].

In contrast, under the VASS semantics, all these integer game problems are shown to be undecidable for dimensions ≥ 2 in [1], even for simple safety/coverability objectives. (The one-dimensional case is a special case of parity games on one-counter machines, which is PSPACE-complete). A special subcase are *single-sided* VASS games, where just Player 0 can modify counters while Player 1 can only change control-states. This restriction makes the winning set for Player 0 upward-closed, unlike in general VASS games. The paper [14] shows decidability of coverability objectives for single-sided VASS games, using a standard backward fixpoint computation.

Our Contribution. First we show how instances of the single-sided VASS parity game can be reduced to the multidimensional energy parity game, and vice-versa. I.e., energy games correspond to the single-sided subcase of VASS games. Notice that, since parity conditions are closed under complement, it is merely a convention that Player 0 (and not Player 1) is the one that can change the counters.

Our main result is the decidability of single-sided VASS parity games for general partial configurations, and thus in particular for the concrete and abstract versions described above. The winning set for Player 0 is upward-closed (wrt. the natural multiset ordering on configurations), and it can be computed (i.e., its finitely many minimal elements). Our algorithm uses the Valk-Jantzen construction [15] and a technique similar to Karp-Miller graphs, and finally reduces the problem to instances of the abstract parity problem under the energy semantics, i.e., to the unknown initial credit problem in multidimensional energy parity games, which is decidable by [7].

From the above connection between single-sided VASS parity games and multidimensional energy parity games, it follows that the winning sets of multidimensional energy parity games are also computable. I.e., one can compute the *Pareto frontier* of the minimal initial energy credit vectors required to win the energy parity game. This solves the problem left open in [6, 7].

Our results imply further decidability results in the following two areas: semantic equivalence checking and model-checking. Weak simulation preorder between a finite-state system and a general VASS can be reduced to a parity game on a single-sided VASS, and is therefore decidable. Combined with the previously known decidability of the reverse direction [2], this implies decidability of weak simulation equivalence. This contrasts with the undecidability of weak *bisimulation* equivalence between VASS and finite-state systems [11]. The model-checking problem for VASS is decidable for many linear-time temporal logics [10], but undecidable even for very restricted branching-time logics [8]. We show the decidability of model-checking for a restricted class of VASS with a large fragment of the modal μ-calculus. Namely we consider VASS where some states do not perform any updates on the counters, and these states are used to guard the for-all-successors modal operators in this fragment of the μ-calculus, allowing us to reduce the model-checking problem to a parity game on single-sided VASS.

2 Integer Games

Preliminaries. We use $\mathbb{N}$ and $\mathbb{Z}$ to denote the sets of natural numbers (including 0) and integers respectively. For a set A, we define $|A|$ to be the cardinality of A. For a function $f : A \mapsto B$ from a set A to a set B, we use $f[a \leftarrow b]$ to denote the function f' such that $f(a) = b$ and $f'(a') = f(a')$ if $a' \neq a$. If f is partial, then $f(a) = \perp$ means that f is undefined for a. In particular $f[a \leftarrow \perp]$ makes the value of a undefined. We define $dom(f) := \{a |\, f(a) \neq \perp\}$.

Model. We assume a finite set $\mathcal{C}$ of *counters*. An *integer game* is a tuple $\mathcal{G} = \langle Q, T, \kappa \rangle$ where Q is a finite set of *states*, T is a finite set of *transitions*, and $\kappa : Q \mapsto \{0, 1, 2, \ldots, k\}$ is a *coloring* function that assigns to each $q \in Q$ a natural number in the interval $[0..k]$ for some pre-defined k. The set Q is partitioned into two sets Q_0 (states of Player 0) and Q_1 (states of Player 1). A transition $t \in T$ is a triple $\langle q_1, op, q_2 \rangle$ where $q_1, q_2 \in Q$ are states and op is an operation of one of the following three forms (where $c \in \mathcal{C}$ is a counter): (i) c++ increments the value of c by one; (ii) c- - decrements the value of c by one; (iii) *nop* does not change the value of any counter. We define $\texttt{source}(t) = q_1$, $\texttt{target}(t) = q_2$, and $\texttt{op}(t) = op$. We say that $\mathcal{G}$ is *single-sided* in case $op = nop$ for all transitions $t \in T$ with $\texttt{source}(t) \in Q_1$. In other words, in a single-sided game, Player 1 is not allowed to changes the values of the counters, but only the state.

Partial Configurations. A *partial counter valuation* $\vartheta : \mathcal{C} \mapsto \mathbb{Z}$ is a partial function from the set of counters to $\mathbb{Z}$. We also write $\vartheta(c) = \perp$ if $c \notin dom(\vartheta)$. A *partial configuration*

γ is a pair $\langle q, \vartheta \rangle$ where $q \in Q$ is a state and ϑ is a partial counter valuation. We will also consider *nonnegative partial configurations*, where the partial counter valuation takes values in $\mathbb{N}$ instead of $\mathbb{Z}$. We define $\texttt{state}(\gamma) := q$, $\texttt{val}(\gamma) := \vartheta$, and $\kappa(\gamma) := \kappa(\texttt{state}(\gamma))$. We generalize assignments from counter valuations to configurations by defining $\langle q, \vartheta \rangle [c \leftarrow x] = \langle q, \vartheta[c \leftarrow x] \rangle$. Similarly, for a configuration γ and $c \in \mathcal{C}$ we let $\gamma(c) := \texttt{val}(\gamma)(c)$, $dom(\gamma) := dom(\texttt{val}(\gamma))$ and $|\gamma| := |dom(\gamma)|$. For a set of counters $C \subseteq \mathcal{C}$, we define $\Theta^C := \{\gamma | \, dom(\gamma) = C\}$, i.e., it is the set of configurations in which the defined counters are exactly those in C. We use Γ^C to denote the restriction of Θ^C to nonnegative partial configurations. We partition Θ^C into two sets Θ_0^C (configurations belonging to Player 0) and Θ_1^C (configurations belonging to Player 1), such that $\gamma \in \Theta_i^C$ iff $dom(\gamma) = C$ and $\texttt{state}(\gamma) \in Q_i$ for $i \in \{0, 1\}$. A configuration is *concrete* if $dom(\gamma) = \mathcal{C}$, i.e., $\gamma \in \Theta^{\mathcal{C}}$ (the counter valuation $\texttt{val}(\gamma)$ is defined for all counters); and it is *abstract* if $dom(\gamma) = \emptyset$, i.e., $\gamma \in \Theta^{\emptyset}$ (the counter valuation $\texttt{val}(\gamma)$ is not defined for any counter). In the sequel, we occasionally write Θ instead of $\Theta^{\mathcal{C}}$, and Θ_i instead of $\Theta_i^{\mathcal{C}}$ for $i \in \{0, 1\}$. The same notations are defined over nonnegative partial configurations with Γ, and Γ_i^C and Γ_i for $i \in \{0, 1\}$. For a nonnegative partial configuration $\gamma = \langle q, \vartheta \rangle \in \Gamma$, and set of counters $C \subseteq \mathcal{C}$ we define the restriction of γ to C by $\gamma' = \gamma | C = \langle q', \vartheta' \rangle$ where $q' = q$ and $\vartheta'(c) = \vartheta(c)$ if $c \in C$ and $\vartheta'(c) = \perp$ otherwise.

Energy Semantics. Under the energy semantics, an integer game induces a transition relation $\longrightarrow_{\mathcal{E}}$ on the set of partial configurations as follows. For partial configurations $\gamma_1 = \langle q_1, \vartheta_1 \rangle$, $\gamma_2 = \langle q_2, \vartheta_2 \rangle$, and a transition $t = \langle q_1, op, q_2 \rangle \in T$, we have $\gamma_1 \xrightarrow{t}_{\mathcal{E}} \gamma_2$ if one of the following three cases is satisfied: (i) $op = c$++ and either both $\vartheta_1(c) = \perp$ and $\vartheta_2(c) = \perp$ or $\vartheta_1(c) \neq \perp$, $\vartheta_2(c) \neq \perp$ and $\vartheta_2 = \vartheta_1[c \leftarrow \vartheta_1(c) + 1]$; (ii) $op = c$--, and either both $\vartheta_1(c) = \perp$ and $\vartheta_2(c) = \perp$ or $\vartheta_1(c) \neq \perp$, $\vartheta_2(c) \neq \perp$ and $\vartheta_2 = \vartheta_1[c \leftarrow \vartheta_1(c) - 1]$; (iii) $op = nop$ and $\vartheta_2 = \vartheta_1$. Hence we apply the operation of the transition only if the relevant counter value is defined (otherwise, the counter remains undefined). Notice that, for a partial configuration γ_1 and a transition t, there is at most one γ_2 with $\gamma_1 \xrightarrow{t}_{\mathcal{E}} \gamma_2$. If such a γ_2 exists, we define $t(\gamma_1) := \gamma_2$; otherwise we define $t(\gamma_1) := \perp$. We say that t is *enabled* at γ if $t(\gamma) \neq \perp$. We observe that, in the case of energy semantics, t is not enabled only if $\texttt{state}(\gamma) \neq \texttt{source}(t)$.

VASS Semantics. The difference between the energy and VASS semantics is that counters in the case of VASS range over the natural numbers (rather than the integers), i.e. the VASS semantics will be interpreted over nonnegative partial configurations. Thus, the transition relation $\longrightarrow_{\mathcal{V}}$ induced by an integer game $\mathcal{G} = \langle Q, T, \kappa \rangle$ under the VASS semantics differs from the one induced by the energy semantics in the sense that counters are not allowed to assume negative values. Hence $\longrightarrow_{\mathcal{V}}$ is the restriction of $\longrightarrow_{\mathcal{E}}$ to nonnegative partial configurations. Here, a transition $t = \langle q_1, c\text{--}, q_2 \rangle \in T$ is enabled from $\gamma_1 = \langle q_1, \vartheta_1 \rangle$ only if $\vartheta_1(c) > 0$ or $\vartheta_1(c) = \perp$. We assume without restriction that at least one transition is enabled from each partial configuration (i.e., there are no deadlocks) in the VASS semantics (and hence also in the energy semantics). Below, we use $\texttt{sem} \in \{\mathcal{E}, \mathcal{V}\}$ to distinguish the energy and VASS semantics.

Runs. A *run* ρ in semantics $\texttt{sem}$ is an infinite sequence $\gamma_0 \xrightarrow{t_1}_{\texttt{sem}} \gamma_1 \xrightarrow{t_2}_{\texttt{sem}} \cdots$ of transitions between concrete configurations. A *path* π in $\texttt{sem}$ is a finite sequence $\gamma_0 \xrightarrow{t_1}_{\texttt{sem}} \gamma_1 \xrightarrow{t_2}_{\texttt{sem}} \cdots \gamma_n$ of transitions between concrete configurations. We say that

ρ (resp. π) is a γ-run (resp. γ-path) if $\gamma_0 = \gamma$. We define $\rho(i) := \gamma_i$ and $\pi(i) := \gamma_i$. We assume familiarity with the logic LTL. For an LTL formula ϕ we write $\rho \models_{\mathcal{G}} \phi$ to denote that the run ρ in $\mathcal{G}$ satisfies ϕ. For instance, given a set β of concrete configurations, we write $\rho \models_{\mathcal{G}} \Diamond\beta$ to denote that there is an i with $\gamma_i \in \beta$ (i.e., a member of β eventually occurs along ρ); and write $\rho \models_{\mathcal{G}} \Box\Diamond\beta$ to denote that there are infinitely many i with $\gamma_i \in \beta$ (i.e., members of β occur infinitely often along ρ).

Strategies. A *strategy* of Player $i \in \{0,1\}$ in sem (or simply an i-strategy in sem) σ_i is a mapping that assigns to each path $\pi = \gamma_0 \xrightarrow{t_1}_{\texttt{sem}} \gamma_1 \xrightarrow{t_2}_{\texttt{sem}} \cdots \gamma_n$ with $\texttt{state}(\gamma_n) \in Q_i$, a transition $t = \sigma_i(\pi)$ with $t(\gamma_n) \neq \bot$ in sem. We use $\Sigma_i^{\texttt{sem}}$ to denote the sets of i-strategies in sem. Given a concrete configuration γ, $\sigma_0 \in \Sigma_0^{\texttt{sem}}$, and $\sigma_1 \in \Sigma_1^{\texttt{sem}}$, we define $\texttt{run}(\gamma, \sigma_0, \sigma_1)$ to be the unique run $\gamma_0 \xrightarrow{t_1}_{\texttt{sem}} \gamma_1 \xrightarrow{t_2}_{\texttt{sem}} \cdots$ such that (i) $\gamma_0 = \gamma$, (ii) $t_{i+1} = \sigma_0(\gamma_0 \xrightarrow{t_1}_{\texttt{sem}} \gamma_1 \xrightarrow{t_2}_{\texttt{sem}} \cdots \gamma_i)$ if $\texttt{state}(\gamma_i) \in Q_0$, and (iii) $t_{i+1} = \sigma_1(\gamma_0 \xrightarrow{t_1}_{\texttt{sem}} \gamma_1 \xrightarrow{t_2}_{\texttt{sem}} \cdots \gamma_i)$ if $\texttt{state}(\gamma_i) \in Q_1$. For $\sigma_i \in \Sigma_i^{\texttt{sem}}$, we write $[i, \sigma_i, \texttt{sem}] : \gamma \models_{\mathcal{G}} \phi$ to denote that $\texttt{run}(\gamma, \sigma_i, \sigma_{1-i}) \models_{\mathcal{G}} \phi$ for all $\sigma_{1-i} \in \Sigma_{1-i}^{\texttt{sem}}$. In other words, Player i has a *winning strategy*, namely σ_i, which ensures that ϕ will be satisfied regardless of the strategy chosen by Player $1-i$. We write $[i, \texttt{sem}] : \gamma \models_{\mathcal{G}} \phi$ to denote that $[i, \sigma_i, \texttt{sem}] : \gamma \models_{\mathcal{G}} \phi$ for some $\sigma_i \in \Sigma_i^{\texttt{sem}}$.

Instantiations. Two nonnegative partial configurations γ_1, γ_2 are said to be *disjoint* if (i) $\texttt{state}(\gamma_1) = \texttt{state}(\gamma_2)$, and (ii) $dom(\gamma_1) \cap dom(\gamma_2) = \emptyset$ (notice that we require the states to be equal). For a set of counters $C \subseteq \mathcal{C}$, and disjoint partial configurations γ_1, γ_2, we say that γ_2 is a *C-complement* of γ_1 if $dom(\gamma_1) \cup dom(\gamma_2) = C$, i.e., $dom(\gamma_1)$ and $dom(\gamma_2)$ form a partitioning of the set C. If γ_1 and γ_2 are disjoint then we define $\gamma_1 \oplus \gamma_2$ to be the nonnegative partial configuration $\gamma := \langle q, \vartheta \rangle$ such that $q := \texttt{state}(\gamma_1) = \texttt{state}(\gamma_2)$, $\vartheta(c) := \texttt{val}(\gamma_1)(c)$ if $\texttt{val}(\gamma_1)(c) \neq \bot$, $\vartheta(c) := \texttt{val}(\gamma_2)(c)$ if $\texttt{val}(\gamma_2)(c) \neq \bot$, and $\vartheta(c) := \bot$ if both $\texttt{val}(\gamma_1)(c) = \bot$ and $\texttt{val}(\gamma_2)(c) = \bot$. In such a case, we say that γ is a *C-instantiation* of γ_1. For a nonnegative partial configuration γ we write $[\![\gamma]\!]_C$ to denote the set of C-instantiations of γ. We will consider the special case where $C = \mathcal{C}$. In particular, we say that γ_2 is a *complement* of γ_1 if γ_2 is a $\mathcal{C}$-complement of γ_1, i.e., $\texttt{state}(\gamma_2) = \texttt{state}(\gamma_1)$ and $dom(\gamma_1) = \mathcal{C} - dom(\gamma_2)$. We use $\bar{\gamma}$ to denote the set of complements of γ. If $\gamma_2 \in \overline{\gamma_1}$, we say that $\gamma = \gamma_1 \oplus \gamma_2$ is an *instantiation* of γ_1. Notice that γ in such a case is concrete. For a nonnegative partial configuration γ we write $[\![\gamma]\!]$ to denote the set of instantiations of γ. We observe that $[\![\gamma]\!] = [\![\gamma]\!]_{\mathcal{C}}$ and that $[\![\gamma]\!] = \{\gamma\}$ for any concrete nonnegative configuration γ.

Ordering. For nonnegative partial configurations γ_1, γ_2, we write $\gamma_1 \sim \gamma_2$ if $\texttt{state}(\gamma_1) = \texttt{state}(\gamma_2)$ and $dom(\gamma_1) = dom(\gamma_2)$. We write $\gamma_1 \sqsubseteq \gamma_2$ if $\texttt{state}(\gamma_1) = \texttt{state}(\gamma_2)$ and $dom(\gamma_1) \subseteq dom(\gamma_2)$. For nonnegative partial configurations $\gamma_1 \sim \gamma_2$, we write $\gamma_1 \preceq \gamma_2$ to denote that $\texttt{state}(\gamma_1) = \texttt{state}(\gamma_2)$ and $\texttt{val}(\gamma_1)(c) \leq \texttt{val}(\gamma_2)(c)$ for all $c \in dom(\gamma_1) = dom(\gamma_2)$. For a nonnegative partial configuration γ, we define $\gamma\!\uparrow := \{\gamma' \mid \gamma \preceq \gamma'\}$ to be the upward closure of γ, and define $\gamma\!\downarrow := \{\gamma' \mid \gamma' \preceq \gamma\}$ to be the downward closure of γ. Notice that $\gamma\!\uparrow = \gamma\!\downarrow = \{\gamma\}$ for any abstract configuration γ. For a set $\beta \subseteq \Gamma^C$ of nonnegative partial configurations, let $\beta\!\uparrow := \cup_{\gamma \in \beta} \gamma\!\uparrow$. We say that β is upward-closed if $\beta\!\uparrow = \beta$. For an upward-closed set $\beta \subseteq \Gamma^C$, we use $min(\beta)$ to denote the (by Dickson's Lemma unique and finite) set of minimal elements of β.

Winning Sets of Partial Configurations. For a nonnegative partial configuration γ, we write $[i,\mathtt{sem}] : \gamma \models_{\mathcal{G}} \phi$ to denote that $\exists\gamma' \in [\![\gamma]\!].[i,\mathtt{sem}] : \gamma' \models_{\mathcal{G}} \phi$, i.e., Player i is winning from some instantiation γ' of γ. For a set $C \subseteq \mathcal{C}$ of counters, we define $\mathcal{W}[\mathcal{G},\mathtt{sem},i,C](\phi) := \left\{\gamma \in \Gamma^C \mid [\mathtt{sem},i] : \gamma \models_{\mathcal{G}} \phi\right\}$. If $\mathcal{W}[\mathcal{G},\mathtt{sem},i,C](\phi)$ is upward-closed, we define the *Pareto frontier* as $\mathtt{Pareto}[\mathcal{G},\mathtt{sem},i,C](\phi) := \mathit{min}\left(\mathcal{W}[\mathcal{G},\mathtt{sem},i,C](\phi)\right)$.

Properties. We show some useful properties of the ordering on nonnegative partial configurations. Note that for nonnegative partial configurations, we will not make distinctions between the energy semantics and the VASS semantics; this is due to the fact that in nonnegative partial configurations and in their instantiations we only consider positive values for the counters. For the energy semantics, as we shall see, this will not be a problem since we will consider winning runs where the counter never goes below 0. We now show *monotonicity* and (under some conditions) "reverse monotonicity" of the transition relation wrt. $\preceq$. We write $\gamma_1 \longrightarrow_{\mathtt{sem}} \gamma_2$ if there exists t such that $\gamma_1 \xrightarrow{t}_{\mathtt{sem}} \gamma_2$.

Lemma 1. *Let γ_1, γ_2, and γ_3 be nonnegative partial configurations. If (i) $\gamma_1 \longrightarrow_{\mathcal{V}} \gamma_2$, and (ii) $\gamma_1 \preceq \gamma_3$, then there is a γ_4 such that $\gamma_3 \longrightarrow_{\mathcal{V}} \gamma_4$ and $\gamma_2 \preceq \gamma_4$. Furthermore, if (i) $\gamma_1 \longrightarrow_{\mathcal{V}} \gamma_2$, and (ii) $\gamma_3 \preceq \gamma_1$, and (iii) $\mathcal{G}$ is single-sided and (iv) $\gamma_1 \in \Gamma_1$, then there is a γ_4 such that $\gamma_3 \longrightarrow_{\mathcal{V}} \gamma_4$ and $\gamma_4 \preceq \gamma_2$.*

We consider a version of the *Valk-Jantzen* lemma [15], expressed in our terminology.

Lemma 2. *[15] Let $C \subseteq \mathcal{C}$ and let $U \subseteq \Gamma^C$ be upward-closed. Then, $\mathit{min}(U)$ is computable if and only if, for any nonnegative partial configuration γ with $\mathit{dom}(\gamma) \subseteq C$, we can decide whether $[\![\gamma]\!]_C \cap U \neq \emptyset$.*

3 Game Problems

Here we consider the parity winning condition for the integer games defined in the previous section. First we establish a correspondence between the VASS semantics when the underlying integer game is single-sided, and the energy semantics in the general case. We will show how instances of the single-sided VASS parity game can be reduced to the energy parity game, and vice-versa. Figure 1 depicts a summary of our results. For either semantics, an instance of the problem consists of an integer game $\mathcal{G}$ and a partial configuration γ. For a given set of counters $C \subseteq \mathcal{C}$, we will consider the *C-version* of the problem where we assume that $\mathit{dom}(\gamma) = C$. In particular, we will consider two special cases: (i) the *abstract* version in which we assume that γ is abstract (i.e., $\mathit{dom}(\gamma) = \emptyset$), and (ii) the *concrete* version in which we assume that γ is concrete (i.e., $\mathit{dom}(\gamma) = \mathcal{C}$). The abstract version of a problem corresponds to the *unknown initial credit problem* [6, 7], while the concrete one corresponds to deciding if a given initial credit is sufficient or, more generally, computing the Pareto frontier (left open in [6, 7]).

Winning Conditions. Assume an integer game $\mathcal{G} = \langle Q, T, \kappa\rangle$ where $\kappa : Q \mapsto \{0,1,2,\ldots,k\}$. For a partial configuration γ and $i : 0 \leq i \leq k$, the relation $\gamma \models_{\mathcal{G}} (\mathtt{color} = i)$ holds if $\kappa(\mathtt{state}(\gamma)) = i$. The formula simply checks the color of the state of γ. The formula $\gamma \models_{\mathcal{G}} \overline{\mathtt{neg}}$ holds if $\mathtt{val}(\gamma)(c) \geq 0$ for all $c \in \mathit{dom}(\gamma)$.

The formula states that the values of all counters are nonnegative in γ. For $i : 0 \leq i \leq k$, the predicate $even(i)$ holds if i is even. Define the path formula $\texttt{Parity} := \bigvee_{(0\leq i\leq k)\wedge even(i)} \left((\Box\Diamond(\texttt{color} = i)) \wedge \left(\bigwedge_{i<j\leq k} \Diamond\Box\neg(\texttt{color} = j)\right)\right)$. The formula states that the highest color that appears infinitely often along the path is even.

Energy Parity. Given an integer game $\mathcal{G}$ and a partial configuration γ, we ask whether $[0,\mathcal{E}] : \gamma \models_{\mathcal{G}} \texttt{Parity} \wedge (\Box\overline{\texttt{neg}})$, i.e., whether Player 0 can force a run in the energy semantics where the parity condition is satisfied and at the same time the counters remain nonnegative. The abstract version of this problem is equivalent to the unknown initial credit problem in classical energy parity games [6, 7], since it amounts to asking for the *existence* of a threshold for the initial counter values from which Player 0 can win. The nonnegativity objective $(\Box\overline{\texttt{neg}})$ justifies our restriction to nonnegative partial configurations in our definition of the instantiations and hence of the winning sets.

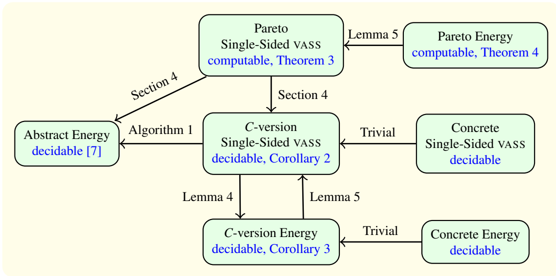

Fig. 1. Problems considered in the paper and their relations. For each property, we state the lemma that show its decidability/computability. The arrows show the reductions of problem instances that we show in the paper.

Theorem 1. *[7] The abstract energy parity problem is decidable.*

The winning set $\mathcal{W}[\mathcal{G},\mathcal{E},0,C](\texttt{Parity} \wedge \Box\overline{\texttt{neg}})$ is upward-closed for $C \subseteq \mathcal{C}$. Intuitively, if Player 0 can win the game with a certain value for the counters, then any higher value for these counters also allows him to win the game with the same strategy. This is because both the possible moves of Player 1 and the colors of configurations depend only on the control-states.

Lemma 3. *For any $C \subseteq \mathcal{C}$, the set $\mathcal{W}[\mathcal{G},\mathcal{E},0,C](\texttt{Parity} \wedge \Box\overline{\texttt{neg}})$ is upward-closed.*

Since this winning set is upward-closed, it follows from Dickson's Lemma that it has finitely many minimal elements. These minimal elements describe the Pareto frontier of the minimal initial credit needed to win the game. In the sequel we will show how to compute this set $\texttt{Pareto}[\mathcal{G},\mathcal{E},0,C](\texttt{Parity} \wedge \Box\overline{\texttt{neg}}) := min\left(\mathcal{W}[\mathcal{G},\mathcal{E},0,C](\texttt{Parity} \wedge \Box\overline{\texttt{neg}})\right)$; cf. Theorem 4.

VASS Parity. Given an integer game $\mathcal{G}$ and a nonnegative partial configuration γ, we ask whether $[0,\mathcal{V}] : \gamma \models_{\mathcal{G}} \texttt{Parity}$, i.e., whether Player 0 can force a run in the VASS semantics where the parity condition is satisfied. (The condition $\Box\overline{\texttt{neg}}$ is always trivially satisfied in VASS.) In general, this problem is undecidable as shown in [1], even for simple coverability objectives instead of parity objectives.

Theorem 2. *[1] The VASS Parity Problem is undecidable.*

We will show that decidability of the VASS parity problem is regained under the assumption that $\mathcal{G}$ is single-sided. In [14] it was already shown that, for a single-sided VASS game with reachability objectives, it is possible to compute the set of winning configurations. However, the proof for parity objectives is much more involved.

Correspondence of Single-Sided VASS *Games and Energy Games.* We show that single-sided VASS parity games can be reduced to energy parity games, and vice-versa. The following lemma shows the direction from VASS to energy.

Lemma 4. *Let $\mathcal{G}$ be a single-sided integer game and let γ be a nonnegative partial configuration. Then $[0,\mathcal{V}] : \gamma \models_{\mathcal{G}} \texttt{Parity}$ iff $[0,\mathcal{E}] : \gamma \models_{\mathcal{G}} \texttt{Parity} \wedge \Box\overline{\texttt{neg}}$.*

Hence for a single-sided $\mathcal{G}$ and any set $C \subseteq \mathcal{C}$, we have $\mathcal{W}[\mathcal{G},\mathcal{V},0,C](\texttt{Parity}) = \mathcal{W}[\mathcal{G},\mathcal{E},0,C](\texttt{Parity} \wedge \Box\overline{\texttt{neg}})$. Consequently, using Lemma 3 and Theorem 1, we obtain the following corollary.

Corollary 1. *Let $\mathcal{G}$ be single-sided and $C \subseteq \mathcal{C}$.*

1. *$\mathcal{W}[\mathcal{G},\mathcal{V},0,C](\texttt{Parity})$ is upward-closed.*
2. *The C-version single-sided* VASS *parity problem is reducible to the C-version energy parity problem.*
3. *The* abstract *single-sided* VASS *parity problem (i.e., where $C = \emptyset$) is decidable.*

The following lemma shows the reverse reduction from energy parity games to single-sided VASS parity games.

Lemma 5. *Given an integer game $\mathcal{G} = \langle Q, T, \kappa\rangle$, one can construct a single-sided integer game $\mathcal{G}' = \langle Q', T', \kappa'\rangle$ with $Q \subseteq Q'$ such that $[0,\mathcal{E}] : \gamma \models_{\mathcal{G}} \texttt{Parity} \wedge \Box\overline{\texttt{neg}}$ iff $[0,\mathcal{V}] : \gamma \models_{\mathcal{G}'} \texttt{Parity}$ for every nonnegative partial configuration γ of $\mathcal{G}$.*

Proof sketch. Since $\mathcal{G}'$ needs to be single-sided, Player 1 cannot change the counters. Thus the construction forces Player 0 to simulate the moves of Player 1. Whenever a counter drops below zero in $\mathcal{G}$ (and thus Player 0 loses), Player 0 cannot perform this simulation in $\mathcal{G}'$ and is forced to go to a losing state instead. $\square$

Computability Results. The following theorem (shown in Section 4) states our main computability result. For single-sided VASS parity games, the minimal elements of the winning set $\mathcal{W}[\mathcal{G},\mathcal{V},0,C](\texttt{Parity})$ (i.e., the Pareto frontier) are computable.

Theorem 3. *If $\mathcal{G}$ is single-sided then $\texttt{Pareto}[\mathcal{G},\mathcal{V},0,C](\texttt{Parity})$ is computable.*

In particular, this implies decidability.

Corollary 2. *For any set of counters $C \subseteq \mathcal{C}$, the C-version single-sided* VASS *parity problem is decidable.*

From Theorem 3 and Lemma 5 we obtain the computability of the Pareto frontier of the minimal initial credit needed to win general energy parity games.

Theorem 4. *$\texttt{Pareto}[\mathcal{G},\mathcal{E},0,C](\texttt{Parity} \wedge \Box\overline{\texttt{neg}})$ is computable for any game $\mathcal{G}$.*

Corollary 3. *The C-version energy parity problem is decidable.*

4 Solving Single-Sided VASS Parity Games (Proof of Theorem 3)

Consider a single-sided integer game $\mathcal{G} = \langle Q, T, \kappa \rangle$ and a set $C \subseteq \mathcal{C}$ of counters. We will show how to compute the set $\texttt{Pareto}[\mathcal{G}, \mathcal{V}, 0, C](\texttt{Parity})$. We reduce the problem of computing the Pareto frontier in the single-sided VASS parity game to solving the abstract energy parity game problem, which is decidable by Theorem 1.

We use induction on $k = |C|$. As we shall see, the base case is straightforward. We perform the induction step in two phases. First we show that, under the induction hypothesis, we can reduce the problem of computing the Pareto frontier to the problem of solving the C-version single-sided VASS parity problem (i.e., we need only to consider individual nonnegative partial configurations in Γ^C). In the second phase, we introduce an algorithm that translates the latter problem to the abstract energy parity problem.

Base Case. Assume that $C = \emptyset$. In this case we are considering the abstract single-sided VASS parity problem. Recall that $\gamma \uparrow = \{\gamma\}$ for any γ with $dom(\gamma) = \emptyset$. Since $C = \emptyset$, it follows that $\texttt{Pareto}[\mathcal{G}, \mathcal{V}, 0, C](\texttt{Parity}) = \big\{\gamma | \, (dom(\gamma) = \emptyset) \wedge \big([0, \mathcal{V}] : \gamma \models_{\mathcal{G}} \texttt{Parity}\big)\big\}$. In other words, computing the Pareto frontier in this case reduces to solving the abstract single-sided VASS parity problem, which is decidable by Corollary 1.

From Pareto Sets to VASS *Parity.* Assuming the induction hypothesis, we reduce the problem of computing the set $\texttt{Pareto}[\mathcal{G}, \mathcal{V}, 0, C](\texttt{Parity})$ to the C-version single-sided VASS parity problem, i.e., the problem of checking whether $[0, \mathcal{V}] : \gamma \models_{\mathcal{G}} \texttt{Parity}$ for some $\gamma \in \Gamma^C$ when the underlying integer game is single-sided. To do that, we will instantiate the Valk-Jantzen lemma as follows. We instantiate $U \subseteq \Gamma^C$ in Lemma 2 to be $\mathcal{W}[\mathcal{G}, \mathcal{V}, 0, C](\texttt{Parity})$ (this set is upward-closed by Corollary 1 since $\mathcal{G}$ is single-sided). Take any nonnegative partial configuration γ with $dom(\gamma) \subseteq C$. We consider two cases. First, if $dom(\gamma) = C$, then we are dealing with the C-version single-sided VASS parity game which will show how to solve in the sequel. Second, consider the case where $dom(\gamma) = C' \subset C$. By the induction hypothesis, we can compute the (finite) set $\texttt{Pareto}[\mathcal{G}, \mathcal{V}, 0, C'](\texttt{Parity}) = min(\mathcal{W}[\mathcal{G}, \mathcal{V}, 0, C'](\texttt{Parity}))$. Then to solve this case, we use the following lemma.

Lemma 6. *For all nonnegative partial configurations γ such that $dom(\gamma) = C' \subset C$, we have $[\![\gamma]\!]_C \cap \mathcal{W}[\mathcal{G}, \mathcal{V}, 0, C](\texttt{Parity}) \neq \emptyset$ iff $\gamma \in \mathcal{W}[\mathcal{G}, \mathcal{V}, 0, C'](\texttt{Parity})$.*

Hence checking $[\![\gamma]\!]_C \cap \mathcal{W}[\mathcal{G}, \mathcal{V}, 0, C](\texttt{Parity}) \neq \emptyset$ amounts to simply comparing γ with the elements of the finite set $\texttt{Pareto}[\mathcal{G}, \mathcal{V}, 0, C'](\texttt{Parity})$, because $\mathcal{W}[\mathcal{G}, \mathcal{V}, 0, C'](\texttt{Parity})$ is upward-closed by Corollary 1.

From VASS *Parity to Abstract Energy Parity.* We introduce an algorithm that uses the induction hypothesis to translate an instance of the C-version single-sided VASS parity problem to an equivalent instance of the abstract energy parity problem.

The following definition and lemma formalize some consequences of the induction hypothesis. First we define a relation that allows us to directly classify some nonnegative partial configurations as winning for Player 1 (resp. Player 0).

Definition 1. *Consider a nonnegative partial configuration γ and a set of nonnegative partial configurations β. We write $\beta \lhd \gamma$ if: (i) for each $\hat{\gamma} \in \beta$, $dom(\hat{\gamma}) \subseteq C$ and $|\gamma| = |\hat{\gamma}| + 1$, and (ii) for each $c \in dom(\gamma)$ there is a $\hat{\gamma} \in \beta$ such that $\hat{\gamma} \preceq \gamma[c \leftarrow \perp]$.*

Lemma 7. *Let* $\beta = \bigcup_{C' \subseteq C, |C'|=|C|-1} \texttt{Pareto}[\mathcal{G}, \mathcal{V}, 0, C'](\texttt{Parity})$ *be the Pareto frontier of minimal Player* 0 *winning nonnegative partial configurations with one counter in C undefined. Let* $\{c_i, \ldots, c_j\} = \mathcal{C} - C$ *be the counters outside C.*

1. *For every* $\hat{\gamma} \in \beta$ *with* $\{c\} = C - dom(\hat{\gamma})$ *there exists a minimal finite number* $v(\hat{\gamma})$ *s.t.* $[\![\hat{\gamma}[c \leftarrow v(\hat{\gamma})]]\!] \cap \mathcal{W}[\mathcal{G}, \mathcal{V}, 0, \mathcal{C}](\texttt{Parity}) \neq \emptyset$.
2. *For every* $\hat{\gamma} \in \beta$ *there is a number* $u(\hat{\gamma})$ *s.t.* $\hat{\gamma}[c \leftarrow v(\hat{\gamma})][c_i \leftarrow u(\hat{\gamma}), \ldots, c_j \leftarrow u(\hat{\gamma})] \in \mathcal{W}[\mathcal{G}, \mathcal{V}, 0, \mathcal{C}](\texttt{Parity})$*, i.e., assigning value* $u(\hat{\gamma})$ *to counters outside C is sufficient to make the nonnegative configuration winning for Player* 0.
3. *If* $\gamma \in \Gamma^C$ *is a Player* 0 *winning nonnegative partial configuration, i.e.,* $[\![\gamma]\!] \cap \mathcal{W}[\mathcal{G}, \mathcal{V}, 0, \mathcal{C}](\texttt{Parity}) \neq \emptyset$*, then* $\beta \lhd \gamma$.

The third part of this lemma implies that if $\neg(\beta \lhd \gamma)$ then we can directly conclude that γ is not winning for Player 0 (and thus winning for Player 1) in the parity game.

Now we are ready to present the algorithm (Algorithm 1).

Input and output of the algorithm. The algorithm inputs a single-sided integer game $\mathcal{G} = \langle Q, T, \kappa \rangle$, and a nonnegative partial configuration γ where $dom(\gamma) = C$. To check whether $[0, \mathcal{V}] : \gamma \models_{\mathcal{G}} \texttt{Parity}$, it constructs an instance of the abstract energy parity problem. This instance is defined by a new integer game $\mathcal{G}^{out} = \langle Q_{out}, T_{out}, \kappa_{out} \rangle$ with counters in $\mathcal{C} - C$, and a nonnegative partial configuration γ^{out}. Since we are considering the abstract version of the problem, the configuration γ^{out} is of the form $\gamma^{out} = \langle q^{out}, \vartheta_{out} \rangle$ where $dom(\vartheta_{out}) = \emptyset$. The latter property means that γ^{out} is uniquely determined by the state q^{out} (all counter values are undefined). Lemma 9 relates $\mathcal{G}$ with the newly constructed $\mathcal{G}^{out}$.

Operation of the algorithm. The algorithm performs a forward analysis similar to the classical Karp-Miller algorithm for Petri nets. We start with a given nonnegative partial configuration, explore its successors, create loops when previously visited configurations are repeated and define a special operation for the case when configurations strictly increase. The algorithm builds the graph of the game $\mathcal{G}^{out}$ successively (i.e., the set of states Q^{out}, the set of transitions T^{out}, and the coloring of states κ). Additionally, for bookkeeping purposes inside the algorithm and for reasoning about the correctness of the algorithm, we define a labeling function λ on the set of states and transitions in $\mathcal{G}^{out}$ such that each state in $\mathcal{G}^{out}$ is labeled by a nonnegative partial configuration in Γ^C, and each transition in $\mathcal{G}^{out}$ is labeled by a transition in $\mathcal{G}$.

The algorithm first computes the Pareto frontier $\texttt{Pareto}[\mathcal{G}, \mathcal{V}, 0, C'](\texttt{Parity})$ for all counter sets $C' \subseteq \mathcal{C}$ with $|C'| = |C| - 1$. This is possible by the induction hypothesis. It stores the union of all these sets in β (line 1). At line 2, the algorithms initializes the set of transitions T^{out} to be empty, creates the first state q^{out}, defines its coloring to be the same as that of the state of the input nonnegative partial configuration γ, labels it by γ, and then adds it to the set of states Q^{out}. At line 4 it adds q^{out} to the set of states of Player 0 or Player 1 (depending on where γ belongs), and at line 6 it adds q^{out} to the set `ToExplore`. The latter contains the set of states that have been created but not yet analyzed by the algorithm.

After the initialization phase, the algorithm starts iterating the *while*-loop starting at line 7. During each iteration, it picks and removes a new state q from the set `ToExplore`

Algorithm 1: Building an instance of the abstract energy parity problem.

Input: $\mathcal{G} = \langle Q, T, \kappa \rangle$: Single-Sided Integer Game; $\gamma \in \Gamma^C$ with $|C| = k > 0$.

Output: $\mathcal{G}^{out} = \langle Q^{out}, T^{out}, \kappa^{out} \rangle$: integer game;
$q^{out} \in Q^{out}$; $\gamma^{out} = \langle q^{out}, \vartheta_{out} \rangle$ where $dom(\vartheta_{out}) = \emptyset$; $\lambda : Q_{out} \cup T_{out} \mapsto \Gamma^C \cup T$

1 $\beta \leftarrow \bigcup_{(C' \subseteq C) \wedge |C'| = |C|-1} \texttt{Pareto}[\mathcal{G}, \mathcal{V}, 0, \mathcal{C}'](\texttt{Parity})$;

2 $T^{out} \leftarrow \emptyset$; $\texttt{new}(q^{out})$; $\kappa(q^{out}) \leftarrow \kappa(\gamma)$; $\lambda(q^{out}) \leftarrow \gamma$; $Q_{out} \leftarrow \{q^{out}\}$;

3 **if** $\lambda(q^{out}) \in \Gamma_0$ **then** $Q_0^{out} \leftarrow \{q^{out}\}$; $Q_1^{out} \leftarrow \emptyset$;

4 **else** $Q_1^{out} \leftarrow \{q^{out}\}$; $Q_0^{out} \leftarrow \emptyset$;

5 ;

6 $\texttt{ToExplore} \leftarrow \{q^{out}\}$;

7 **while** $\texttt{ToExplore} \neq \emptyset$ **do**

8 Pick and remove a $q \in \texttt{ToExplore}$;

9 **if** $\neg(\beta \lhd \lambda(q))$ **then**

10 $\kappa^{out}(q) \leftarrow 1$; $T^{out} \leftarrow T^{out} \cup \{\langle q, nop, q \rangle\}$

11 **else if** $\exists q'. (q', q) \in (T^{out})^* \wedge (\lambda(q') \prec \lambda(q))$ **then**

12 $\kappa^{out}(q) \leftarrow 0$; $T^{out} \leftarrow T^{out} \cup \{\langle q, nop, q \rangle\}$

13 **else for** *each* $t \in T$ *with* $t(\lambda(q)) \neq \bot$ **do**

14 **if** $\exists q'. (q', q) \in (T^{out})^* . \lambda(q') = t(\lambda(q))$ **then**

15 $T^{out} \leftarrow T^{out} \cup \{\langle q, \texttt{op}(t), q' \rangle\}$; $\lambda(\langle q, \texttt{op}(t), q' \rangle) \leftarrow t$

16 **else**

17 $\texttt{new}(q')$; $\kappa(q') \leftarrow \kappa(t(\lambda(q)))$; $\lambda(q') \leftarrow t(\lambda(q))$;

18 **if** $\lambda(q') \in \Gamma_0$ **then** $Q_0^{out} \leftarrow Q_0^{out} \cup \{q'\}$;

19 **else** $Q_1^{out} \leftarrow Q_1^{out} \cup \{q'\}$;

20 ;

21 $T^{out} \leftarrow T^{out} \cup \{\langle q, \texttt{op}(t), q' \rangle\}$; $\lambda(\langle q, \texttt{op}(t), q' \rangle) \leftarrow t$;

22 $\texttt{ToExplore} \leftarrow \texttt{ToExplore} \cup \{q'\}$;

23 ;

24 ;

(line 8). First, it checks two special conditions under which the game is made immediately losing (resp. winning) for Player 0.

Condition 1: If $\neg(\beta \lhd \lambda(q))$ (line 9), then we know by Lemma 7 (item 3) that the nonnegative partial configuration $\lambda(q)$ is not winning for Player 0 in $\mathcal{G}$.

Therefore, we make the state q losing for Player 0 in $\mathcal{G}^{out}$. To do that, we change the color of q to 1 (any odd color will do), and add a self-loop to q. Any continuation of a run from q is then losing for Player 0 in $\mathcal{G}^{out}$.

Condition 2: If Condition 1 did not hold then the algorithm checks (at line 11) whether there is a *predecessor* q' of q in $\mathcal{G}^{out}$ with a label $\lambda(q')$ that is *strictly* smaller than the label $\lambda(q)$ of q, i.e., $\lambda(q') \prec \lambda(q)$. (Note that we are not comparing q to arbitrary other states in $\mathcal{G}^{out}$, but only to predecessors.) If that is the case, then the state q is made winning for Player 0 in $\mathcal{G}^{out}$. To do that, we change the color of q to 0 (any even color will do), and add a self-loop to q. The intuition for making q winning for Player 0 is as follows. Since $\lambda(q') \prec \lambda(q)$, the path from $\lambda(q')$ to $\lambda(q)$ increases the value of at least one of the defined counters (those in C), and will not decrease the other counters in C (though it might have a negative effect on the undefined counters in $\mathcal{C} - C$). Thus, if a run in $\mathcal{G}$ iterates this path sufficiently many times, the value of at least one counter in C will be pumped and becomes sufficiently high to allow Player 0 to win the parity game on $\mathcal{G}$, provided that the counters in $\mathcal{C} - C$ are initially instantiated with sufficiently high values. This follows from the property $\beta \lhd \lambda(q)$ and Lemma 7 (items 1 and 2).

If none of the tests for Condition1/Condition2 at lines 9 and 11 succeeds, the algorithm continues expanding the graph of $\mathcal{G}^{out}$ from q. It generates all successors of q by applying each transition $t \in T$ in $\mathcal{G}$ to the label $\lambda(q)$ of q (line 13). If the result $t(\lambda(q))$ is defined then there are two possible cases. The first case occurs if we have previously encountered (and added to Q^{out}) a state q' whose label equals $t(\lambda(q))$ (line 14). Then we add a transition from q back to q' in $\mathcal{G}^{out}$, where the operation of the new transition is the same operation as that of t, and define the label of the new transition to be t. Otherwise (line 17), we create a new state q', label it with the nonnegative configuration $t(\lambda(q))$ and assign it the same color as $t(\lambda(q))$. At line 19 q^{out} is added to the set of states of Player 0 or Player 1 (depending on where γ belongs). We add a new transition between q and q' with the same operation as t. The new transition is labeled with t. Finally, we add the new state q' to the set of states to be explored.

Lemma 8. *Algorithm 1 will always terminate.*

Lemma 8 implies that the integer game $\mathcal{G}^{out}$ is finite (and hence well-defined). The following lemma shows the relation between the input and output games $\mathcal{G}, \mathcal{G}^{out}$.

Lemma 9. $[0, \mathcal{V}] : \gamma \models_{\mathcal{G}} \texttt{Parity}$ *iff* $[0, \mathcal{E}] : \gamma^{out} \models_{\mathcal{G}^{out}} \texttt{Parity} \wedge \Box\overline{\texttt{neg}}$.

Proof sketch. The left to right implication is easy. Given a Player 0 winning strategy in $\mathcal{G}$, one can construct a winning strategy in $\mathcal{G}^{out}$ that uses the same transitions, modulo the labeling function $\lambda()$. The condition $\Box\overline{\texttt{neg}}$ in $\mathcal{G}^{out}$ is satisfied since the configurations in $\mathcal{G}$ are always nonnegative and the parity condition is satisfied since the colors seen in corresponding plays in $\mathcal{G}^{out}$ and $\mathcal{G}$ are the same.

For the right to left implication we consider a Player 0 winning strategy σ_0 in $\mathcal{G}^{out}$ and construct a winning strategy σ_0' in $\mathcal{G}$. The idea is that a play π in $\mathcal{G}$ induces a play π' in $\mathcal{G}^{out}$ by using the same sequence of transitions, but removing all so-called *pumping sequences*, which are subsequences that end in **Condition 2**. Then σ_0' acts on history π like σ_0 on history π'. For a play according to σ_0' there are two cases. Either it will eventually reach a configuration that is sufficiently large (relative to β) such that a winning strategy is known by induction hypothesis. Otherwise it contains only finitely many pumping sequences and an infinite suffix of it coincides with an infinite suffix of a play according to σ_0 in $\mathcal{G}^{out}$. Thus it sees the same colors and satisfies `Parity`. $\Box$

Since γ^{out} is abstract and the abstract energy parity problem is decidable (Theorem 1) we obtain Theorem 3.

The termination proof in Lemma 8 relies on Dickson's Lemma, and thus there is no elementary upper bound on the complexity of Algorithm 1 or on the size of the constructed energy game $\mathcal{G}^{out}$. The algorithm in [4] for the fixed initial credit problem in pure energy games without the parity condition runs in d-exponential time (resp. $(d-1)$-exponential time for offsets encoded in unary) for dimension d, and is thus not elementary either. As noted in [4], the best known lower bound is EXPSPACE hardness, easily obtained via a reduction from the control-state reachability (i.e., coverability) problem for Petri nets.

5 Applications to Other Problems

5.1 Weak simulation preorder between VASS and finite-state systems

Weak simulation preorder [9] is a semantic preorder on the states of labeled transition graphs, which can be characterized by weak simulation games. A configuration of the game is given by a pair of states (q_1, q_0). In every round of the game, Player 1 chooses a labeled step $q_1 \xrightarrow{a} q_1'$ for some label a. Then Player 0 must respond by a move which is either of the form $q_0 \xrightarrow{\tau^* a \tau^*} q_0'$ if $a \neq \tau$, or of the form $q_0 \xrightarrow{\tau^*} q_0'$ if $a = \tau$ (the special label τ is used to model internal transitions). The game continues from configuration (q_1', q_0'). A player wins if the other player cannot move and Player 0 wins every infinite play. One says that q_0 *weakly simulates* q_1 iff Player 0 has a winning strategy in the weak simulation game from (q_1, q_0). States in different transition systems can be compared by putting them side-by-side and considering them as a single transition system.

We use $\langle Q, T, \Sigma, \lambda \rangle$ to denote a labeled VASS where the states and transitions are defined as in Section 2, Σ is a finite set of labels and $\lambda : T \mapsto \Sigma$ assigns labels to transitions.

It was shown in [2] that it is decidable whether a finite-state labeled transition system weakly simulates a labeled VASS. However, the decidability of the reverse direction was open. (The problem is that the weak $\xrightarrow{\tau^* a \tau^*}$ moves in the VASS make the weak simulation game infinitely branching.) We now show that it is also decidable whether a labeled VASS weakly simulates a finite-state labeled transition system. In particular this implies that weak simulation equivalence between a labeled VASS and a finite-state labeled transition system is decidable. This is in contrast to the undecidability of weak *bisimulation* equivalence between VASS and finite-state systems [11].

Theorem 5. *It is decidable whether a labeled* VASS *weakly simulates a finite-state labeled transition system.*

Proof sketch. Given a labeled VASS and a finite-state labeled transition system, one constructs a single-sided VASS parity game s.t. the VASS weakly simulates the finite system iff Player 0 wins the parity game. The idea is to take a controlled product of the finite system and the VASS s.t. every round of the weak simulation game is encoded by a single move of Player 1 followed by an arbitrarily long sequence of moves by Player 0. The move of Player 1 does not change the counters, since it encodes a move in the finite system, and thus the game is single-sided. Moreover, one enforces that every sequence of consecutive moves by Player 0 is finite (though it can be arbitrarily long), by assigning an odd color to Player 0 states and a higher even color to Player 1 states.

5.2 μ-Calculus model checking VASS

While model checking VASS with linear-time temporal logics (like LTL and linear-time μ-calculus) is decidable [8, 10], model checking VASS with most branching-time logics (like EF, EG, CTL and the modal μ-calculus) is undecidable [8]. However, we show that Theorem 3 yields the decidability of model checking single-sided VASS with a guarded fragment of the modal μ-calculus. We consider a VASS $\langle Q, T \rangle$ where the states, transitions and semantics are defined as in Section 2, and reuse the notion of partial configurations and the transition relation defined for the VASS semantics on integer games. We specify properties on such VASS in the positive μ-calculus L_μ^{pos} whose atomic propositions q refer to control-states $q \in Q$ of the input VASS.

The syntax of the positive μ-calculus L_μ^{pos} is given by the following grammar: $\phi ::= q \mid X \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Diamond\phi \mid \Box\phi \mid \mu X.\phi \mid \nu X.\phi$ where $q \in Q$ and X belongs to a countable set of variables $\mathcal{X}$. The semantics of L_μ^{pos} is defined as usual (see [3]). To each closed formula ϕ in L_μ^{pos} (i.e., without free variables) it assigns a subset of concrete configurations $[\![\phi]\!]$.

The model-checking problem of VASS with L_μ^{pos} can then be defined as follows. Given a VASS $\mathcal{S} = \langle Q, T\rangle$, a closed formula ϕ of L_μ^{pos} and an initial configuration γ_0 of $\mathcal{S}$, do we have $\gamma_0 \in [\![\phi]\!]$? If the answer is yes, we will write $\mathcal{S}, \gamma_0 \models \phi$. The more general *global model-checking problem* is to compute the set $[\![\phi]\!]$ of configurations that satisfy the formula. The general unrestricted version of this problem is undecidable.

Theorem 6. *[8] The model-checking problem of* VASS *with* L_μ^{pos} *is undecidable.*

One way to solve the μ-calculus model-checking problem for a given Kripke structure is to encode the problem into a parity game [12]. The idea is to construct a parity game whose states are pairs, where the first component is a state of the structure and the second component is a subformula of the given μ-calculus formula. States of the form $\langle q, \Box\phi\rangle$ or $\langle q, \phi \wedge \psi\rangle$ belong to Player 1 and the remainder belong to Player 0. The colors are assigned to reflect the nesting of least and greatest fixpoints. We can adapt this construction to our context by building an integer game from a formula in L_μ^{pos} and a VASS $\mathcal{S}$, as stated by the next lemma.

Lemma 10. *Let $\mathcal{S}$ be a* VASS, *γ_0 a concrete configuration of $\mathcal{S}$ and ϕ a closed formula in L_μ^{pos}. One can construct an integer game $\mathcal{G}(\mathcal{S}, \phi)$ and an initial concrete configuration γ_0' such that $[0, \mathcal{V}] : \gamma_0' \models_{\mathcal{G}(\mathcal{S},\phi)}$* `Parity` *if and only if $\mathcal{S}, \gamma_0 \models \phi$.*

Now we show that, under certain restrictions on the considered VASS and on the formula from L_μ^{pos}, the constructed integer game $\mathcal{G}(\mathcal{S}, \phi)$ is single-sided, and hence we obtain the decidability of the model-checking problem from Theorem 3. First, we reuse the notion of single-sided games from Section 2 in the context of VASS, by saying that a VASS $\mathcal{S} = \langle Q, T\rangle$ is single-sided iff there is a partition of the set of states Q into two sets Q_0 and Q_1 such that $op = nop$ for all transitions $t \in T$ with $\texttt{source}(t) \in Q_1$. The guarded fragment L_μ^{sv} of L_μ^{pos} for single-sided VASS is then defined by guarding the $\Box$ operator with a predicate that enforces control-states in Q_1. Formally, the syntax of L_μ^{sv} is given by the following grammar: $\phi ::= q \mid X \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Diamond\phi \mid Q_1 \wedge \Box\phi \mid \mu X.\phi \mid \nu X.\phi$, where Q_1 stands for the formula $\bigvee_{q \in Q_1} q$. By analyzing the construction of Lemma 10 in this restricted case, we obtain the following lemma.

Lemma 11. *If $\mathcal{S}$ is a single-sided* VASS *and $\phi \in L_\mu^{sv}$ then the game $\mathcal{G}(\mathcal{S}, \phi)$ is equivalent to a single-sided game.*

By combining the results of the last two lemmas with Corollary 1, Theorem 3 and Corollary 2, we get the following result on model checking single-sided VASS.

Theorem 7.

1. *Model checking L_{μ}^{sv} over single-sided* VASS *is decidable.*
2. *If $\mathcal{S}$ is a single-sided* VASS *and ϕ is a formula of L_{μ}^{sv} then $[\![\phi]\!]$ is upward-closed and its set of minimal elements is computable.*

6 Conclusion and Outlook

We have established a connection between multidimensional energy games and single-sided VASS games. Thus our algorithm to compute winning sets in VASS parity games can also be used to compute the minimal initial credit needed to win multidimensional energy parity games, i.e., the Pareto frontier.

It is possible to extend our results to integer parity games with a mixed semantics, where a subset of the counters follow the energy semantics and the rest follow the VASS semantics. If such a mixed parity game is single-sided w.r.t. the VASS counters (but not necessarily w.r.t. the energy counters) then it can be reduced to a single-sided VASS parity game by our construction in Section 3. The winning set of the derived single-sided VASS parity game can then be computed with the algorithm in Section 4.

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