Convergence of a residual based artificial viscosity finite element method[☆]

Murtazo Nazarov*

Abstract

We present a residual based artificial viscosity finite element method to solve conservation laws. The Galerkin approximation is stabilized by only residual based artificial viscosity, without any least-squares, SUPG, or streamline diffusion terms. We prove convergence of the method, applied to a scalar conservation law in two space dimensions, toward an unique entropy solution for implicit time stepping schemes.

Keywords: nonlinear conservation laws, finite element method, convergence, shock-capturing, artificial viscosity.

1. Introduction

The purpose of this paper is to study the convergence properties of a stabilized finite element method for solving nonlinear scalar conservation equations. The method is a stripped-down version of the Streamline-Diffusion-Shock-Capturing (SDSC) method analyzed in [15, 16]. The novelty of the present approach is that the streamline diffusion part of the method is entirely disregarded; the only stabilizing mechanism present in the algorithm is a residual-based nonlinear viscous regularization. The main result of the paper is that the method is convergent, i.e. the sequence of approximate solutions converges to the entropy solution under grid refinement. The analysis is based on the convergence theory of measure-valued solutions by DiPerna [5]. The three ingredients of the proof are as follows: (1) uniform boundedness in L^∞; (2) weak consistency with every entropy inequalities; (3) strong consistency with the initial data. That the streamline diffusion method augmented with a residual-based shock-capturing mechanism is convergent has been known since the groundbreaking work of Szepessy et al. [20, 19, 16, 21]. The novel idea defended in the present paper is that Streamline Diffusion [13], and more generally linear stabilization, [2, 4, 8], is not necessary to guaranty convergence to the entropy solution; the residual-based viscous regularization is actually the key ingredient of the method.

The idea of constructing a residual-based stabilization is not new, see e.g. the early work of Hughes and Tezduyar [1, 12, 22] and Johnson, Hansbo and Szepessy [13, 15, 11]. These so-called shock-capturing techniques were initially introduced with the sole purpose of correcting some defects of the streamline diffusion and SUPG methods as it was observed soon after their introduction that these techniques could not suppress the Gibbs phenomenon. That shock-capturing techniques do not need linear stabilization to work properly is an idea that has gained momentum only recently, see e.g. [18]. For instance Guermond et. al. [9, 10] used an entropy residual to construct an artificial viscosity, and the resulting method has been shown numerically to work properly on nonlinear conservation equations, including the compressible Euler equations, without invoking any linear stabilization mechanism. It has even been shown in [6] that some linear stabilization techniques have adverse effects on nonlinear conservation equations with nonconvex fluxes. More specifically, it is shown in [6] that by adding some linear stabilization to a convergent shock-capturing technique one can obtain a method that converges to a weak solution that is not the entropy solution, i.e. the action of linear stabilization is counter-productive.

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*Correspondence to: Department of Mathematics and Institute for Applied Mathematics and Computational Science, Texas A&M University, College Station, TX 77843-3368

Email address: murtazo@math.tamu.edu (Murtazo Nazarov)

URL: http://www.math.tamu.edu/~murtazo (Murtazo Nazarov)

The paper is organized as follows. In Section 2 we introduce some definitions, and we give a short description of scalar conservation equations and entropy inequalities. The discrete scheme under consideration in the present paper is also introduced in this section. The time stepping is implicit and the space is approximated using continuous finite elements. Well-posedness of the discrete problem is established by using Brouwer's fixed point theorem. The main convergence result of the paper is presented in Section 3. It is proved therein that the finite element approximation described in Section 2 converges strongly in $L^p_{\text{loc}}(\mathbb{R}^d\times\mathbb{R}_+)$, $1 \leq p < \infty$, to the unique solution as the meshsize and time step tend to zero. This convergence result is established by proving that the sequence of approximations is uniformly bounded in $L^\infty(\mathbb{R}^d\times\mathbb{R}_+)$, weakly consistent with all entropy inequalities, and strongly consistent with the initial data. Some concluding remarks are reported in Section 4.

2. Governing equations and the method

2.1. Theory

We consider the following scalar conservation equation

$$\begin{aligned} \partial_t u + \nabla\cdot\boldsymbol{f}(u) &= 0, \qquad && (\boldsymbol{x}, t) \in \mathbb{R}^d \times \mathbb{R}_+ \\ u(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}), && \boldsymbol{x} \in \mathbb{R}^d, \end{aligned} \tag{2.1}$$

where $\boldsymbol{f} \in C^1_b(\mathbb{R}, \mathbb{R}^d)$ is a smooth flux with continuous and bounded derivatives. To avoid unnecessary technicalities due to boundary conditions we assume that the support of the initial data $u_0 \in L^\infty(\mathbb{R}^d)$ is compact in $\mathbb{R}^d$. As usual we call entropy solution of (2.1) the unique member of $L^\infty(\mathbb{R}^d\times\mathbb{R}_+)$ that satisfies (2.1) weakly and is entropy admissible, i.e., the following holds for every $\phi \in C^\infty_0([0, +\infty)\times\mathbb{R}^d)$:

$$\int_0^{+\infty}\int_{\mathbb{R}^d} u(\partial_t\phi + \boldsymbol{f}(u)\cdot\nabla\phi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_{\mathbb{R}^d} u_0\phi(\boldsymbol{x}, 0)\,\mathrm{d}\boldsymbol{x} = 0, \tag{2.2}$$

and the following holds for every convex entropy $\eta \in C^1(\mathbb{R}; \mathbb{R})$ and every non-negative test function $\phi \in C^\infty_0(\mathbb{R}^d\times\mathbb{R}_+)$:

$$\int_0^{+\infty}\int_{\mathbb{R}^d} (\eta(u)\partial_t\phi + \boldsymbol{q}_\eta(u)\cdot\nabla\phi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \geq 0, \tag{2.3}$$

where q_η is the entropy flux associated with η, i.e., $\boldsymbol{q}_\eta(u) = \int_0^u \eta'(s)\boldsymbol{f}'(s)ds$.

The objective of the present work is to construct a numerical approximation of the entropy solution of (2.1) using a finite element method. We will regularize the numerical method by an artificial viscosity based on the residual of the PDE. The key ingredient for proving convergence of the method is the following theorem by DiPerna [5]:

Theorem 1. *Let $\{U^h\}_{h>0}$ be a sequence of functions in $L^\infty(\mathbb{R}^d\times\mathbb{R}_+; \mathbb{R})$. Assume that the initial condition $u_0 \in L^\infty(\mathbb{R}^d)\cap BV(\mathbb{R}^d)$ is compactly supported. Moreover, assume that the following conditions hold:*

1. *Uniformly boundedness: There is a constant C such that*

$$\|U^h\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}_+)} \leq C, \quad \forall h > 0. \tag{2.4}$$

2. *Weak consistency with all entropy inequalities in the distribution sense:*

$$\limsup_{h\to 0}\left(\partial_t\eta(U^h) + \nabla\cdot\boldsymbol{q}_\eta(U^h)\right) \leq 0, \tag{2.5}$$

for every convex entropy $\eta \in C^3(\mathbb{R}; \mathbb{R})$.

3. *Strong consistency with the initial condition:*

$$\lim_{t\to 0+}\limsup_{h\to 0}\|U^h(\cdot, t) - u_0\|_{L^1(\mathbb{R}^d)} = 0. \tag{2.6}$$

Then, if in addition $d = 2$, the sequence $\{U^h\}_{h>0}$ converges strongly in $L^p_{\text{loc}}(\mathbb{R}^d\times\mathbb{R}_+)$, $1 \leq p < \infty$, to the unique solution u of (2.1) as $h \to 0$.

Remark 2.1. Note that the uniform boundedness condition and the weak consistency with all the entropy inequalities imply that the sequence is also consistent with the conservation equation. More precisely upon defining the smooth convex functions $\eta_1(v)$ and $\eta_2(v)$ so that $\eta_1(v) = 2C - v$ if $v \le \frac{3}{2}C$ and $\eta_2(v) = v - 2C$ if $v \ge -\frac{3}{2}C$, respectively, we infer that $\eta_1(U^h) = 2C - U^h$ and $\eta_2(U^h) = U^h - 2C$ (the exact definitions of the extensions of η_1 and η_2 in the range $[\frac{3}{2}C, +\infty)$ and $(-\infty, -\frac{3}{2}C]$, respectively, are not important for our argumentation). The condition (2.5) implies in turn that $0 \le \limsup_{h\to 0} \left(\partial_t U^h + \nabla\cdot \boldsymbol{f}(U^h)\right) \le 0$, which proves the sequence $\{U^h\}$ is an approximation of the conservation equation.

2.2. Finite element setting in space

Due to u_0 being compactly supported, the support of the entropy solution u of (2.1) is compact in $\mathbb{R}^d \times [0, T^\infty]$. This implies that there exists $M < +\infty$ be so that $u(\boldsymbol{x}, t) = 0$ for all $|\boldsymbol{x}| \ge M$ and all $t \in [0, T^\infty]$.

Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular mesh family of $\mathbb{R}^d$. Each mesh is assumed to be composed of affine simplices. For every $K \in \mathcal{T}$, the diameter of K is denoted h_K and we set $h := \max_{K\in\mathcal{T}_h} h_K$. We then introduce the following finite element space

$$V_h = \{v \in H^1(\mathbb{R}^d) : v \in C^0(\mathbb{R}^d_n),\ v|_K \in \mathcal{P}_1(K),\ v(\boldsymbol{x}) = 0 \text{ for } |\boldsymbol{x}| \ge 2M\},$$

where $\mathcal{P}_1(K)$ is the set of d-variate polynomials over K of total degree at most 1.

The Lagrange interpolation operator in V_h is denoted π_h. The following standard interpolation estimates are known to hold:

Lemma 1. *There is a uniform constant C, such that the following holds for all $w \in W^{s,p}(E)$:*

$$\|w - \pi w\|_{W^{k,\infty}(E)} \le Ch^{s-k}|w|_{W^{s,\infty}(E)}, \quad s = 1, 2,\ k = 0, 1, \tag{2.7}$$

$$\|w - \pi w\|_{H^k(E)} \le Ch^{2-k}|w|_{H^2(E)}, \quad k = 0, 1, \tag{2.8}$$

$$\|w\|_{L^\infty(E)} \le Ch^{-\frac{d}{2}}\|w\|_{L^2(E)}, \tag{2.9}$$

$$\|\nabla w\|_{L^p(E)} \le Ch^{-1}\|w\|_{L^p(E)}, \quad 1 \le p \le \infty, \tag{2.10}$$

where the domain E can be either $\mathbb{R}^d$ or $K \in \mathcal{T}_h$.

2.3. Residual-based viscosity method

Let T^∞ be some finite time and let $0 = t_0 < t_1 < ... < t_N = T^\infty$ be a sequence of discrete time steps with associated time intervals $I_n = (t_{n-1}, t_n]$ of length $\tau_n = t_n - t_{n-1}$, $n = 1, 2, \cdots, N$. We assume that the mesh in time is quasi-uniform, i.e. $\max \tau_n / \min \tau_n$ is uniformly bounded for all possible time sequences. We define $\tau := \max \tau_n$.

Let $u_{0,h}$ be some reasonable approximation of u_0; for instance, set $u_{0,h} = \pi_h u_0$ if u_0 is continuous, or use the L^2-projection of u_0 onto V_h if u_0 is not continuous. Then set $U_h^0 = u_{0,h}$ and consider the following stabilized finite element approximation of the conservation equation (2.1) with implicit Euler time stepping: for $n = 1, 2, \cdots, N$ find $U_n^h \in V_h$ such that

$$\int_{\mathbb{R}^d} \frac{U_n^h - U_{n-1}^h}{\tau_n} v \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} \nabla\cdot\boldsymbol{f}(U_n^h) v \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} \varepsilon_n(U^h)\nabla U_n^h\cdot\nabla v \,\mathrm{d}\boldsymbol{x} = 0, \tag{2.11}$$

for all test functions $v \in V_h$. The artificial viscosity $\varepsilon_n(U_h)$ is defined as follows:

$$\varepsilon_n(U^h)|_K = C_\varepsilon h_K^\alpha |R_n(U^h)|_K|, \qquad \forall K \in \mathcal{T}_h, \tag{2.12}$$

$$R_n(U^h) = \frac{U_n^h - U_{n-1}^h}{\tau_n} + \nabla\cdot\boldsymbol{f}(U_n^h), \tag{2.13}$$

where $C_\varepsilon > 0$ is a user-defined $O(1)$ constant, and $\alpha \in [\frac{3}{2}, 2)$.

The key differences between (2.11) and the method studied in [14, 16, 19, 21] is that no linear stabilization is invoked in (2.11) and the nonlinear viscosity is scaled differently. More precisely, the method in [14, 16, 19, 21] uses time-dependent test functions and a space-time formulation that requires to use a discontinuous Galerkin approximation in time. The Galerkin formulation in [14, 19, 21, 16] is augmented with a streamline diffusion as follows:

$$\int_{t_n}^{t_{n+1}} \int_{R^d} \left(\partial_t U^h + \nabla\cdot \boldsymbol{f}(U^h)\right)\left(v + \tau_{SD}\left(\partial_t v + \sum_{i=1}^{2} \boldsymbol{f}_i'(U^h)\partial_{\boldsymbol{x}_i} v\right)\right) \mathrm{d}\boldsymbol{x}\, \mathrm{d}t$$
$$+ \int_{R^d} (U^h(t_n^+) - U(t_n^-))v^+ \,\mathrm{d}\boldsymbol{x} + \text{nonlinear viscosity} = 0. \quad (2.14)$$

where $\tau_{SD} \sim h$. We show in the present paper that the method still converges toward the entropy solution when the streamline diffusion terms are disregarded, i.e. $\tau_{SD} = 0$. Moreover, the scaling of the nonlinear viscosity in [14, 16] is such that it requires the exponent α to be less that 1, making the method excessively diffusive. More precisely the viscosity used in [14, 16] is proportional to $h_K^{\alpha}|R_n(U^h)|_K|/(|\nabla U^h|_K| + \epsilon)$ where $\alpha < 1$ and ϵ is a small regularization parameter. This scaling is also used in [3] (note in passing that it is remarkable that convergence to the entropy solution is proved in [3] directly from a priori estimates). Since we do not scale the residual by the gradient of U^h, our scaling allows the range $\alpha \in [\frac{3}{2}, 2)$, which is far less diffusive than that used in [14, 16, 3]. Note that the viscosity used in [19, 21], being proportional to $h_K^{\alpha}|R_n(U^h)|_K|(1 + |\boldsymbol{f}'(U^h)|_K)$ with $\alpha \in [\frac{3}{2}, 2)$, scales like ours with the exception of the factor $(1 + |\boldsymbol{f}'(U^h)|_K)$ that makes this viscosity potentially larger than that we propose. In conclusion, the originality of our formulation and of the analysis presented in the next section is that by removing the linear stabilization and by greatly reducing the scaling of the non linear viscosity, we still can prove convergence to the entropy solution. This result is original to the best of our knowledge.

2.4. Well-posedness of the discrete problem

Since the discrete problem (2.11) is nonlinear, due to the implicit time stepping, we must make sure that there is a unique solution at each time step. This is done by using the following variant of Brouwer's fixed point theorem (see eg. [17], [7, p. 279]):

Lemma 2. *Let $B : V_h \to V_h$ be a continuous mapping where V_h is equipped with the L^2-norm. Assume that the following property: there exists $r > 0$ such that*

$$(Bu, u) > 0, \qquad \forall u \in V_h \text{ with } \|u\|_{V_h} = r. \quad (2.15)$$

where $(\cdot,\cdot)$ denotes the L^2-inner product in V_h. Then there exists $w \in V_h$ with $\|w\|_{V_h} \le r$ such that $Bw \equiv 0$.

Let us define the following space:

$$V = \{v \in H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) : v(\boldsymbol{x}) = 0 \text{ for all } |\boldsymbol{x}| \ge 2M\}. \quad (2.16)$$

Then, integration by parts of the integral involving the nonlinear term $\nabla\cdot\boldsymbol{f}(U)$, with $U \in V$, is handled by using the following result:

Lemma 3. *Assume that the mapping $\boldsymbol{f} : \mathbb{R} \longrightarrow \mathbb{R}^d$ is uniformly Lipschitz, then*

$$\int_{\mathbb{R}^d} \nabla\cdot\boldsymbol{f}(U)\,\eta'(U)\,\mathrm{d}\boldsymbol{x} = 0, \qquad \forall\eta \in C^1(\mathbb{R};\mathbb{R}), \qquad \forall U \in V. \quad (2.17)$$

Proof. Upon setting $\boldsymbol{q}_\eta(u) = \int_0^s \boldsymbol{f}'(s)\eta'(s)\,\mathrm{d}s$, the following sequence of equalities holds:

$$\int_{\mathbb{R}^d} \nabla\cdot\boldsymbol{f}(U)\,\eta'(U)\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \eta'(U)\boldsymbol{f}'(s)\cdot\nabla U\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \boldsymbol{q}_\eta'(U)\cdot\nabla U\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \nabla\cdot\boldsymbol{q}_\eta(U)\,\mathrm{d}\boldsymbol{x} = 0,$$

which proves the desired result. □

In the following C denotes a generic positive constant that is uniform with respect to the meshsize and the time step.

Proposition 1. *There exists a unique solution to* (2.11) *at each time step provided the time step is small enough.*

Proof. Let u_0 be a member of V_h. Let $u \in V_h$ and let us define $Bu \in V_h$ by

$$(Bu, v) = \int_{\mathbb{R}^d} \frac{u - u_0}{\tau} v \, \mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} \nabla\cdot\boldsymbol{f}(u) v \, \mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} \varepsilon(u) \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x}, \tag{2.18}$$

where $\varepsilon(u) = C_\varepsilon h^\alpha \left| \frac{u-u_0}{\tau} + \nabla\cdot\boldsymbol{f}(u) \right|$. Let us now verify that (2.15) holds. We set $u = v$ in (2.18), and we obtain the following lower bound owing to Lemma 3:

$$\begin{aligned}(Bu, u) &= \int_{\mathbb{R}^d} \frac{u - u_0}{\tau} u \, \mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} \nabla\cdot\boldsymbol{f}(u) u \, \mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} \varepsilon(u) \nabla u \cdot \nabla u \, \mathrm{d}\boldsymbol{x} \\ &\geq \int_{\mathbb{R}^d} \frac{1}{2\tau} \left(|u|^2 - |u_0|^2 + (u - u_0)^2 \right) \mathrm{d}\boldsymbol{x} \geq \frac{1}{2\tau} \|u\|_{L^2}^2 - \frac{1}{2\tau} \|u_0\|_{L^2}^2.\end{aligned}$$

Therefore, $(Bu, u) > 0$ for all $u \in V_h$ with $\|u\|_{L^2} = 2\|u_0\|_{L^2}$. Note that we used that $V_h \subset V$.

Now, we have to show that $B(u)$ is a continuous operator with respect to the topology induced by the L^2-norm in V_h. Let $h = \min_{K \in \mathcal{T}_h} h_K$ and let $u, w \in V_h$. Using standard inverse inequalities, we infer that

$$\begin{aligned}\|Bu - Bw\|_{L^2} \leq& \tau^{-1} \|u - w\|_{L^2} + Ch^{-1} \|\boldsymbol{f}'(w)\|_{L^2} \|u - w\|_{L^2} + Ch^{-1} \|u\|_{L^2} \|\boldsymbol{f}'(u) - \boldsymbol{f}'(w)\|_{L^2} \\ &+ \|\epsilon(u)\|_{L^\infty} Ch^{-2} \|u - w\|_{L^2} + Ch^{-2} \|w\|_{L^\infty} \|\epsilon(u) - \epsilon(w)\|_{L^2}.\end{aligned}$$

We now estimate $\|\varepsilon(u)\|_{L^\infty}$ and $\|\varepsilon(u) - \varepsilon(w)\|_{L^2}$. Upon introducing the CFL number $\lambda = \tau \|\boldsymbol{f}'\|_{L^\infty} h^{-1}$, we have

$$\|\varepsilon(u)\|_{L^\infty} \leq Ch^{-\frac{d}{2}} \tau^{-1} ((1 + \lambda) \|u\|_{L^2} + \|u_0\|_{L^2}).$$

Similarly, using the inequality $\||a| - |b|\| \leq |a - b|$, we control $\|\varepsilon(u) - \varepsilon(w)\|_{L^2}$ as follows:

$$\begin{aligned}\|\varepsilon(u) - \varepsilon(w)\|_{L^2} &\leq \|\tau^{-1}(u - w) + (\boldsymbol{f}'(u) - \boldsymbol{f}'(w)) \cdot \nabla w + \boldsymbol{f}'(u) \cdot \nabla (u - w)\|_{L^2} \\ &\leq \|\tau^{-1} \|u - w\|_{L^2} + Ch^{-1} (\|\boldsymbol{f}'(u) - \boldsymbol{f}'(w)\|_{L^2} \|w\|_{L^2} + \|\boldsymbol{f}'\|_{L^\infty} \|u - w\|_{L^2}).\end{aligned}$$

Putting together the above estimates we obtain

$$\|Bu - Bw\|_{L^2} \leq C(h, \tau, \|u\|_{L^2}, \|u_0\|_{L^2}) (\|u - w\|_{L^2} + \|\boldsymbol{f}'(u) - \boldsymbol{f}'(w)\|_{L^2}).$$

Let u be fixed and consider a sequence w_n converging to u in $L^2(\mathbb{R}^d)$ as $n \to +\infty$. Since the L^2- and L^∞-norms are equivalent in V_h, up to some $h^{\frac{d}{2}}$ constant, the sequence w_n converges to u in the L^∞-norm. We conclude then that $\|\boldsymbol{f}'(u) - \boldsymbol{f}'(w_n)\|_{L^2}$ goes to zero when $\|u - w_n\|_{L^2}$ owing to $\boldsymbol{f}'$ being continuous, i.e. $\boldsymbol{f} \in C_b^1(\mathbb{R}; \mathbb{R})$. It immediately follows that $\|Bu - Bw_n\|_{L^2}$ converge to zero when $n \to \infty$, thereby proving that B is continuous in V_h. □

3. Convergence analysis

Let us denote $U^h \in L^\infty([0, T^\infty]; V_h)$ the function with value in V_h that is piecewise constant in time so that $U^h(t)_{|(t_n, t_{n+1}]} = U_{n+1}^h$ for all $n = 0, \ldots, N$. We establish in this section that the sequence $\{U^h\}$ converges to the unique entropy solution of (2.1).

3.1. Proof of Uniform boundedness

We start by deriving a standard L^2-estimate

Lemma 4 (L^2-estimate)**.** *There is uniform constant C so that*

$$\begin{aligned}\|U^h\|_{L^\infty((0,T^\infty);L^2(\mathbb{R}^d))}^2 &+ \sum_{n=1}^{N} \tau_n^2 \|D_\tau U_n^h\|_{L^2(\mathbb{R}^d)}^2 \\ &+ \sum_{n=1}^{N} \tau_n \int_{\mathbb{R}^d} \varepsilon_n(U^h) |\nabla U_n^h|^2 \, \mathrm{d}\boldsymbol{x} \leq \|u_0\|_{L^2(\mathbb{R}^d)}^2,\end{aligned} \tag{3.1}$$

where we have defined the approximate derivative $D_\tau U_n^h = \frac{1}{\tau_n}(U_n^h - U_{n-1}^h)$.

Proof. Let us use the test function $v = U_n^h$ in (2.11):

$$
\begin{aligned}
0 = \int_{\mathbb{R}^d} \frac{U_n^h - U_{n-1}^h}{\tau_n} U_n^h \,\mathrm{d}\boldsymbol{x} + &\int_{\mathbb{R}^d} \nabla\cdot \boldsymbol{f}(U_n^h) U_n^h \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} \varepsilon_n(U^h)\nabla U_n^h \cdot \nabla U_n^h \,\mathrm{d}\boldsymbol{x} \\
= &\int_{\mathbb{R}^d} \frac{1}{2\tau_n}\left(|U_n^h|^2 - |U_{n-1}^h|^2 + (U_n^h - U_{n-1}^h)^2\right) \mathrm{d}\boldsymbol{x} \\
+ &\int_{\mathbb{R}^d} \nabla\cdot \boldsymbol{f}(U_n^h) U_n^h \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} \varepsilon_n(U^h)|\nabla U_n^h|^2 \,\mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{3.2}
$$

Using Lemma 3 we infer that the second integral vanishes. By summing the above identity from $n = 1$ to N, we derive the desired L^2-stability estimate. □

We shall need the following lemma to prove the uniform boundedness in L^∞ of the approximate sequence $\{U_n^h\}_{n=0,\ldots,N}$:

Lemma 5 (Szepessy [19])**.** *There is a uniform constant constant $C > 0$ such that the following holds for all $p = 2m$, $m = 1, 2, 3, \ldots$ and for all $U \in V_h$:*

$$
\int_{\mathbb{R}^d} \nabla U \cdot \nabla \pi(U^{p-1}) \,\mathrm{d}\boldsymbol{x} \geq \frac{C}{p^2} \sum_{K\in\mathcal{T}} \int_K |\nabla U|^2 U^{p-2} \,\mathrm{d}\boldsymbol{x}. \tag{3.3}
$$

We are now in measure to establish the uniform boundedness estimate.

Lemma 6. *There is a uniform constant $C > 0$, such that the solution U^h of* (2.11) *satisfies*

$$
\|U^h\|_{L^\infty(\mathbb{R}^d\times(0,T^\infty))} \leq (1 + Ch^{\frac{1}{4}(2-\alpha)} \log(1/h))\|u_0\|_{L^\infty(\mathbb{R}^d)} \tag{3.4}
$$

Proof. By taking $v = \pi(U_n^h)^{p-1}$ in (2.11) with $p = 2m$, $m = 1, 2, 3, \cdots$ we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{U_n^h - U_{n-1}^h}{\tau_n} (U_n^h)^{p-1} \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} \varepsilon_n(U^h)\nabla U_n^h \cdot \nabla\pi(U_n^h)^{p-1} \,\mathrm{d}\boldsymbol{x} \\
&- \int_{\mathbb{R}^d} R(U_n^h)((U_n^h)^{p-1} - \pi(U_n^h)^{p-1}) \,\mathrm{d}\boldsymbol{x} = 0,
\end{aligned}
\tag{3.5}
$$

where we used $\int_{\mathbb{R}^d} \nabla\cdot \boldsymbol{f}(U_n^h)(U_n^h)^{p-1} \,\mathrm{d}\boldsymbol{x} = 0$, owing to Lemma 3. Recall that the residual is computed as follows: $R_n(U^h) = D_\tau U_n^h + \nabla\cdot \boldsymbol{f}(U_n^h)$.

We estimate first the term involving the time increment. For this purpose we introduce the function $\eta(z) = \frac{1}{p} z^p$ and we use the following Taylor expansion:

$$
\eta(U_n^h) - \eta(U_{n-1}^h) = (U_n^h - U_{n-1}^h)\eta'(U_n^h) - \frac{1}{2}(U_n^h - U_{n-1}^h)^2 \eta''(\xi). \tag{3.6}
$$

where $\xi \in [U_{n-1}^h, U_n^h]$. The convexity of η implies that

$$
\int_{\mathbb{R}^d} \frac{U_n^h - U_{n-1}^h}{\tau_n} (U_n^h)^{p-1} \,\mathrm{d}\boldsymbol{x} \geq \frac{1}{p\tau_n} \int_{\mathbb{R}^d} ((U_n^h)^p - (U_{n-1}^h)^p) \,\mathrm{d}\boldsymbol{x} \tag{3.7}
$$

We now estimate the term $I := \left|\int_{\mathbb{R}^d} R(U_n^h)((U_n^h)^{p-1} - \pi(U_n^h)^{p-1}) \,\mathrm{d}\boldsymbol{x}\right|$. Using standard interpolation estimates from Lemma 1, we infer that

$$
\begin{aligned}
I &\leq Ch^2 \sum_{K\in\mathcal{T}_h} \int_K R(U_n^h)\|D^2(U_n^h)^{p-1}\|_{L^\infty(K)} \,\mathrm{d}\boldsymbol{x} \\
&\leq Cp^2h^2 \sum_{K\in\mathcal{T}_h} \int_K |R(U_n^h)|\|\nabla U_n^h\|^2_{L^\infty(K)} \|U_n^h\|^{p-3}_{L^\infty(K)} \,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Upon using the definition of the viscosity $|\varepsilon_n(U^h)|$, the inequality (3.3), and separating the regions where $|U_n^h|$ is larger or smaller than 1, we deduce that

$$\begin{aligned} I \le\,& CC_\epsilon^{-1} p^2 h^{2-\alpha} \left(\sum_{K\in\mathcal{T}_h} \int_{K\cap\{|U|>1\}} \varepsilon_n(U^h)|\nabla U_n^h|^2 \|U_n^h\|_{L^\infty(K)}^{p-2} \,\mathrm{d}\boldsymbol{x} \right. \\ &\left. + \sum_{K\in\mathcal{T}_h} \int_{K\cap\{|U|\le 1\}} \varepsilon_n(U^h)|\nabla U_n^h|^2 \,\mathrm{d}\boldsymbol{x} \right) \\ \le\,& Cp^4 h^{2-\alpha} \int_{\mathbb{R}^d} \varepsilon_n(U^h)\nabla U_n^h\cdot\nabla\pi(U_n^h)^{p-1}\,\mathrm{d}\boldsymbol{x} + C' p^2 h^{2-\alpha} \int_{\mathbb{R}^d} \varepsilon_n(U^h)|\nabla U_n^h|^2 \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Now we collect all terms and we obtain

$$\begin{aligned} &\frac{1}{p}\|U_n^h\|_{L^p}^p + \tau_n(1 - Cp^4h^{2-\alpha}) \int_{\mathbb{R}^d} \varepsilon_n(U^h)\nabla U_n^h\cdot\nabla\pi((U_n^h)^{p-1})\,\mathrm{d}\boldsymbol{x} \\ &\qquad \le \frac{1}{p}\|U_{n-1}^h\|_{L^p}^p + C'\tau_n p^2 h^{2-\alpha} \int_{\mathbb{R}^d} \varepsilon_n(U^h)|\nabla U_n^h|^2\,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{3.8}$$

After summing the above estimate from $n = 1$ to N, assuming that $p^2 \le p^4 \le Ch^{\alpha-2}$, and using the energy estimates (3.1), we finally arrive at

$$\|U_N^h\|_{L^p(\mathbb{R}^d)}^p \le \|u_0\|_{L^p(\mathbb{R}^d)}^p + Ch^{2(2-\alpha)}\|u_0\|_{L^2(\mathbb{R}^d)}^2.$$

which means that there is a uniform constant C_0 so that the following holds for h small enough:

$$\|U_N^h\|_{L^p(\mathbb{R}^d)} \le 2^{\frac{1}{p}}\|u_0\|_{L^\infty(\mathbb{R}^d)}, \qquad p \le C_0 h^{\frac{1}{4}(-2+\alpha)}, \quad p \text{ even}.$$

Let p_h be the largest even integer so that $p_h \le C_0 h^{\frac{1}{4}(-2+\alpha)} < p_h + 2$. There is another constant C_1 so that $h^{\frac{1}{4}(-2+\alpha)} \le C_1 p_h$. Then using a standard inverse estimate together with the assumption $\alpha < 2$, we infer that

$$\begin{aligned} \|U_N^h\|_{L^\infty(\mathbb{R}^d)} &\le (Ch)^{-\frac{d}{p_h}} \|U_N^h\|_{L^p(\mathbb{R}^d)}^p \le (Ch)^{-\frac{d}{p_h}} 2^{\frac{1}{p_h}} \|u_0\|_{L^\infty(\mathbb{R}^d)} \le (C'h)^{-\frac{d}{p_h}}\|u_0\|_{L^\infty(\mathbb{R}^d)} \\ &\le e^{-\frac{d}{p_h}\log(C'h)}\|u_0\|_{L^\infty(\mathbb{R}^d)} \le e^{-dC_1 h^{\frac{1}{4}(2-\alpha)}\log(C'h)}\|u_0\|_{L^\infty(\mathbb{R}^d)} \\ &\le (1 + C_2 h^{\frac{1}{4}(2-\alpha)}\log(1/h))\|u_0\|_{L^\infty(\mathbb{R}^d)}. \end{aligned}$$

This completes the proof. □

3.2. Proof of consistency with all entropy inequalities

The purpose of this section is to prove that the sequence of functions $\{U^h\}_{h>0}$ is entropy consistent.

Lemma 7. *Let x be a fixed number in $(1, \alpha + \frac{1}{2})$ and assume that the following condition holds uniformly: $h^x \le \tau$. Then the following inequality holds*

$$\limsup_{\tau, h\to 0} -\int_{\mathbb{R}^d\times(0,T^\infty)} \left(\eta(U^h)\partial_t\varphi + \boldsymbol{q}_\eta(U^h)\cdot\nabla\varphi\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le 0. \tag{3.9}$$

for every convex entropy $\eta \in C^3(\mathbb{R};\mathbb{R})$ ($\boldsymbol{q}_\eta$ being the associated entropy flux) and for all $\varphi \in C_0^\infty(\mathbb{R}^d\times(0,T^\infty);\mathbb{R}_+)$.

Proof. Let $\eta \in C^3(\mathbb{R};\mathbb{R})$ be a convex entropy with entropy flux $\boldsymbol{q}_\eta$, and let $\varphi \in C_0^\infty(\mathbb{R}^d\times(0,T^\infty);\mathbb{R}_+)$ be a smooth positive test function. Upon setting $D := \mathbb{R}^d\times(0,T^\infty)$, using the test function $v = \pi(\eta'(U^h)\varphi)$ in (2.11) and integrating over time, we obtain

$$\begin{aligned} &\sum_{n=1}^N \int_{\mathbb{R}^d} D_\tau U_n^h \eta'(U_n^h)\,\mathrm{d}\boldsymbol{x} \int_{t_{n-1}}^{t_n} \varphi(t)\,\mathrm{d}t + \int_D \nabla\cdot\boldsymbol{f}(U^h))\eta'(U^h)\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \\ &\qquad - \int_D \varepsilon_n(U^h)\nabla U^h\cdot\nabla\pi(\eta'(U^h)\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_D R(U^h)(\eta'\varphi - \pi(\eta'(U^h)\varphi))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \end{aligned} \tag{3.10}$$

We reorganize the term involving the time increments of U^h by using the Taylor expansion (3.6) as follows:

$$\sum_{n=1}^{N}\int_{\mathbb{R}^d} D_\tau U_n^h \eta'(U_n^h)\,\mathrm{d}\boldsymbol{x}\int_{t_{n-1}}^{t_n}\varphi(t)\,\mathrm{d}t := \sum_{n=1}^{N}\int_{\mathbb{R}^d} D_\tau U_n^h \eta'(U_n^h)\,\mathrm{d}\boldsymbol{x}\tau_n\varphi(t_{n-1}) + R_0$$
$$= \sum_{n=1}^{N}\int_{\mathbb{R}^d}\left(\eta(U_n^h)-\eta(U_{n-1}^h)+\frac{1}{2}(U_n^h-U_{n-1}^h)^2\eta''(\xi_n)\right)\varphi(t_{n-1})\,\mathrm{d}\boldsymbol{x} + R_0$$
$$\geq -\int_D \eta(U^h)\partial_t\varphi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + R_0.$$

Then, we can rewrite (3.10) as follows:

$$-\int_D \eta(U^h)\partial_t\varphi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_D \boldsymbol{q}_\eta(\eta(U^h))\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq -R_0$$
$$-\int_D \varepsilon_n(U^h)\nabla U^h\cdot\nabla\pi(\eta'(U^h)\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_D R(U^h)(\eta'\varphi-\pi(\eta'(U^h)\varphi))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \quad (3.11)$$

The rest of the proof consists of bounding from above the three terms in the right-hand of the above inequality that we denote R_0, R_1 and R_2, respectively.

Bound on R_0: Using the L^∞ estimate (3.4) together with the fact that φ is a smooth function, the remainder R_0 is handled as follows:

$$-R_0 = \sum_{n=1}^{N}\int_{\mathbb{R}^d} D_\tau U_n^h\eta'(U_n^h)\left(\tau_n\varphi(\boldsymbol{x},t_{n-1}) - \int_{t_{n-1}}^{t_n}\varphi(\boldsymbol{x},t)\,\mathrm{d}t\right)\mathrm{d}\boldsymbol{x}$$
$$\leq C\sum_{n=1}^{N}\tau_n^2\|D_\tau U_n^h\|_{L^1(\mathbb{R}^d)} \leq C'\sum_{n=1}^{N}\tau_n^2\|D_\tau U_n^h\|_{L^2(\mathbb{R}^d)}.$$

The energy estimate (3.1) in turn implies that

$$|R_0| \leq C\max_{0\leq n\leq N}\tau_n^{\frac{1}{2}} \leq C\tau^{\frac{1}{2}}. \quad (3.12)$$

Bound on R_1: Let us evaluate the remainder $R_1 := \int_D \varepsilon_n(U^h)\nabla U^h\cdot\nabla\pi(\eta'(U^h)\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$. Let $P\varphi$ be the L^2-projection of φ onto the set of discontinuous functions that are piecewise constant over the mesh $\mathcal{T}_h$, i.e., $P\varphi_{|K} = \mathrm{meas}(K)^{-1}\int_K\varphi\,\mathrm{d}\boldsymbol{x}$. Then, R_1 is decomposed as follows:

$$R_1 = -\sum_{n=1}^{N}\tau_n\sum_{K\in\mathcal{T}_h}\int_K \varepsilon(U_n^h)\nabla U_n^h\cdot\nabla\left(\pi(\eta'(U_n^h)(\varphi-P\varphi))\right)\mathrm{d}\boldsymbol{x}$$
$$-\sum_{n=1}^{N}\tau_n\sum_{K\in\mathcal{T}_h}\int_K \varepsilon(U_n^h)\nabla U_n^h\cdot\nabla\pi(\eta'(U_n^h)P\varphi)\,\mathrm{d}\boldsymbol{x} := R_{11}+R_{12}.$$

The first component of the residual, R_{11}, is estimated as follows:

$$|R_{11}| \leq \sum_{n=1}^{N}\tau_n\sum_{K\in\mathcal{T}_h}\int_K \varepsilon_n(U^h)|\nabla U_n^h|\,\|\nabla(\eta'(U_n^h)((\varphi-P\varphi))\|_{L^\infty(K)}\,\mathrm{d}\boldsymbol{x}$$
$$\leq \sum_{n=1}^{N}\tau_n\sum_{K\in\mathcal{T}_h}\int_K \varepsilon_n(U^h)|\nabla U_n^h|\,\|\eta''(U_n^h)\nabla U^h(\varphi-P\varphi)+\eta'(U_n^h)\nabla(\varphi-P\varphi)\|_{L^\infty(K)}\,\mathrm{d}\boldsymbol{x}$$
$$\leq C\sum_{n=1}^{N}\tau_n\sum_{K\in\mathcal{T}_h}\int_K \varepsilon_n(U^h)(h|\nabla U_n^h|^2+|\nabla U_n^h|)\|\nabla\varphi\|_{L^\infty(K)}\,\mathrm{d}\boldsymbol{x}$$
$$\leq Ch + C'\sum_{n=1}^{N}\tau_n\sum_{K\in\mathcal{T}_h}\varepsilon(U_n^h)|\nabla U_n^h|\,\mathrm{d}\boldsymbol{x},$$

where we have used that $\|\varphi - P\varphi\|_{L^\infty(K)} \leq Ch\|\nabla\varphi\|_{L^\infty(K)}$, $\|\eta''\|_{L^\infty([-2\gamma,2\gamma])}$, $\|\eta'\|_{L^\infty([-2\gamma,2\gamma])}$, $\gamma = \limsup_h \|U^h\|_{L^\infty}$, are bounded and the stability estimate (3.1). We consider two possible cases to estimate the second integral of the last inequality. Let us assume that $\tau \geq h^x$ and let us define $\Gamma := h^{-\frac{\alpha}{2}+\frac{1}{4}}$. Then we have

$$\begin{aligned}\sum_{n=1}^N \tau_n \sum_{\{|\nabla U_n^h|\leq\Gamma\}} \int_K \varepsilon(U_n^h)|\nabla U_n^h| \,\mathrm{d}\boldsymbol{x} &= \sum_{n=1}^N \tau_n \sum_{\{|\nabla U_n^h|\leq\Gamma\}} \int_K h^\alpha \left|D_\tau U_n^h + \nabla\cdot \boldsymbol{f}(U_n^h)\right| \Gamma \,\mathrm{d}\boldsymbol{x} \\ &\leq \sum_{n=1}^N \tau_n h^\alpha \|D_\tau U_n^h\|_{L^1(\mathbb{R}^d)}\Gamma + Ch^\alpha\Gamma^2 \leq C\left(\tau^{-\frac{1}{2}} h^\alpha \Gamma + h^\alpha\Gamma^2\right).\end{aligned}$$

The second case is handled as follows:

$$\sum_{n=1}^N \tau_n \sum_{\{|\nabla U_n^h|>\Gamma\}} \int_K \varepsilon_n(U^h)|\nabla U_n^h| \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \sum_{n=1}^N \tau_n \int_{\mathbb{R}^d} \varepsilon(U_n^h)\frac{|\nabla U_n^h|^2}{\Gamma} \,\mathrm{d}\boldsymbol{x} \leq C\Gamma^{-1}.$$

By combining the above estimates and given that $h^x \leq \tau$ with $1 \leq x$ by assumption, we derive the following bound

$$|R_{11}| \leq C(h^{\frac{\alpha-x}{2}+\frac{1}{4}} + h^{\frac{1}{2}}).$$

We now deal with the term R_{12}. Using the definition of $P\varphi$, which recall is piece-wise constant, together with Lemma 8, we infer that

$$R_{12} = -\sum_{n=1}^N \tau_n \sum_{K\in\mathcal{T}_h} \int_K P\varphi\varepsilon_n(U^h)\nabla U_n^h\cdot\nabla\pi(\eta'(U_n^h)) \,\mathrm{d}\boldsymbol{x} \leq 0.$$

In conclusion we have

$$R_1 \leq C(h^{\frac{\alpha-x}{2}+\frac{1}{4}} + h^{\frac{1}{2}}). \tag{3.13}$$

Bound on R_2: We now estimate the third integral in (3.11). Using standard interpolation error estimates we obtain the following bound:

$$\begin{aligned}|R_2| &= \left|\int_D R(U^h)(\eta'\varphi - \pi(\eta'\varphi) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \\ &\leq \sum_{n=1}^N \tau_n \sum_{K\in\mathcal{T}_h} \int_K Ch^2|R(U_n^h)|(|\nabla U_n^h|^2 + |\nabla U_n^h| + 1)\|\varphi\|_{W^{2,\infty}(K)} \,\mathrm{d}\boldsymbol{x} \\ &\leq Ch^2 \sum_{n=1}^N \tau_n \sum_{K\in\mathcal{T}_h} \int_K |R(U_n^h)|(|\nabla U_n^h|^2 + 1) \,\mathrm{d}\boldsymbol{x} \\ &\leq Ch^{2-\alpha} \sum_{n=1}^N \tau_n \sum_{K\in\mathcal{T}_h} \int_K \varepsilon_n(U^h)|\nabla U_n^h|^2 \,\mathrm{d}\boldsymbol{x} + Ch^2 \sum_{n=1}^N \tau_n \int_{\mathbb{R}^d} |R(U_n^h)| \,\mathrm{d}\boldsymbol{x}\end{aligned}$$

We assume that $\tau \geq h^x$ and we introduce the quantity $\Gamma := h^{-\frac{\alpha}{3}}$, which is larger than 1 when h is small enough, and we proceed as in the estimate of R_{11} by distinguishing cells where $|\nabla U_n^h| \leq \Gamma$ from those where $|\nabla U_n^h| > \Gamma$.

$$h^2 \sum_{n=1}^N \tau_n \sum_{|\nabla U_n^h|\leq\Gamma} \int_K |R(U_n^h)| \,\mathrm{d}\boldsymbol{x} \leq h^2 \sum_{n=1}^N \tau_n \int_{\mathbb{R}^d} (|D_\tau U_n^h| + C\Gamma) \,\mathrm{d}\boldsymbol{x} \leq C\left(\tau^{-\frac{1}{2}} h^2 + h^2\Gamma\right).$$

$$\begin{aligned}h^2 \sum_{n=1}^N \tau_n \sum_{|\nabla U_n^h|>\Gamma} \int_K |R(U_n^h)| \,\mathrm{d}\boldsymbol{x} &\leq Ch^2 \sum_{n=1}^N \tau_n \sum_{|\nabla U_n^h|>\Gamma} \int_K |R(U_n^h)| \frac{|\nabla U_n^h|^2}{\Gamma^2} \,\mathrm{d}\boldsymbol{x} \\ &\leq Ch^{2-\alpha}\Gamma^{-2} \sum_{n=1}^N \tau_n \int_{\mathbb{R}^d} h^\alpha |R(U_n^h)|\,|\nabla U_n^h|^2 \,\mathrm{d}\boldsymbol{x} \leq Ch^{2-\alpha}\Gamma^{-2}.\end{aligned}$$

With the assumption $h^x \leq \tau$, this finally gives

$$|R_2| \leq C(h^{2-\frac{x}{2}} + h^{2-\frac{\alpha}{3}}) \tag{3.14}$$

By using the estimates (3.12)-(3.13)-(3.14) we finally infer that

$$-\int_{\mathbb{R}^d\times(0,T^\infty)} \Big(\eta(U^h)\partial_t\varphi - \boldsymbol{q}_\eta(U^h)\cdot\nabla\varphi\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq C(\tau^{\frac{1}{2}} + h^{\frac{\alpha-x}{2}+\frac{1}{4}} + h^{\frac{1}{2}} + h^{2-\frac{x}{2}} + h^{2-\frac{\alpha}{3}})$$
$$\leq C(\tau^{\frac{1}{2}} + h^{\frac{\alpha-x}{2}+\frac{1}{4}} + h^{\frac{1}{2}}).$$

which implies the desired result owing to the assumptions. □

Remark 3.1. Whether the condition $h^x \leq \tau$ is necessary is not clear. We suspect that it is only a technical difficulty that could be removed by deriving better a priori estimates. The convergence rate in the entropy inequality is maximal with this condition, .i.e. $h^{\frac{1}{2}}$, for $x = \alpha - \frac{1}{2}$, which gives $x \to \frac{3}{2}$ when $\alpha \to 2$.

Lemma 8. *The following inequality holds*

$$\nabla V\cdot\nabla(\pi(\eta'(V))) \geq 0, \tag{3.15}$$

for all $V \in V_h$ and fall all convex entropy $\eta \in C^2(\mathbb{R};\mathbb{R})$.

Proof. See Proposition 5.3 in [19] and Lemma 3.3 in [21]. □

3.3. Proof of strong consistency with the initial condition

We finish this section by proving strong the consistency with the initial data.

Lemma 9. *Provided the assumptions of Lemma 7 hold, The sequence of approximate solutions $\{U^h\}$ is strongly consistent with the initial data, i.e.*

$$\lim_{t\to 0+}\limsup_{h\to 0} \|U^h(\cdot,t) - u_0\|_{L^2(\mathbb{R}^d)} = 0. \tag{3.16}$$

Proof. Let us first assume for the time being that the following conditions hold

$$\lim_{t\to 0+}\limsup_{h\to 0}\int_{\mathbb{R}^d} |U^h(\boldsymbol{x},t)|^2\,\mathrm{d}\boldsymbol{x} \leq \|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{3.17}$$

$$\lim_{t\to 0+}\lim_{h\to 0}\int_{\mathbb{R}^d} (U^h(\boldsymbol{x},t) - u_0(\boldsymbol{x}))\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0, \quad \varphi \in C^\infty(\mathbb{R}^d\times[0,T^\infty)), \tag{3.18}$$

Then, we prove the consistency with the initial condition as follows. Let $\{\varphi_\epsilon\}_{\epsilon>0}$ be sequence of smooth functions that converges to u_0 strongly in $L^2(\mathbb{R}^d)$ as ϵ goes to zero. Then

$$\lim_{t\to 0}\limsup_{h\to 0}\int_{\mathbb{R}^d} (U^h(\cdot,t) - u_0)^2\,\mathrm{d}\boldsymbol{x}$$
$$= \lim_{t\to 0}\limsup_{h\to 0}\int_{\mathbb{R}^d} \Big((U^h(\cdot,t))^2 - u_0^2 - 2\varphi_\epsilon(U^h - u_0)\Big) + 2(\varphi_\epsilon - u_0)(U^h - u_0)\,\mathrm{d}\boldsymbol{x}$$
$$\leq 4\|\varphi_\epsilon - u_0\|_{L^2(\mathbb{R}^d)}\|u_0\|_{L^2(\mathbb{R}^d)}.$$

Since the above estimate holds true for all $\epsilon > 0$, we obtain the desired result by passing to the limit on ϵ, i.e., $\lim_{t\to 0+}\lim_{h\to 0}\|U^h(\cdot,t) - u_0\|_{L^2(\mathbb{R}^d)} = 0$.

We now need to verify that (3.17) and (3.18) hold. Clearly (3.17) is a consequence of the estimate (3.4). We need to work a little bit more to establish (3.18). Let $\varphi \in C_b^\infty(\mathbb{R}^d,\mathbb{R})$ and let t be some fixed time. Let n be the integer so that $t \in (t_{n-1}, t_n]$ where $t_n = \sum_{i=0}^n \tau_i$. Then $U^h(t) = U_n^h$ by definition and

$$\int_{\mathbb{R}^d} (U_n^h - U_0^h)\varphi\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} (U_n^h - U_0^h)(\varphi - \pi\varphi)\,\mathrm{d}\boldsymbol{x}$$
$$-\sum_{i=0}^{n-1}\tau_i\int_{\mathbb{R}^d}\Big(\nabla\cdot\boldsymbol{f}(U_i^h)\pi\varphi + \varepsilon_i(U^h)\nabla U_n^h\cdot\nabla\pi\varphi\Big)\,\mathrm{d}\boldsymbol{x}.$$

We need to show that the right-hand side goes to zero as $h \to 0$ and $t \to 0$. The first term does not pose any particular difficulty.

$$\left|\int_{\mathbb{R}^d} (U_n^h - U_0^h)(\varphi - \pi\varphi)\,\mathrm{d}\boldsymbol{x}\right| \leq Ch\|u_0\|_{L^2(\mathbb{R}^d)}\|\varphi\|_{H^1(\mathbb{R}^d)}.$$

For the second term, one integration by parts implies that

$$\left|\int_{\mathbb{R}^d} \nabla\cdot \boldsymbol{f}(U_i^h)\pi\varphi\,\mathrm{d}\boldsymbol{x}\right| = \left|\int_{\mathbb{R}^d} \boldsymbol{f}(U_i^h)\nabla\pi\varphi\,\mathrm{d}\boldsymbol{x}\right| \leq C\|U_i^h\|_{L^\infty(\mathbb{R}^d)}\|\varphi\|_{W^{1,\infty}(\mathbb{R}^d)}.$$

Then

$$\left|\sum_{i=0}^{n-1}\tau_i \int_{\mathbb{R}^d} \nabla\cdot \boldsymbol{f}(U_i^h)\pi\varphi\,\mathrm{d}\boldsymbol{x}\right| \leq t_n C(\varphi)\|u_0\|_{L^\infty(\mathbb{R}^d)}.$$

The last term is handled as follows:

$$\left|\sum_{i=0}^{n-1}\tau_i \int_{\mathbb{R}^d} \varepsilon_i(U^h)\nabla U_n^h\cdot\nabla\pi\varphi\,\mathrm{d}\boldsymbol{x}\right| \leq C\|\varphi\|_{W^{1,\infty}(\mathbb{R}^d)}\left|\sum_{i=0}^{n-1}\tau_i \int_{\mathbb{R}^d} \varepsilon_i(U^h)|\nabla U_n^h\cdot|\,\mathrm{d}\boldsymbol{x}\right|.$$

By proceeding as in the derivation of the estimate for R_{11} in the proof of Lemma 7, we infer that

$$\left|\sum_{i=0}^{n-1}\tau_i \int_{\mathbb{R}^d} \varepsilon_i(U^h)\nabla U_n^h\cdot\nabla\pi\varphi\,\mathrm{d}\boldsymbol{x}\right| \leq C(\varphi)(t_n^{\frac{1}{2}}h^{\frac{1}{4}} + h^{\frac{1}{2}}).$$

The desired result follows readily. □

4. Conclusions

We have introduced a stripped-down version of the Streamline-Diffusion-Shock-Capturing (SDSC) method analyzed in [15, 16]. The only stabilizing mechanism present in our algorithm is a residual-based nonlinear viscous regularization. The main result of the paper is that the method is convergent, i.e. the sequence of approximate solutions converges to the entropy solution under grid refinement. The idea defended in the present paper is that Streamline Diffusion, and more generally linear stabilization, is not necessary to guaranty convergence to the entropy solution; the residual-based viscous regularization is actually the key ingredient of the method. In that sense, our conclusion is similar in spirit to that in [6].

The analysis is done for the implicit Euler time stepping. We are currently extending the methodology presented in this paper to explicit time stepping of first- and second-order accuracy.

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