OPTIMAL BOUNDS FOR A LAGRANGE INTERPOLATION INEQUALITY FOR PIECEWISE LINEAR CONTINUOUS FINITE ELEMENTS IN TWO SPACE DIMENSIONS[§]

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Abstract. In this paper the interpolation inequality of Szepessy [12, Lemma 4.2] is revisited. The lower bound in the above reference is proven to be proportional to p^{-2}, where p is a polynomial degree, that goes fast to zero as p increases. We prove that the lower bound is proportional to $\ln^2 p$ which is an increasing function. Moreover, we prove that this estimate is sharp.

Key words. inequality, Lagrange interpolation estimates, finite elements, scalar conservation laws, convergence

AMS subject classifications. 65M60

1. Introduction. The stabilized finite element method is one of the most investigated numerical methods for over three decades and it goes back to early work of Brooks and Hughes [1] and Johnson et al. [8]. The finite element discretization is stabilized by adding a residual based nonlinear term, the so-called streamline diffusion term. This approach is then successfully applied to many problems including compressible and incompressible fluid flows, see e.g. Hughes and Tezduyar [6], Johnson and Saranen [10], Hughes et al. [4]. Soon after the introduction of this method it was observed that the streamline diffusion method could not fully suppress the Gibbs phenomenon. The so-called discontinuous or shock-capturing techniques have been developed by Hughes et al. [5], Hughes and Mallet [3], Tezduyar and Park [15], Johnson and Szepessy [7], where the main idea was to introduce another dissipation in the direction normal to the gradient of the transport quantity.

However, no theoretical justification of showing improvements of solutions from these shock-capturing methods was available until Johnson et al. [9]. In Johnson et al. [9] the authors proposed to construct the shock-capturing term as a certain artificial viscosity with a residual-based viscosity coefficient. The convergence of the streamline diffusion method augmented with a residual-based shock-capturing mechanism has been known since the groundbreaking work of Szepessy [13, 12], Johnson et al. [9], Szepessy [14]. Szepessy's proof is based on the theory of measure-valued solutions introduced by DiPerna [2]. The three ingredients of the proof are as follows: (1) uniform boundedness in L^∞; (2) weak consistency with all entropy inequalities; (3) strong consistency with the initial data. One should remark that the nonlinear shock-capturing term is the main term that is needed to achieve the convergence. Recently, it has been shown by Nazarov [11] that by disregarding the streamline diffusion term entirely and by having only residual based shock-capturing or artificial diffusion one can prove the convergence to the unique entropy solution.

The first condition of DiPerna, the uniform boundedness in L^∞, or the so-called maximum principle, is one of the key points for the convergence of numerical methods and analysis of nonlinear hyperbolic equations. In the above references, the L^∞-bound

[§]The second author is supported by Award No. KUS-C1-016-04, made by King Abdullah University of Science and Technology (KAUST). Draft version, October 27, 2014

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is proved using interpolation estimates of Szepessy [12, Lemmas 3.4, 4.2]. The main contribution of this paper is to improve Lemma 4.2 of Szepessy [12] and derive the optimal estimate in terms of polynomial degree, p. The original estimate gives the lower bound proportional to the negative square of p (p^{-2}), which obviously goes to zero as p increases. We prove that the lower bound proportional to the square of logarithm of p ($\ln^2 p$), which is increasing function, moreover we show that this lower bound is sharp. This result gives us that convergence results of finite element discretizations of scalar conservation laws can be obtained by using less amount of viscosity than it has been used in the literature.

The paper is organized as follows: the problem formulation and the main results are stated in §2; the proof of the lower estimate is given in §3; and finally the proof of the upper estimate is provided in §4.

2. Main results. Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular mesh family of $\mathbb{R}^2$, $K \in \mathcal{T}_h$ be an element of this mesh. We introduce the following finite element space

$$V = \{v \in H^1(\mathbb{R}^2) : v \in \mathcal{C}^0(\mathbb{R}^2),\, v|_K \in \mathcal{P}_1(K)\},$$

where $\mathcal{P}_1(K)$ is the set of two-variate polynomials over K of total degree at most 1. The standard Lagrange interpolation operator in V is denoted by π.

The following lemma is an essential ingredient for the L^∞-bound that is used in the above literature:

Lemma 2.1. *There is a uniform constant $C > 0$ such that the following inequality holds for all $p = 2m$, $m = 1, 2, 3, \ldots$ and all $U \in V$:*

$$\int_{\mathcal{T}_h} \nabla U \cdot \nabla \pi(U^{p-1})\,\mathrm{d}\boldsymbol{x} \geq C M(p) \sum_{K\in\mathcal{T}_h} \int_K |\nabla U|^2 U^{p-2}\,\mathrm{d}\boldsymbol{x}, \tag{2.1}$$

where $M(p)$ is a constant depending only on p.

In Szepessy [12, Lemma 4.2], Johnson et al. [9, Lemma 4.2], and Szepessy [14, Lemma 3.3], the function $M(p)$ is defined to be $M(p) = 1/p^2$, which gives a very weak estimate, since $1/p^2$ goes fast to zero as $p \to \infty$. The main contribution of this paper is to improve the estimate (2.1) significantly. It is stated as the following theorem:

Theorem 2.2. *There is a uniform constant $C > 0$ such that the inequality* (2.1) *holds for $M(p) = \ln^2 p$, $p = 2m$, $m = 1, 2, 3, \ldots$ and all $U \in V$.*

The function $M(p) = \ln^2 p$ is an increasing function. Here we formulate the second contribution of this paper that shows that this constant is optimal:

Theorem 2.3. *Assume $\mathcal{T}_0$ is a triangulation which consists of only one right triangle K_0. Then there exists a function $U \in V$ such that*

$$\int_{K_0} \nabla U \cdot \nabla \pi(U^{p-1})\,\mathrm{d}\boldsymbol{x} < C_0 M(p) \int_{K_0} |\nabla U|^2 U^{p-2}\,\mathrm{d}\boldsymbol{x}, \tag{2.2}$$

for all $p = 2m$, $m = 1, 2, 3, \ldots$, where $C_0 > 0$ is a constant independent of p.

In other words Theorem 2.3 proves that the inequality (2.1) is sharp, i.e there is no another function $M(p) \neq \ln^2 p$ such that satisfies the inequality (2.1) and $\lim_{p\to\infty} M(p)/\ln^2 p = \infty$.

We give the details of the proof of theorems in the following sections.

3. Proof of Theorem 2.2. Let us consider a right triangle $K \in \mathcal{T}_h$ given as in Fig 3.1(a), with sides of length a and b. The function U in this triangle is defined as

$$U = y_1 + \frac{x_1}{a}(y_2 - y_1) + \frac{x_2}{b}(y_3 - y_1),$$

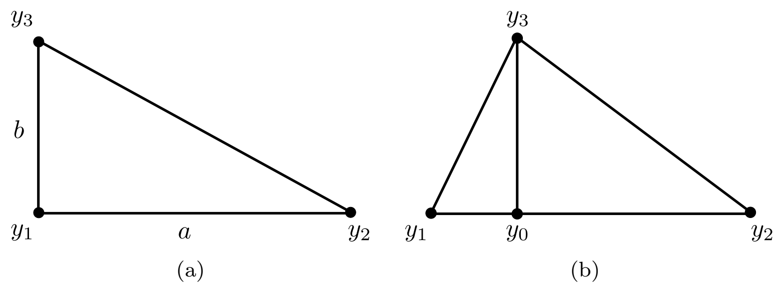

FIG. 3.1. *The local element of the triangulation in 2D*

where y_i, $i = 1, 2, 3$ are the nodal values of the function, x_1 and x_2 are the coordinate directions, $(x_1, x_2) \in \mathbb{R}^2$.

Since $\nabla U \cdot \nabla \pi(U^{p-1})$ and $|\nabla U|^2$ are constant on K therefore it is sufficient to prove the inequality for one triangle. For this purpose, let us define two functions $F_{K,p}$ and S_p such as

$$F_{K,p}(y_1, y_2, y_3) = \frac{\int_K \nabla U \cdot \nabla \pi(U^{p-1})\, \mathrm{d}\boldsymbol{x}}{|\nabla U|^2} \quad \text{and} \quad S_p(y_1, y_2, y_3) = \int_K U^{p-2}\, \mathrm{d}\boldsymbol{x}.$$

A simple algebra gives that $\nabla U = \left(\frac{1}{a}(y_2 - y_1), \frac{1}{b}(y_3 - y_1)\right)$, and $\nabla \pi(U^{p-1}) = \left(\frac{1}{a}(y_2^{p-1} - y_1^{p-1}), \frac{1}{b}(y_3^{p-1} - y_1^{p-1})\right)$,

$$F_{K,p}(y_1, y_2, y_3) = \frac{\frac{1}{a^2}(y_2 - y_1)(y_2^{p-1} - y_1^{p-1}) + \frac{1}{b^2}(y_3 - y_1)(y_3^{p-1} - y_1^{p-1})}{\frac{1}{a^2}(y_2 - y_1)^2 + \frac{1}{b^2}(y_3 - y_1)^2} \cdot \frac{ab}{2}$$

By defining the constants $C_{a,b}$, $C'_{a,b}$ and the function $F_p(y_1, y_2, y_3)$ such as

$$C_{a,b} := \frac{ab}{2} \min\left(\frac{a^2}{b^2}, \frac{b^2}{a^2}\right) \qquad C'_{a,b} := \frac{ab}{2} \max\left(\frac{a^2}{b^2}, \frac{b^2}{a^2}\right).$$

$$F_p(y_1, y_2, y_3) := \frac{(y_2 - y_1)(y_2^{p-1} - y_1^{p-1}) + (y_3 - y_1)(y_3^{p-1} - y_1^{p-1})}{(y_2 - y_1)^2 + (y_3 - y_1)^2},$$

we get the following relation:

$$C_{a,b} \cdot F_p(y_1, y_2, y_3) \le F_{K,p}(y_1, y_2, y_3) \le C'_{a,b} \cdot F_p(y_1, y_2, y_3). \tag{3.1}$$

Then, the proof of the theorem consists of minimizing the following relation

$$\frac{F_p(y_1, y_2, y_3)}{S_p(y_1, y_2, y_3)},$$

where $|y_i| \le 1$, $i = 1, 2, 3$, and S_p is the integral of U^{p-2} over the triangle K:

$$\begin{aligned} S_p(y_1, y_2, y_3) &= \int_K U^{p-2}\, \mathrm{d}\boldsymbol{x} \\ &= \int_0^b \int_0^{a(1-\frac{x_2}{b})} \left(y_1 + \frac{x_1}{a}(y_2 - y_1) + \frac{x_2}{b}(y_3 - y_1)\right)^{p-2} \mathrm{d}x_1\, \mathrm{d}x_2 \\ &= \frac{ab}{p(p-1)} \frac{1}{y_2 - y_1} \left[\frac{y_3^p - y_2^p}{y_3 - y_2} - \frac{y_3^p - y_1^p}{y_3 - y_1}\right]. \end{aligned}$$

Note, that the function S_p has the following symmetry property:

$$S_p(1,x,y) = S_p(x,1,y) = S_p(x,y,1), \quad \forall x,y.$$

We consider two possible cases: first we consider $y_1 = 1$ and perform the analysis, then we continue with the case when $y_2 = 1$.

Remark 3.1. The following proof is valid to any triangle: consider the element in Fig 3.1(b). We can easily construct two right triangles by setting an inner altitude and apply the following discussion for each of them.

Case 1. Let us consider the case when $y_1 = 1$. We denote $x = y_2$, $y = y_3$, $F_{1p}(x,y) := F_p(1,x,y)$, i.e.

$$F_{1p}(x,y) = \frac{(1-x)(1-x^{p-1}) + (1-y)(1-y^{p-1})}{(1-x)^2 + (1-y)^2},$$

and $S_p(x,y) = S_p(1,x,y)$.

The result for this case follows from the following lemma.

Lemma 3.1. *The following estimate holds*

$$\frac{F_{1p}(x,y)}{S_p(x,y)} \geq \frac{p-1}{2ab}, \quad -1 \leq x,y \leq 1.$$

Proof. First, let us consider the case $x \leq 0$ or $y \leq 0$.

For any real numbers u and v the following holds

$$\frac{u^p - v^p}{u-v} = \sum_{i=0}^{p-1} u^i v^{p-1-i} = \sum_{i=0}^{p-1} u^{p-1-i} v^i. \tag{3.2}$$

Now, using the fact that $\sum_{i=0}^{p-2} x^i = \frac{1-x^{p-1}}{1-x} \geq \frac{1}{2}$, we easily get

$$\begin{aligned} F_{1p}(x,y) &= \frac{(1-x)(1-x^{p-1}) + (1-y)(1-y^{p-1})}{(1-x)^2 + (1-y)^2} \\ &= \frac{(1-x)^2 \sum_{i=0}^{p-2} x^i + (1-y)^2 \sum_{i=0}^{p-2} y^i}{(1-x)^2 + (1-y)^2} \geq \frac{1}{2}. \end{aligned} \tag{3.3}$$

On the other hand if $x \leq 0$ and $-1 \leq y \leq 1$ using (3.2) and by noting that $1 - x^{p-1-2j} + y(1 - x^{p-2-2j}) \leq 2$ we obtain

$$\begin{aligned} S_p(x,y) &= \frac{ab}{p(p-1)} \frac{1}{1-x} \left[\frac{1-y^p}{1-y} - \frac{y^p - x^p}{y-x} \right] \leq \frac{ab}{p(p-1)} \sum_{i=0}^{p-1} y^i (1 - x^{p-1-i}) \\ &= \frac{ab}{p(p-1)} \left[\sum_{j=0}^{p/2-1} y^{2j} \left(1 - x^{p-1-2j} + y(1 - x^{p-2-2j})\right) \right] \\ &\leq \frac{ab}{p(p-1)} \cdot 2 \cdot \sum_{j=0}^{p/2-1} y^{2j} \leq \frac{ab}{p-1}. \end{aligned}$$

Analogously, we get the same estimate for the case when $y \leq 0$ and $-1 \leq x \leq 1$. Therefore,

$$S_p(x,y) \leq \frac{ab}{p-1}, \quad x \leq 0 \text{ or } y \leq 0. \tag{3.4}$$

Using (3.4) and (3.3) we get the desired estimate of the proof for $x \leq 0$ or $y \leq 0$.

Then, let us consider the case when $x \geq 0, y \geq 0$. We have

$$S_p(x,y) = \frac{ab}{p(p-1)} \frac{1}{y-x} \left[\frac{1-y^p}{1-y} - \frac{1-x^p}{1-x} \right] \leq \frac{ab}{p(p-1)} \sum_{i=0}^{p-2} x^i \sum_{j=0}^{p-2-i} y^j$$

$$\leq \frac{ab}{p(p-1)} \sum_{i=0}^{p-2} x^i (p-1-i) \leq \frac{ab}{p} \sum_{i=0}^{p-2} x^i.$$

Analogously, we get

$$S_p(x,y) \leq \frac{ab}{p} \sum_{i=0}^{p-2} y^i.$$

From the last estimates of $S_p(x,y)$ and the formula (3.2) it follows that

$$\begin{aligned} \left[(1-x)^2 + (1-y)^2\right] S_p(x,y) &= (1-x)^2 S_p(x,y) + (1-y)^2 S_p(x,y) \\ &\leq \frac{ab}{p} \left[(1-x)^2 \sum_{i=0}^{p-2} x^i + (1-y)^2 \sum_{i=0}^{p-2} y^i \right] \\ &= \frac{ab}{p} \left[(1-x)(1-x^{p-1}) + (1-y)(1-y^{p-1}) \right] \\ &= \frac{ab}{p} \left[(1-x)^2 + (1-y)^2 \right] F_{1p}(x,y). \end{aligned}$$

We get that $S_p(x,y) \leq \frac{ab}{p} F_{1p}(x,y)$ or

$$\frac{F_{1p}(x,y)}{S_p(x,y)} \geq \frac{p}{ab} > \frac{p-1}{2ab}, \quad 0 \leq x, y < 1.$$

The proof of the lemma is completed here. □

Case 2. Assume $y_2 = 1$, $|y_1|, |y_3| \leq 1$. For simplicity, we denote $x = y_1$ and $y = y_3$, $F_p(x,y) := F_p(x,1,y)$ and $S_p(x,y) := S_p(x,1,y)$ for the rest of the proof. Then, the functions take the following form

$$F_p(x,y) = \frac{(1-x)(1-x^{p-1}) + (x-y)(x^{p-1} - y^{p-1})}{(1-x)^2 + (x-y)^2}, \tag{3.5}$$

$$S_p(x,y) = \frac{ab}{p(p-1)} \frac{1}{x-y} \left[\frac{1-x^p}{1-x} - \frac{1-y^p}{1-y} \right]. \tag{3.6}$$

Let us introduce the following auxiliary function:

$$G_p(x,y) := \left(x^{p-1} - y^{p-1}\right) \left(\frac{1-x^p}{1-x} - \frac{1-y^p}{1-y} \right)^{-1}; \; 0 \leq x < 1, \; -1 \leq y < 1, \; x \neq y. \tag{3.7}$$

We remark, that the function $S_p(x,y)$ has a continuous extension on the lines $x = y$, $x = 1$, $y = 1$ and at the point $x = y = 1$. For instance on the line $y = x$ we have

$$S_p(x,y) = \frac{ab}{p(p-1)} \frac{d}{dx} \left[\frac{1-x^p}{1-x} \right] = \frac{-px^{p-1}(1-x) + (1-x^p)}{(1-x^p)}; \quad 0 \leq x \leq 1,$$

and on the line $y = 1$

$$S_p(x,y) = \frac{ab}{p(p-1)}\frac{1}{x-1}\left[\frac{1-x^p}{1-x} - p\right]; \quad 0 \le x \le 1,$$

and at the point $(x, y) = (1, 1)$

$$S_p(x,y) = \frac{ab}{p(p-1)}p(p-1) = ab.$$

The function $G_p(x, y)$ which is defined in (3.7) is symmetric, i.e. $G_p(x, y) = G_p(y, x)$, and has a continuous extension at the closed unit square $\{(x, y) : 0 \le x, y \le 1\}$. For instance, at points $(x, 1)$, $0 \le x < 1$ it is defined as

$$G_p(x,1) = \left(x^{p-1} - 1\right)\left(\frac{1-x^p}{1-x} - p\right)^{-1}; \quad 0 \le x < 1,$$

and at the point $(1, 1)$

$$G_p(1,1) = (p-1)\left(\frac{p(p-1)}{2}\right)^{-1} = \frac{2}{p}.$$

Using these properties, the continuous extension at points $x = y$, $0 \le x < 1$ is obtained using the relation (3.2). In fact,

$$\begin{aligned} G_p(x,y) &= \left((x-y)\sum_{j=0}^{p-2} x^{p-j-2}y^j\right)\left(\sum_{i=0}^{p-1} x^i - \sum_{i=0}^{p-1} y^i\right)^{-1} \\ &= \left((x-y)\sum_{j=0}^{p-2} x^{p-j-2}y^j\right)\left(\sum_{i=1}^{p-1}(x-y)\sum_{j=0}^{i-1} x^{i-1-j}y^j\right)^{-1} \\ &= \left(\sum_{j=0}^{p-2} x^{p-j-2}y^j\right)\left(\sum_{i=1}^{p-1}\sum_{j=0}^{i-1} x^{i-1-j}y^j\right)^{-1}. \end{aligned}$$

From this representation it follows that the function G_p has a continuous extension at the closed unit square $(x, y) : 0 \le x, y \le 1$.

The following relations for the functions $F_p(x, y)$, $S_p(x, y)$ and $G_p(x, y)$ are used in the below analysis:

$$\begin{aligned} F_p(x,y) = &\frac{(1-x)^2(1-y)}{(1-x)^2 + (x-y)^2}\frac{p(p-1)}{ab}S_p(x,y) \\ &+ \left[1 - \frac{(1-x)^2(1-y)}{(1-x)^2+(x-y)^2}\right]\frac{x^{p-1} - y^{p-1}}{x-y}, \end{aligned}$$

and for any $0 \le x < 1$, $-1 \le y < 1$, $x \ne y$

$$\begin{aligned} \frac{F_p(x,y)}{S_p(x,y)} = \frac{p(p-1)}{ab}\Bigg\{&\frac{(1-x)^2(1-y)}{(1-x)^2+(x-y)^2} \\ &+ \left[1 - \frac{(1-x)^2(1-y)}{(1-x)^2+(x-y)^2}\right]G_p(x,y)\Bigg\}. \end{aligned} \tag{3.8}$$

The relation between the functions F_p, S_p and G_p (3.8) is important in the proof. In some quadrants the first term of the expression inside the brackets is enough to use, while the second term gives very crucial estimates that we are going to discuss below.

Next, let us define another auxiliary function $\psi(p)$ as the solution of the following equation

$$\psi(p)e^{\psi(p)} = p, \quad p > 0.$$

The existence of this solution comes from the monotonicity of the function ψe^{ψ}. From the definition of $\psi(p)$ it follows that $\psi(p) \to \infty$ as $p \to \infty$. By taking a logarithm and then a limit from the last equation we get that

$$\frac{\psi(p)}{\ln p}\left(1 + \frac{\ln \psi(p)}{\psi(p)}\right) = 1, \text{ and } \lim_{p\to\infty} \frac{\psi(p)}{\ln p} = \lim_{p\to\infty} \left(1 + \frac{\ln \psi(p)}{\psi(p)}\right)^{-1} = 1,$$

i.e. the function $\psi(p)$ has the same asymptotic rate as $\ln p$. Below, we shall prove that the lower bound that we are looking for is proportional to the square of $\psi(p)$, that will then complete the proof.

Let us now define a number

$$C_p = \inf\left\{\frac{F_p(x,y)}{S_p(x,y)} : -1 \le x, y < 1\right\}, \quad p = 2, 4, \ldots.$$

The rest of the proof studies the behavior of the sequence $\{C_p\}_{p=2}^{\infty}$ when $p \to \infty$. To simplify the discussion, we split the unit square $-1 \le x, y \le 1$ into four quadrants and consider each of them separately.

Quadrant I: $0 \le x, y \le 1$. Note that on this quadrant we have that

$$0 \le \frac{(1-x)^2(1-y)}{(1-x)^2 + (x-y)^2} \le 1.$$

First, we prove the following property of the function G_p, that will be used for the proof for this quadrant.

LEMMA 3.2. *The function $G_p(x,y)$ increases with respect to y, i.e.*

$$G_p(x,0) \le G_p(x,y).$$

Proof. Since $G_p(x,y) = (x^{p-1} - y^{p-1})\Big(\sum_{i=1}^{p-1}(x^i - y^i)\Big)^{-1}$, and

$$\frac{\partial G_p}{\partial y}(x,y) = \left(-(p-1)y^{p-2}\sum_{i=1}^{p-1}(x^i - y^i) + (x^{p-1} - y^{p-1})\sum_{i=1}^{p-1} iy^{i-1}\right)\left(\sum_{i=1}^{p-1}(x^i - y^i)\right)^{-2},$$

therefore, for the proof of the lemma it is enough to establish that the numerator of the last equality is non-negative. The numerator can be written as

$$\sum_{i=1}^{p-1}(p-1)y^{p-2}(y^i - x^i) - \sum_{i=1}^{p-1} iy^{i-1}(y^{p-1} - x^{p-1})$$

$$= \sum_{i=1}^{p-1} \left[(p-1)y^{p-2}(y^i - x^i) - iy^{i-1}(y^{p-1} - x^{p-1}) \right].$$

Lemma A.1, presented in the Appendix A, proves that each terms of the last sum is non-negative. The lemma is proved here. □

The following lemma gives the relation between two auxiliary functions $G_p(x, y)$ and $\psi(p)$.

LEMMA 3.3. *For every $p = 2, 4, \dots$ and $0 \le x, y \le 1$ such that either $1 - \frac{\psi(p)}{p} < x$ or $1 - \frac{\psi(p)}{p} < y$ the following estimate holds:*

$$G_p(x, y) \ge \frac{\psi^2(p)}{p^2} e^{-\frac{\psi^2(p)}{2(p-\psi(p))}}. \tag{3.9}$$

Proof. Assume that $x > 1 - \frac{\psi(p)}{p}$. According to Lemma 3.2 we obtain

$$G_p(x, y) \ge G_p(x, 0) = \frac{(1-x)x^{p-1}}{(1-x^p) - (1-x)} = \frac{(1-x)x^{p-1}}{x(1-x^{p-1})}.$$

By the symmetry of $G_p(x, y)$ and Lemma 3.2 it follows that the function $G_p(x, 0)$ increases at the interval $0 \le x \le 1$. Therefore,

$$G_p(x, 0) \ge G_p\Big(1 - \frac{\psi(p)}{p}, 0\Big); \quad 1 - \frac{\psi(p)}{p} \le x \le 1.$$

Let us now estimate the value

$$G_p\Big(1 - \frac{\psi(p)}{p}, 0\Big) = \left(\frac{\psi(p)}{p}\left(1 - \frac{\psi(p)}{p}\right)^{p-2}\right)\left(1 - \left(1 - \frac{\psi(p)}{p}\right)^{p-1}\right)^{-1}.$$

Using Lemma A.3 from Appendix A we obtain:

$$\left(1 - \frac{\psi(p)}{p}\right)^{p-2} \ge \left(1 - \frac{\psi(p)}{p}\right)^{p} \ge e^{-\psi(p)\left[1 + \frac{\psi(p)}{2(p-\psi(p))}\right]}$$
$$= e^{-\psi(p)} e^{-\frac{\psi^2(p)}{2(p-\psi(p))}} = \frac{\psi(p)}{p} e^{-\frac{\psi^2(p)}{2(p-\psi(p))}}.$$

Thus,

$$G_p\Big(1 - \frac{\psi(p)}{p}, 0\Big) \ge \left(\frac{\psi^2(p)}{p^2} e^{-\frac{\psi^2(p)}{2(p-\psi(p))}}\right)\left(1 - \left(1 - \frac{\psi(p)}{p}\right)^{p-1}\right)^{-1}$$
$$\ge \frac{\psi^2(p)}{p^2} e^{-\frac{\psi^2(p)}{2(p-\psi(p))}}.$$

By the symmetry of G_p, the same estimate is true for $y < 1 - \frac{\psi(p)}{p}$. □

We now formulate the main result for this quadrant.

THEOREM 3.4. *For the sequence of the numbers*

$$C'_p = \inf\left\{\frac{F_p(x, y)}{S_p(x, y)} : 0 \le x, y \le 1\right\}, \; p = 2, 4, \dots$$

there exists a constant $d > 0$ independent of p such that the following estimate holds

$$C'_p \geq d \cdot \psi^2(p), \quad p = 2, 4, \dots. \tag{3.10}$$

Proof. Consider two possible situations for the terms of the relation (3.8)

$$(a)\ \frac{(1-x)^2(1-y)}{(1-x)^2+(x-y)^2} \geq \frac{\psi^2(p)}{2p^2}; \quad (b)\ \frac{(1-x)^2(1-y)}{(1-x)^2+(x-y)^2} < \frac{\psi^2(p)}{2p^2}.$$

In case (a), form (3.8) it easily follows that

$$\frac{F_p(x,y)}{S_p(x,y)} \geq \left(1-\frac{1}{p}\right)\frac{1}{2ab}\psi^2(p). \tag{3.11}$$

In case (b), let us consider two sets of points $y \leq x$ and $x < y$.

(i) $\{(x,y) : 0 \leq y \leq x \leq 1\}$. Since

$$\begin{aligned}\frac{(1-x)^2(1-y)}{(1-x)^2+(x-y)^2} &\geq \frac{(1-x)^2(1-y)}{(1-x)^2+(1-y)^2} \\ &= \frac{(1-x)^2 \cdot \frac{1}{1-y}}{\left(\frac{1-x}{1-y}\right)^2+1} \geq \frac{1}{2}\frac{(1-x)^2}{1-y} \geq \frac{(1-x)^2}{2},\end{aligned}$$

we get that $1 - x < \frac{\psi(p)}{p}$, i.e. $x > 1 - \frac{\psi(p)}{p}$.

(ii) $\{(x,y) : 0 \leq x < y \leq 1\}$. For this set the following inequality holds

$$\frac{(1-x)^2(1-y)}{(1-x)^2+(x-y)^2} \geq \frac{1-y}{2},$$

which gives us $\frac{1}{2}(1-y) < \frac{\psi^2(p)}{2p^2}$ and therefore $y > 1 - \frac{\psi(p)}{p}$.

Thanks to Lemma 3.3 for either $x > 1 - \frac{\psi(p)}{p}$ or $y > 1 - \frac{\psi(p)}{p}$ we have

$$\begin{aligned}\frac{F_p(x,y)}{S_p(x,y)} &\geq \frac{p(p-1)}{ab}\left(1-\frac{\psi^2(p)}{2p^2}\right)G_p(x,y) \\ &\geq \frac{p(p-1)}{ab}\left(1-\frac{\psi^2(p)}{2p^2}\right)e^{-\frac{\psi^2(p)}{2(p-\psi(p))}}\frac{\psi^2(p)}{p^2}.\end{aligned} \tag{3.12}$$

The desired inequality (3.10) is obtained from the inequalities (3.11) in case (a), and (3.12) in case (b), and by setting

$$d = \min_{p=2,4,\dots}\left\{\left(1-\frac{1}{p}\right)\frac{1}{2ab}\left(1-\frac{\psi^2(p)}{2p^2}\right)e^{-\frac{\psi^2(p)}{2(p-\psi(p))}}\right\}. \tag{3.13}$$

□

Now, we continue with the remaining quadrants. Below, the estimates on Quadrants II and III are obtained easily, while it is rather technical for Quadrant IV.

Quadrants II and III: $x \leq 0$.

For these quadrants we easily get

$$F_p(x,y) = \frac{(1-x)(1-x^{p-1})+(x-y)(x^{p-1}-y^{p-1})}{(1-x)^2+(x-y)^2} \geq \frac{(1-0)(1-0)+0}{2^2+2^2} = \frac{1}{8},$$

and by (3.4) we have

$$\frac{F_p(x,y)}{S_p(x,y)} \geq \frac{p-1}{8ab}.$$

Quadrant IV: $0 \leq x \leq 1; -1 \leq y \leq 0$.

Let us consider the following set

$$E = \{(x,y) : 0 \leq x \leq 1; -1 \leq y \leq 0\}.$$

Note, that for $x = 1$ the desired result follows from Lemma 3.1 in Case 1. The idea is now to split the set E into the following subsets and consider each subset separately:

$$\begin{aligned} E_1 &= \left\{(x,y) : 0 \leq x \leq \frac{1}{2}; -1 \leq y \leq 0\right\}, \\ E_{1p} &= \left\{(x,y) : \frac{1}{2} < x \leq 1 - \frac{\psi(p)}{p}; -1 \leq y \leq 0\right\}, \\ E_{2p} &= \left\{(x,y) : 1 - \frac{\psi(p)}{p} < x < 1; -1 \leq y \leq 0\right\}. \end{aligned}$$

It turns out that $\frac{F_p(x,y)}{S_p(x,y)}$ minimizes at the region $\frac{1}{2} < x < 1$. Therefore, for this region we discuss each term of the expression inside the brackets of (3.8) and take advantage of the properties of $G_p(x,y)$.

I. Assume $(x,y) \in E_1$. From the inequality

$$\frac{1}{2} \leq (1-x)^2 + (x-y)^2 \leq 4, \quad (x,y) \in E \tag{3.14}$$

we get that

$$\begin{aligned} F_p(x,y) &= \frac{(1-x)(1-x^{p-1}) + (x-y)(x^{p-1} - y^{p-1})}{(1-x)^2 + (x-y)^2} \\ &\geq \frac{1}{4}(1-x)(1-x^{p-1}) \geq \frac{1}{16}, \end{aligned}$$

and by (3.4) we get:

$$\frac{F_p(x,y)}{S_p(x,y)} \geq \frac{p-1}{16ab}, \quad (x,y) \in E_1. \tag{3.15}$$

II. Assume $(x,y) \in E_{1p}$. Using the inequality (3.14) we get

$$\frac{(1-x)^2}{4} \leq \frac{(1-x)^2(1-y)}{(1-x)^2 + (x-y)^2} \leq 4(1-x)^2, \quad (x,y) \in E. \tag{3.16}$$

It follows that

$$\frac{(1-x)^2(1-y)}{(1-x)^2 + (x-y)^2} \leq 1, \quad (x,y) \in E_{1p}. \tag{3.17}$$

By using the last inequality (3.17) and the fact that $G_p(x,y) \geq 0$, for $p = 2, 4, \ldots$, $(x,y) \in E$, from the relation (3.8) we get the following estimate

$$\frac{F_p(x,y)}{S_p(x,y)} \geq \frac{p(p-1)}{ab} \frac{(1-x)^2(1-y)}{(1-x)^2 + (x-y)^2} \geq \frac{p(p-1)}{ab} \frac{1}{4}(1-x)^2$$

$$\geq \frac{p(p-1)}{4ab} \cdot \frac{\psi^2(p)}{p^2} = \Big(1 - \frac{1}{p}\Big) \cdot \frac{1}{4ab}\psi^2(p).$$

Thus, we obtain that

$$\frac{F_p(x,y)}{S_p(x,y)} \geq \Big(1 - \frac{1}{p}\Big)\frac{1}{4ab}\psi^2(p), \quad (x,y) \in E_{1p}. \tag{3.18}$$

III. Assume $(x,y) \in E_{2p}$. Let us consider the region $0 < 1-x \leq \frac{\psi(p)}{p}$, $-1 \leq y \leq 0$. First of all, let us show that for every $p = 4, 6, \dots$

$$\frac{1-x^p}{1-x} - \frac{3}{2} \geq 0, \quad (x,y) \in E_{2p}. \tag{3.19}$$

Since the function $\frac{1-x^p}{1-x} = 1 + \cdots + x^{p-1}$, $0 \leq x \leq 1$, is increasing, it is sufficient to show that

$$\frac{1-x_p^p}{1-x_p} \geq \frac{3}{2}, \quad x_p = 1 - \frac{\psi(p)}{p}.$$

By virtue of the inequality $(1-\frac{\psi(p)}{p})^p < e^{-p\frac{\psi(p)}{p}} = e^{-\psi(p)} = \frac{\psi(p)}{p}$, we obtain that $\frac{1-x_p^p}{1-x_p} \geq (1-\frac{\psi(p)}{p})/(\frac{\psi(p)}{p}) = \frac{p}{\psi(p)} - 1$, and hence, if $\frac{p}{\psi(p)} - 1 \geq \frac{3}{2}$, $p = 4, 6, \dots$, then (3.19) holds. In fact, note that $\frac{p}{\psi(p)} \geq \frac{5}{2}$ when $p \geq 4$. Therefore, (3.19) is true.

By virtue of (3.19) for all $(x,y) \in E_{2p}$ we have the following inequalities:

$$\begin{aligned} &(1-y)(1-x^p) - (1-x)(1-y^p) > 0, \\ &\qquad (1-x)(1-y)(x^{p-1} - y^{p-1}) \geq (1-x)(1-y)x^{p-1}, \\ &(1-y)(1-x^p) - (1-x)(1-y^p) \leq (1-y)(1-x^p). \end{aligned} \tag{3.20}$$

Next, let us show that the following inequality holds:

$$G_p(x,y) \geq \frac{1}{3}G_p(x,0), \quad (x,y) \in E_{2p}. \tag{3.21}$$

When $p = 2$ the inequality (3.21) passes into $1 \geq \frac{1}{3}$. Consider the case when $p \geq 4$. Using the inequalities (3.20) we get

$$\begin{aligned} G_p(x,y) - \frac{1}{3}G_p(x,0) &= \frac{(1-x)(1-y)(x^{p-1} - y^{p-1})}{(1-y)(1-x^p) - (1-x)(1-y^p)} - \frac{1}{3}\frac{(1-x)x^{p-2}}{1-x^{p-1}} \\ &\geq \frac{(1-x)x^{p-1}}{1-x^p} - \frac{1}{3}\frac{(1-x)x^{p-2}}{1-x^{p-1}} \\ &= \frac{2}{3}\frac{(1-x)^2x^{p-2}}{(1-x^{p-1})(1-x^p)} \cdot \Big(\frac{1-x^p}{1-x} - \frac{3}{2}\Big). \end{aligned}$$

From here and the inequality (3.19) we get (3.21).

Let us come back to our main question which is estimating the function $\frac{F_p(x,y)}{S_p(x,y)}$ in the set E_{2p}. From the inequality (3.16) we have

$$\frac{(1-x)^2(1-y)}{(1-x)^2 + (x-y)^2} \leq 4\frac{\psi^2(p)}{p^2}, \quad (x,y) \in E_{2p},$$

and therefore

$$\frac{F_p(x,y)}{S_p(x,y)} \geq \frac{p(p-1)}{ab}\left[1-\frac{(1-x)^2(1-y)}{(1-x)^2+(x-y)^2}\right]G_p(x,y)$$
$$\geq \frac{p(p-1)}{ab}\Big(1-4\frac{\psi^2(p)}{p^2}\Big)\frac{1}{3}G_p(x,0).$$

On the other hand Lemma 3.3 gives us that for all $x > 1-\frac{\psi(p)}{p}$ we have that $G_p(x,0) \geq \frac{\psi^2(p)}{p^2}e^{-\frac{\psi^2(p)}{2(p-\psi(p))}}$. So,

$$\frac{F_p(x,y)}{S_p(x,y)} \geq \Big(1-\frac{1}{p}\Big)\frac{1}{3ab}\Big(1-4\frac{\psi^2(p)}{p^2}\Big)e^{-\frac{\psi^2(p)}{2(p-\psi(p))}}\cdot\psi^2(p). \tag{3.22}$$

Let us define the number

$$d_1 = \min\left\{\min_{p=2,4,\ldots}\frac{p-1}{16ab}\cdot\frac{1}{\psi^2(p)},\ \min_{p=2,4,\ldots}\Big(1-\frac{1}{p}\Big)\frac{1}{3ab}\Big(1-4\frac{\psi^2(p)}{p^2}\Big)e^{-\frac{\psi^2(p)}{2(p-\psi(p))}}\right\}. \tag{3.23}$$

The following theorem follows from estimates (3.15), (3.18) and (3.22)

Theorem 3.5. *The sequence of numbers*

$$C_p'' = \inf\left\{\frac{F_p(x,y)}{S_p(x,y)} : 0 \leq x \leq 1,\ -1 \leq y \leq 0\right\},\ p = 2,4,\ldots$$

satisfies the following estimate

$$C_p'' \geq d_1\cdot\psi^2(p), \quad p = 2,4,\ldots, \tag{3.24}$$

where d_1 is defined in (3.23).

Here we complete the proof of Theorem 2.2.

4. Proof of Theorem 2.3. The idea consists in constructing a piecewise linear function such that the inequality in Theorem 2.3 is satisfies. Let us construct a function U_p by setting $y_1 = 1-\frac{\psi(p)}{p}$, $y_2 = 1$ and $y_3 = 0$:

$$U_p = 1-\frac{\psi(p)}{p}+\frac{x_1}{a}\frac{\psi(p)}{p}-\frac{x_2}{b}\Big(1-\frac{\psi(p)}{p}\Big).$$

Next we recall, that

$$\frac{F_p(x,0)}{S_p(x,0)} = \frac{p(p-1)}{ab}\frac{1}{(1-x)^2+x^2}\left\{(1-x)^2+\frac{1-x}{1-x^{p-1}}x^p\right\}.$$

For the points $x_p = 1-\frac{\psi(p)}{p}$ the following limits hold:

$$\lim_{p\to\infty}\frac{1}{(1-x_p)^2+x_p^2} = \frac{1}{0+(1-0)^2} = 1;$$
$$\lim_{p\to\infty}\frac{p(p-1)}{\psi^2(p)}(1-x_p)^2 = \lim_{p\to\infty}\Big(1-\frac{1}{p}\Big) = 1;$$

and

$$\lim_{p\to\infty}\frac{p(p-1)}{\psi^2(p)}\frac{1-x_p}{1-x_p^{p-1}}\cdot x_p^p$$
$$=\lim_{p\to\infty}\frac{p-1}{p}\frac{p}{\psi(p)}\Big(1-\frac{\psi(p)}{p}\Big)^p\cdot\lim_{p\to\infty}\bigg(1-\Big(1-\frac{\psi(p)}{p}\Big)^{p-1}\bigg)^{-1}=1.$$

Note that, the second limit in the last equality is 1, which is the result of the formula $\lim_{u\to\infty}(1-\frac{1}{u})^u=e^{-1}$. For the first limit we have used:

$$\frac{p}{\psi(p)}\Big(1-\frac{\psi(p)}{p}\Big)^p=e^{\psi(p)}\cdot e^{p\ln\left(1-\frac{\psi(p)}{p}\right)}=e^{\psi(p)}\cdot e^{-p\left\{\frac{\psi(p)}{p}+\frac{\psi^2(p)}{2p^2}+\frac{\psi^3(p)}{3p^3}+\dots\right\}}$$
$$=e^{\psi(p)}\cdot e^{-\left\{\psi(p)+\frac{\psi^2(p)}{2p}+\frac{\psi^3(p)}{3p^2}+\dots\right\}}=e^{-\left\{\frac{\psi^2(p)}{2p}+\frac{\psi^3(p)}{3p^2}+\dots\right\}}.$$

The following equality follows by using the last relation and properties of the function $\psi(p)$: $\frac{\psi(p)}{p}\to 0$, $\frac{\psi^2(p)}{p}\to 0$ as $p\to\infty$:

$$\lim_{p\to\infty}\frac{p}{\psi(p)}\Big(1-\frac{\psi(p)}{p}\Big)^p=1.$$

Thus, we finally obtain

$$\lim_{p\to\infty}\frac{1}{\psi^2(p)}\frac{F_p(x_p,0)}{S_p(x_p,0)}=\frac{2}{ab},$$

that means that there exists p_0 such that for all $p>p_0$

$$\frac{1}{\psi^2(p)}\frac{F_p(x_p,0)}{S_p(x_p,0)}\le\frac{3}{ab}\ \text{ or }\ F_p(x_p,0)\le\frac{3\psi^2(p)}{ab}S_p(x_p,0).$$

Coming back to the inequality (3.1) we get

$$\frac{1}{C'_{a,b}}F_{K,p}(x_p,1,0)\le F_p(x_p,1,0)\le\frac{3\psi^2(p)}{ab}S_p(x_p,1,0),\quad\forall p>p_0.$$

By defining

$$C_0=\max\left\{\frac{3C'_{a,b}}{ab},\max_{p=2,4,\dots,p_0}\left\{\frac{F_p(x_p,1,0)}{\psi^2(p)S_p(x_p,1,0)}\right\}\right\},$$

we get

$$F_{K,p}(x_p,1,0)\le C_0\psi^2(p)S_p(x_p,1,0),\quad p=2,4,\dots.$$

Since $\psi^2(p)\le M(p)=\ln^2 p$, we finally get

$$\frac{F_{K,p}(x_p,1,0)}{S_p(x_p,1,0)}\le C_0M(p),\quad p=2,4,\dots.$$

That shows that for the function U_p the inequality of Theorem 2.3 is satisfied. Here the proof of Theorem 2.3 is completed.

4.1. Final remarks. The estimates on Theorems 3.4 and 3.5 provide the following corollary:

COROLLARY 4.1. *For any $\gamma > 0$, the sequence $\{C'_p\}_{p=2}^{\infty}$ satisfies the following condition*

$$\lim_{p\to\infty} p^{-\gamma} C'_p = 0.$$

Note, that $C_p \le C'_p$, therefore Corollary 4.1 is also true for C_p. In other words, Corollary 4.1 shows that the lower bound $M(p)$ cannot be proportional to any positive power of p.

Acknowledgments. The authors are thankful to Jean-Luc Guermond and Bojan Popov to helpful discussions and remarks.

Appendix A.

LEMMA A.1. *The following equality holds*

$$(p-1)y^{p-2}(y^i - x^i) - iy^{i-1}(y^{p-1} - x^{p-1}) \equiv \sum_{l=i}^{p-2}\sum_{j=0}^{i-1} y^{p-3+i-l}x^j(y^{l-j} - x^{l-j})(y-x),$$

for any $0 \le x, y \le 1$ and $p = 2, 4, \ldots$.

Proof.

$$\begin{aligned}
&(p-1)y^{p-2}(y^i - x^i) - iy^{i-1}(y^{p-1} - x^{p-1}) \\
&\quad= \sum_{l=0}^{p-2} y^{p-2}(y-x)\sum_{j=0}^{i-1} x^j y^{i-1-j} - \sum_{j=0}^{i-1} y^{i-1}(y-x)\sum_{l=0}^{p-2} x^l y^{p-2-l} \\
&\quad= (y-x)\left(\sum_{l=0}^{p-2}\sum_{j=0}^{i-1} x^j y^{p-3+i-j} - \sum_{j=0}^{i-1}\sum_{l=0}^{p-2} x^l y^{p-3+i-l}\right) \\
&\quad= (y-x)\left(\sum_{l=i}^{p-2}\sum_{j=0}^{i-1} x^j y^{p-3+i-l}(y^{l-j} - x^{l-j})\right) \\
&\quad= \sum_{l=i}^{p-2}\sum_{j=0}^{i-1} x^j y^{p-3+i-l}(y^{l-j} - x^{l-j})(y-x).
\end{aligned}$$

∎

LEMMA A.2. *The following inequality holds*

$$x^p < e^{-p(1-x)}, \quad 0 < x < 1,\, p > 0.$$

Proof. From the decomposition of e^u we get

$$e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \ldots + \frac{u^k}{k!} + \ldots < 1 + u + u^2 + \ldots + u^k = \frac{1}{1-u}, \quad 0 < u < 1,$$

or we can rewrite it as $1 - u < e^{-u}$. From this inequality by setting $x = 1 - u$, $u = 1 - x$ we obtain $x < e^{-(1-x)}$, $0 < x < 1$. By raising the last inequality to the power $p > 0$ we obtain the desired estimate. ∎

Lemma A.3. *The following inequality holds*

$$x > e^{-(1-x)\left(1+\frac{1-x}{2x}\right)}, \quad 0 < x < 1.$$

Proof. Since $\ln(1-u) = -\left(u + \frac{u^2}{2} + \frac{u^3}{3} + \dots\right)$, $0 < u < 1$, and

$$u + \frac{u^2}{2} + \frac{u^3}{3} + \dots < u\left(1 + \frac{u}{2} + \frac{u^2}{2} + \frac{u^3}{2} \dots\right) = u\left(1 + \frac{u}{2(1-u)}\right),$$

that is

$$\ln(1-u) > -u\left(1 + \frac{u}{2(1-u)}\right) \text{ or } 1-u > e^{-u\left(1+\frac{u}{2(1-u)}\right)}.$$

Here by substituting $1 - u = x$, $u = 1 - x$ we obtain the desired estimate. □

References.

[1] A.N. Brooks and T.J.R. Hughes. Streamline Upwind/Petrov–Galerkin formulations for convective dominated flows with particular emphasis on the incompressible Navier–Stokes equations. *Comput. Meth. Appl. Mech. Eng.*, 32:199–259, 1982.

[2] R. J. DiPerna. Measure-valued solutions to conservation laws. *Archive for Rational Mechanics and Analysis*, 88:223–270, 1985. 10.1007/BF00752112.

[3] T. J. R. Hughes and M. Mallet. A new finite element formulation for computational fluid dynamics. IV. A discontinuity-capturing operator for multidimensional advective-diffusive systems. *Comput. Methods Appl. Mech. Engrg.*, 58(3): 329–336, 1986. ISSN 0045-7825.

[4] T. J. R. Hughes, L. P. Franca, and M. Mallet. A new finite element formulation for computational fluid dynamics. I. Symmetric forms of the compressible Euler and Navier-Stokes equations and the second law of thermodynamics. *Comput. Methods Appl. Mech. Engrg.*, 54(2):223–234, 1986. ISSN 0045-7825.

[5] T. J. R. Hughes, M. Mallet, and A. Mizukami. A new finite element formulation for computational fluid dynamics. II. Beyond SUPG. *Comput. Methods Appl. Mech. Engrg.*, 54(3):341–355, 1986. ISSN 0045-7825.

[6] T.J.R. Hughes and T.E. Tezduyar. Finite element methods for first-order hyperbolic systems with particular emphasis on the compressible Euler equations. *Computer Methods in Applied Mechanics and Engineering*, 45(1-3):217–284, 1984.

[7] C. Johnson and A. Szepessy. On the convergence of a finite element method for a nonlinear hyperbolic conservation law. *Math. Comp.*, 49(180):427–444, 1987. ISSN 0025-5718. doi: 10.2307/2008320.

[8] C Johnson, U Nävert, and J Pitkäranta. Finite element methods for linear hyperbolic equations. *Comput. Methods Appl. Mech. Eng.*, 45:285–312, 1984.

[9] C. Johnson, A. Szepessy, and P. Hansbo. On the convergence of shock-capturing streamline diffusion finite element methods for hyperbolic conservation laws. *Mathematics of Computation*, 54(189):107–129, 1990.

[10] Claes Johnson and Jukka Saranen. Streamline diffusion methods for the incompressible Euler and Navier-Stokes equations. *Math. Comp.*, 47 (175):1–18, 1986. ISSN 0025-5718. doi: 10.2307/2008079. URL `http://dx.doi.org/10.2307/2008079`.

[11] M. Nazarov. Convergence of a residual based artificial viscosity finite element method. *Computers & Mathematics with Applications*, 64(4):616–626, 2013.

[12] A. Szepessy. Convergence of a shock-capturing streamline diffusion finite element method for a scalar conservation law in two space dimensions. *Mathematics of Computation*, 53:527–545, 1989.

[13] A. Szepessy. An existence result for scalar conservation laws using measure valued solutions. *Communications in Partial Differential Equations*, 14(10):1329–1350, 1989.

[14] A. Szepessy. Convergence of a streamline diffusion finite element method for scalar conservation laws with boundary conditions. *Mathematical Modelling and Numerical Analysis*, 25:749–782, 1991.

[15] T.E. Tezduyar and Y.J. Park. Discontinuity-capturing finite element formulations for nonlinear convection-diffusion-reaction equations. *Computer Methods in Applied Mechanics and Engineering*, 59(3):307 – 325, 1986. ISSN 0045-7825.