Robust High-Order Discontinuous Galerkin Methods: Why? and How?

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Nature is non-linear. Many fundamental physical principles such as conservation of mass, momentum, and energy are mathematically modeled by non-linear time dependent partial differential equations (PDEs). Such non-linear conservation laws describe a broad range of applications in science and engineering, e.g., the prediction of noise and drag from aircrafts, the build up and propagation of tsunamis in oceans, the behavior of gas clouds, and propagation of non-linear acoustic waves in different materials.

Solutions of non-linear conservation laws contain many complex phenomena such as discontinuities, singularities, and turbulence. These phenomena are all time dependent and feature multiple scales in space and time. Because of the multi-scale nature of such problems, no general framework for analytical solutions is available. Further, in many cases, experimental studies are either too costly, too dangerous, or impossible to perform. However, knowledge of such solutions is important for modern industry, science, and medicine. Therefore, we consider the numerical simulation of solutions to time dependent non-linear conservation laws as a key technology.

A wide variety of low-order and high-order numerical methods have been developed over many decades. There is, however, a balancing act when applying either type of method to a given problem:

- Low-order methods offer remarkable robustness but require a very large number of degrees of freedom (DOFs) to properly capture multi-scale non-linear phenomena.
- High-order methods offer great capabilities to accurately capture non-linear phenomena while requiring a moderate number of DOFs. However, they often lack robustness.

In general, when the grid resolution parameter, h, is small a high-order method will always outrun low-order variants due to the favorable asymptotic behavior of the discretization errors. However, the ability to have simulations with small h (i.e. well-resolved approximations) is typically only something that academia can afford.

In practice, marginally resolved simulations of non-linear conservation laws, such as the compressible Navier-Stokes equations or the ideal magnetohydrodynamics (MHD) equations, reveal that high-order methods are prone to aliasing instabilities. This can lead to total failure and breakdown of the algorithms. Aliasing issues are introduced and intensified by a combination of insufficient discrete integration precision, collocation of non-linear terms, polynomial approximations applied to rational functions, and, again, by insufficient grid resolution.

The aim of this talk is to discuss a remedy for such aliasing issues and present its connection to discrete variants of the product and chain rules. As such, we construct nodal high-order numerical methods for non-linear conservation laws that are entropy stable. To do so, we focus on particular approximate derivative operators used to mimic steps from the continuous well-posedness PDE analysis on the discrete level and enhance the robustness of the high-order numerics.