Assignment on iterative solution methods and preconditioning
Indefinite (saddle point) matrices

1 Introduction: The problem

We study the solution of systems of linear equations

\[ Au = b \]  

where \( A(n \times n) \) is a symmetric but indefinite matrix, i.e., \( A \) has both positive and negative (real) eigenvalues. We assume that \( A \) is nonsingular, i.e., there is no zero eigenvalue in its spectrum.

Such matrices arise in the context of many problems, for example in minimax optimal control with equality constraints, mixed finite element method discretizations of second order elliptic problems, linear elasticity for almost incompressible materials and the Stokes problem in fluid mechanics, Maxwell equations, incompressible miscible or immiscible problems of oil and water, parameter identification problems, domain decomposition methods (nonmatching grids, mortar elements etc.

The fact that \( A \) is indefinite requires special iterative solution methods. One such method is the so-called Minimal Residual method MINRES.

The test problem used to produce the matrices is as follows. A homogeneous elastic body occupies a rectangular domain of size \( 10^7 \times 4 \times 10^6 \) in meters and is subject to a vertical load on a part of the top boundary. The value of the Young modulus \( E \) is taken to be \( 4 \times 10^{11} Pa \), the Poisson ratio \( \nu \) is chosen as 0.2 or 0.5, and the body forces \( f \) are zero. Homogeneous Dirichlet boundary conditions are assumed at the bottom of the domain and homogeneous Neumann boundary conditions at the rest of the boundary.

The formulation of the continuous problem as a system of partial differential equations (referred to as the Lamé-Navier equations of elasticity) is as follows,

\[-2\mu \Delta u - \lambda \nabla (\nabla \cdot u) - \mu \nabla \times (\nabla \times u) = f \quad \text{in } \Omega \subset \mathbb{R}^2\]

with some appropriate boundary conditions. Here \( u = [u, v]^T \) is the displacement vector, \( \mu = \frac{E}{2(1 + \nu)} \) and \( \lambda = \mu \frac{2\nu}{1 - 2\nu} \) are the Lamé coefficients and \( E \) and \( \nu \) are Young’s modulus and Poisson’s ratio, respectively.

As can be seen, when \( \nu \to \frac{1}{2} \), then \( \lambda \to \infty \) and the problem becomes very sensitive to small perturbations of the displacement field \( u \). A usual approach to handle (nearly or fully)
incompressible materials is to first regularize the problem by introducing the scaled (hydrostatic) pressure \( p \) as an auxiliary variable,

\[
p = \frac{\lambda}{\mu} \nabla \cdot u
\]  

(3)

and consider the following coupled differential equation problem

\[
\begin{align*}
-2\mu \Delta u - \mu \nabla \times (\nabla \times u) - \mu \nabla p &= f \\
\mu \nabla \cdot u - \frac{\mu^2}{\lambda} p &= 0
\end{align*}
\]

(4)

Problem (4) is next discretized using a stable (Taylor-Hood) pair of finite elements, namely piece-wise quadratic finite element basis functions for the displacements and piece-wise linear finite element basis functions for the pressure. The linear system to be solved is now with a matrix \( A \) of saddle-point form

\[
A = \begin{bmatrix} K & B^T \\ B & -\rho \mu M \end{bmatrix},
\]

where \( K \) and \( M \) are symmetric and positive definite blocks, \( B \) is of full rank and the coefficient \( \rho = \frac{\mu}{\lambda} \) can be either positive or zero. The solution \( U \) of the system \( AU = b \) contains the displacements in \( 'x' \)-direction \( u \) (ordered first), then the displacements in \( 'y' \)-direction \( v \) (ordered second) and finally the additional discrete pressure variable \( p \).

2 Data

You have at your disposal files which contain two series of test data that correspond to Poisson’s ratio \( \nu = 0 \) for compressible materials and \( \nu = 0.5 \) for incompressible materials. The file-names are

- Elast_saddle_6043_0.2.mat Elast_saddle_6043_0.5.mat
- Elast_saddle_23513_0.2.mat Elast_saddle_23513_0.5.mat
- Elast_saddle_93283_0.2.mat Elast_saddle_93283_0.5.mat
- Elast_saddle_370883_0.2.mat Elast_saddle_370883_0.5.mat

The numbers 6043, 23513 etc indicate the size of the matrix \( A \). The values 0.2 and 0.5 indicate the value of the Poisson’s ratio for the corresponding test.

Each file contains the matrices \( A, K \) and the pressure mass matrix \( M \), the right-hand-side vector \( rhs \), \( \rho \) and \( \mu \). The files also include an array called \texttt{Node}, which contains the coordinates of the vertices of the corresponding finite element mesh.

3 Tasks

Problem (1) has to be solved using different iterative methods for the four sizes and iteration counts and performance have to be compared. For the tests you can use Matlab or any other programming environment you find suitable. The examples below are from Matlab.
1. Load consecutively the matrix and the right hand side vector from the files.
   Do some analysis of the matrices at hand. For instance, check the sparsity of the above
   matrices (spy) and their structure. For the smaller-sized matrices compute the complete
   spectra by using the MATLAB function eig and verify that they are indefinite. Observe
   that the eigenvalues of $A$ are contained in two intervals and the upper bound of the interval
   with the negative eigenvalue, as well as the lower bound of the interval with the positive
   eigenvalues approaches 0 as the size of the matrices grow.

2. Solve the systems using the unpreconditioned MINRES method, cf. e.g., [1].
   You can use the implementation of MINRES provided as a black box from some library.
   Alternatively, if you are curious how the MINRES algorithm looks like, you can use the
   included implementation minres0.m. The call to minres0 is as follows
   
   $\text{[it,U,resvec]} = \text{minres0}(A,rhs,\text{eps}_\text{minres})$
   
   where $A$ is the given matrix, $rhs$ is the right hand side vector and $\text{eps}_\text{minres}$ is the
   required stopping tolerance.

   For the numerical experiments choose it as $10^{-6}$ or $10^{-8}$.

   As output, we obtain the number of iterations $it$, the iteratively computed solution $U$ and
   a vector $\text{resvec}$ which contains the norm of the residual vector per iteration. Plotting
   $\text{resnorm}$ enables us to monitor the convergence history of the solution process.

   One way to visualize the solution is as follows:

   $$u = U(1:nnode, 1);$$
   $$v = U(nnode+1:2*nnode,1);$$
   $$p = U(2*nnode+1:end, 1);$$
   $$\text{figure}(1), \text{clf}, \text{plot}(\text{Node}(1,:),\text{Node}(2,:),'o')$$
   $$\text{hold}, \text{plot}(\text{Node}(1,:)+u',\text{Node}(2,:)+v','ro')$$

   Here $\text{nnode} = \text{size}(\text{Node}, 2)$ is the number of the nodes in the discretization mesh. As
   already explained, first 'nnode' elements in the solution vector are the displacements in
   'x'-direction, the second 'nnode' elements are the displacements in 'y'-direction, and the
   remaining elements correspond to the introduced pressure.

3. Solve the system by using a preconditioned MINRES.
   In this case the preconditioner must be symmetric and positive definite.

   As a preconditioner for $A$ use a block-diagonal preconditioner

   $$P_1 = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix},$$

   where $D_1$ has to be an approximation of $K$ and $D_2$ has to be an approximation of the
   (negative) Schur complement matrix $S_A = \rho \mu M + BK^{-1}B^T$.

   Perform the following tests:
(a) \( D_1 = K, D_2 = S_A \) (feasible only for small problem sizes);
(b) \( D_1 = K, D_2 = M \);
(c) \([U_1] = ichol(K), D_1 = U_1^TU_1, [U_2] = ichol(M), D_2 = U_2^TU_2\)

For the smallest size problem plot the spectrum of the preconditioned matrix \( P_1^{-1}K \), which you obtain from MATLAB by solving the generalized eigenvalue problem

\[ E_1 = eig(full(A), full(P_1)); \]

How are the eigenvalues clustered? Can you see how the clustering changes depending on how accurate \( D_1 \) and \( D_2 \) approximate \( K \) and \( S_A \)?

4. Solve the system by using a preconditioned GMRES.

Test with

\[ P_2 = \begin{bmatrix} D_1 & 0 \\ 0 & -D_2 \end{bmatrix} \quad \text{and} \quad P_3 = \begin{bmatrix} D_1 & 0 \\ B & -D_2 \end{bmatrix}. \]

Check the spectrum of \( P_s^{-1}A, \ s = 2, 3 \).

What is the difference? Which preconditioner is better? Which method would you recommend?

5. (Theoretical) Why the pressure mass matrix is a good approximation of the Schur complement \( S_A \)? Do a little theoretical study, a similar approximation is used for the stationary Stokes problem, cf. [2], Chapters 5 and 6.

Remark 1 Regarding the implementation: Do not form the preconditioners explicitly. Instead, provide a function that solves systems with those, for instance,

\[ [u, flag, relres, it, resvec] = minres(A, rhs, tol, maxit, @blockprec, ... \]

You can choose different incomplete Cholesky factorizations if you want to try.
Time separately the construction of the preconditioner (in case of ichol) and the solver.

4 Writing a report on the results

The report has to have the following issues covered:

1. Brief description of the problem, the methods and the expected behaviour.

2. Numerical experiments

Describe the experiments (iteration counts, plots of the residual history). How does the number of iterations grow with the size (expected and obtained computational complexity)? What is the condition number of these matrices (as a function of the size)? Is there a difference with respect to the problem parameter \( \nu \)? Add a discussion on the suggested preconditioners - are they good or not and why. Add any additional knowledge you may have about another preconditioner for this type of problems.
3. Conclusions. Are the preconditioners robust with respect to problem size (discretization parameter) and the problem parameter $\nu$?

A listing of the program code has to be attached to the report.

Success!

References
