Exercise 1
Consider the matrix of size \( n \geq 1 \)
\[
A_n(a) = \begin{bmatrix}
a_1 + a_3/2 & -a_3/2 & a_3/2 & -a_5/2 & \\
-a_3/2 & a_3 + a_5/2 & -a_5/2 & \\
& \ddots & \ddots & \ddots & \\
& & -a_n-1/2 & a_n-1 + a_{n+1}/2 \\
\end{bmatrix}, \quad a_t = a \left( \frac{t}{n+1} \right),
\]
(1)
where \( a : [0,1] \to \mathbb{R} \) is a positive function (choose now a positive function).

1. Write a Matlab procedure for building the matrix \( A_n(a) \) with fixed function \( a \) (the one you have chosen) and \( n \geq 1 \) as a parameter.

2. Apply CG to a system \( A_n(a)x = b \) (choose your data vector \( b \)) for various \( n \) (\( n = 16, 32, 64, 128, \ldots \)) and fixed tolerance \( \epsilon = 10^{-6} \):
   - What do you see concerning the number of required iterations for convergence?

3. Compute \( \lambda_{\min}(A_n(a)) \) for various \( n \) (\( n = 16, 32, 64, 128, \ldots \)):
   - What do you see?
   - Given the fact that \( \lambda_{\min}(A_n(a)) \approx \alpha(a)/n^{s} \), \( \alpha(a) \), \( s \) positive numbers, "invent" a numerical procedure for computing \( s \) and then \( \alpha(a) \).

4. Compare and discuss the results in Item 2 and the value \( s \) that you numerically detected at Item 3.

Exercise 2 Consider the matrix of size \( n \geq 1 \)
\[
A_n(a) = \begin{bmatrix}
a_1 + a_3/2 & -a_3/2 & a_3/2 & -a_5/2 & \\
-a_3/2 & a_3 + a_5/2 & -a_5/2 & \\
& \ddots & \ddots & \ddots & \\
& & -a_n-1/2 & a_n-1 + a_{n+1}/2 \\
\end{bmatrix}, \quad a_t = a \left( \frac{t}{n+1} \right)
\]
(2)
where \( a : [0,1] \to \mathbb{R} \) is a positive function.
1. Prove that the matrix in (2) is the approximation of the boundary value problem
\[
\begin{align*}
-\left( a(x)u_x \right)_x &= f(x) & \text{on } \Omega = (0,1), \\
\text{Dirichlet B.C. on } \partial \Omega,
\end{align*}
\]
by centered Finite Differences of precision order 2 and step-size \( h = (n+1)^{-1} \).

2. Prove that \( A_n(a) \) as in (1) is positive definite.

3. Prove that \( P_n^{-1}A_n(a) \) is similar to a positive definite matrix with \( P_n = A_n(1) \).

4. Prove that any eigenvalue of \( P_n^{-1}A_n(a) \) belongs to \([a_*, a^*]\) with 
   \[ a_* = \min_{x \in [0,1]} a(x) \]
   and 
   \[ a^* = \max_{x \in [0,1]} a(x) \].

**Exercise 3**

Consider the same matrix \( A_n(a) \) of size \( n \geq 1 \) as defined in (1).

1. Consider the matrix \( P_n = A_n(1) \) and prove that it coincides with \( T_n(2 - 2 \cos(s)) \) (i.e. the \( n \)-by-\( n \) Toeplitz matrix generated by the cosine polynomial \( 2 - 2 \cos(s) \)).

2. Apply CG with preconditioner \( P_n \) to a system \( A_n(a)x = b \) (choose your data vector \( b \)) for various \( n \) (\( n = 16, 32, 64, 128, \ldots \)) and fixed tolerance \( \epsilon = 10^{-6} \):
   - What do you see concerning the number of required iterations for convergence?

3. Compute \( \lambda_{\min}(P_n) \) for various \( n \) (\( n = 16, 32, 64, 128, \ldots \)):
   - What do you see?
   Given the fact that \( \lambda_{\min}(P_n) \approx \alpha/n^s \), \( \alpha, s \) positive numbers, "invent" a numerical procedure for computing \( s \) and then \( \alpha \).

4. Compute \( \lambda_{\min}(P_n^{-1}A_n(a)) \), \( \lambda_{\max}(P_n^{-1}A_n(a)) \) for various \( n \) (\( n = 16, 32, 64, 128, \ldots \)):
   - Which is the relation that you see between the computed values \( \lambda_{\min}(P_n^{-1}A_n(a)) \) and \( \min_{x \in [0,1]} a(x) \)?
   - Which is the relation that you see between the computed values \( \lambda_{\max}(P_n^{-1}A_n(a)) \) and \( \max_{x \in [0,1]} a(x) \)?

5. Compare and discuss the results in Item 2 and in Item 4.

**Exercise 4**

Consider the matrix of size \( n \geq 1 \) whose \( j \)-th row is defined as
\[
(0, \ldots, 0, a_{j-1}, -2(a_{j-1} + a_j), a_{j-1} + 4a_j + a_{j+1}, -2(a_j + a_{j+1}), a_{j+1}, 0, \ldots, 0), \tag{4}
\]
with \((j, j)\) position given by \( a_{j-1} + 4a_j + a_{j+1} \) and where \( a_t = a \left( \frac{t}{n+1} \right) \), \( a : [0,1] \to \mathbb{R} \) as in Exercise 1.
1. Write a Matlab procedure for building the matrix $A_n(a)$ with fixed function $a$ (the one you have chosen) and $n \geq 1$ as a parameter.

2. Apply CG to a system $A_n(a)x = b$ (choose your data vector $b$) for various $n$ ($n = 16, 32, 64, 128, \ldots$) and fixed tolerance $\epsilon = 10^{-6}$:
   - What do you see concerning the number of required iterations for convergence?

3. Compute $\lambda_{\text{min}}(A_n(a))$ for various $n$ ($n = 16, 32, 64, 128, \ldots$):
   - What do you see?
   Given the fact that $\lambda_{\text{min}}(A_n(a)) \approx \alpha(a)/n^s$, $\alpha(a), s$ positive numbers, "invent" a numerical procedure for computing $s$ and then $\alpha(a)$.

4. Compare and discuss the results in Item 2 and the value $s$ that you numerically detected at Item 3.