Multigrid methods
Algebraic Multigrid methods
Algebraic Multilevel Iteration methods
Residual correction

\[ Ax = b, \quad x_{exact}, \quad e^{(k)} = x_{exact} - x^{(k)} \]

\[ r^{(k)} = b - Ax^{(k)} \]

Residual equation: \[ Ae^{(k)} = r^{(k)} \]

Residual correction: \[ x^{(k+1)} = x^{(k)} + e^{(k)} \]

Recall: \[ x^{(k+1)} = x^{(k)} + C^{-1}(b - Ax^{(k)}) \]

Error propagation: \[ e^{(k+1)} = (I - C^{-1}A)e^{(k)} \]
Run Jacobi demo...

student/NLA/Demos/Module3/L5
High and low frequencies - nonsmooth, smooth
Main idea: R. Fedorenko (1961), N.S. Bakhvalov (1966)

Reduce the error $e^{(k)} = x_{exact} - x^{(k)}$ on the given (fine) grid by successive residual corrections on a hierarchy of (nested) coarser grids.
Some contributors:

<table>
<thead>
<tr>
<th>Years</th>
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<th>AMG matches</th>
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</table>

Archi Brandt  Jan Mandel  Tom Manteiffel
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An example

PDE: \(-\Delta u = 20 e^{-10((x-0.5)^2+(y-0.5)^2)}\) in \(\Omega = (0, 1) \times (0, 1)\)

\[u = 0\] on \(\partial\Omega\)

Uniform grid with mesh size \(h\), five-point finite difference.

Solution with \(h^{-1} = 50\)

Solution with \(h^{-1} = 25\)
An idea

Fine grid (system to solve):

\[ A \mathbf{u} = \mathbf{b} . \]

Coarse grid (auxiliary system):

\[ A_C \mathbf{u}_C = \mathbf{b}_C . \]

\( \mathbf{u}_C \) may be computed and prolongated (by interpolation) on the fine grid:

\[ \mathbf{u}^{(1)} = p \mathbf{u}_C \]

\( \mathbf{u}^{(1)} \) may serve as initial approximation, i.e., one solves

\[ A (\mathbf{u}^{(1)} + \mathbf{x}) = \mathbf{b} \quad \text{or} \quad A \mathbf{x} = \mathbf{b} - A p A_C^{-1} \mathbf{b}_C . \]
Error on the fine grid after interpolation

\[
\frac{\| \mathbf{u} - \mathbf{u}^{(1)} \|}{\| \mathbf{u} \|} = 0.0019
\]
Let us repeat

\[ \mathbf{A} (\mathbf{u}^{(1)} + \mathbf{x}) = \mathbf{b} \quad \text{or} \quad \mathbf{A} \mathbf{x} = \mathbf{b} - \mathbf{A} p \mathbf{A}^{-1} \mathbf{b}_C = \mathbf{r}^{(1)}. \]

(1) Restrict on the coarse grid:

\[ \mathbf{r}_C = r \mathbf{r}^{(1)}. \]

(2) Solve on the coarse grid:

\[ \mathbf{x}_C^{(2)} = \mathbf{A}_C^{-1} \mathbf{r}_C. \]

(3) Prolongate:

\[ \mathbf{x}^{(2)} = p \mathbf{x}_C^{(2)}; \]

\[ \mathbf{u}^{(2)} = \mathbf{u}^{(1)} + \mathbf{x}^{(2)}. \]
Still working?

Error on the fine grid after interpolation

Repeating the process . . .

\[ \frac{\|u - u^{(1)}\|}{\|u\|} = 0.0019 \]

\[ \frac{\|u - u^{(2)}\|}{\|u\|} = 0.0018 \]
Error controlled through residual

Initial residual (r.h.s.)

After coarse grid correction

\[
\frac{\|b - Ap A_C^{-1} b_C\|}{\|b\|} = 0.7142
\]
Assume (for simplicity) that $\mathbf{b}_C = r \mathbf{b}$.

One has

$$u - u^{(1)} = u - p A_C^{-1} r \mathbf{b}$$

$$= (I - p A_C^{-1} r A) u,$$

$$u - u^{(2)} = (I - p A_C^{-1} r A)^2 u,$$

etc.

Similarly

$$r^{(1)} = \mathbf{b} - A p A_C^{-1} r \mathbf{b}$$

$$= (I - A p A_C^{-1} r) r^{(0)}.$$

$p A_C^{-1} r$ has rank $n_C$ →

$$\rho (I - A p A_C^{-1} r) = \rho (I - p A_C^{-1} r A) \geq 1.$$
Smoother enters the scene

$\mathbf{u} - \mathbf{u}^{(1)}$ and $\mathbf{r}^{(1)}$ very oscillatory

$\rightarrow$ improve $\mathbf{u}^{(1)}$ with a simple iterative method, efficient in smoothing the error & residual.

**Example:** symmetric Gauss-Seidel (SGS)

\[
L \mathbf{u}^{(1+1/2)} = \mathbf{b} - (A - L) \mathbf{u}^{(1)}, \quad (L = \text{low}(A))
\]

\[
U \mathbf{u}^{(2)} = \mathbf{b} - (A - U) \mathbf{u}^{(1+1/2)} . \quad (U = \text{upp}(A))
\]

Same as

\[
\mathbf{u}^{(2)} = \mathbf{u}^{(1)} + M^{-1} \mathbf{r}^{(1)}, \quad M = L D^{-1} U \quad (D = \text{diag}(A))
\]

Thus:

\[
\mathbf{u} - \mathbf{u}^{(2)} = (I - M^{-1} A) (\mathbf{u} - \mathbf{u}^{(1)})
\]

\[
\mathbf{r}^{(2)} = (I - A M^{-1}) \mathbf{r}^{(1)}
\]

One may repeat: \[
\mathbf{r}^{(m+1)} = (I - A M^{-1})^{m} \mathbf{r}^{(1)} .
\]
Smoothing effect

Residual after CG correction

Adding 1 SGS step

Adding 3 SGS steps

Adding 8 SGS steps

\[ \frac{\| r \|}{\| b \|} = 0.7142 \]

\[ \frac{\| r \|}{\| b \|} = 0.0039 \]

\[ \frac{\| r \|}{\| b \|} = 0.0018 \]

\[ \frac{\| r \|}{\| b \|} = 0.0012 \]
Adding now a CG correction

\[
\frac{||r||}{||r_{\text{previous}}||} = 0.746
\]
Adding now a CG correction

\[ \frac{\| r \|}{\| r_{\text{previous}} \|} = 0.746 \]

... and again 1 SGS step

\[ \frac{\| r \|}{\| r_{\text{previous}} \|} = 0.0155 \]
What we learned

For each coarse grid correction:

\[ u - u^{(m+1)} = (I - p A_C^{-1} r A) (u - u^{(m)}) . \]

Cannot work alone because \( \rho \left( I - p A_C^{-1} r A \right) \geq 1 . \)

For each smoothing step

\[ u - u^{(m+1)} = (I - M^{-1} A) (u - u^{(m)}) . \]

Not efficient alone because \( \rho \left( I - M^{-1} A \right) \approx 1 . \)

However

\[
\rho \left( (I - M^{-1} A) (I - p A_C^{-1} r A) (I - M^{-1} A) \right) \ll 1
\]

Rmk: if \( A = A^T \), we assume \( M = M^T \).
1D Interpolation (Prolongation)

- Values at points on the coarse grid map unchanged to the fine grid
- Values at fine-grid points NOT on the coarse grid are the averages of their coarse-grid neighbors
The prolongation operator (1D)

• We may regard $I_{2h}^h$ as a linear operator from $\mathbb{R}^{N/2-1}$ to $\mathbb{R}^{N-1}$

• e.g., for $N=8$,

\[
\begin{pmatrix}
1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
1 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
1 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
1 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
1 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
\end{pmatrix}_{7x3}
\begin{pmatrix}
v_{1h}^2 \\
v_{2h}^2 \\
v_{3h}^2 \\
v_{4h}^2 \\
v_{5h}^2 \\
v_{6h}^2 \\
v_{7h}^2 \\
\end{pmatrix}_{3x1}
= \begin{pmatrix}
v_{1h} \\
v_{2h} \\
v_{3h} \\
v_{4h} \\
v_{5h} \\
v_{6h} \\
v_{7h} \\
\end{pmatrix}_{7x1}
\]

• $I_{2h}^h$ has full rank, and thus null space $\{\phi\}$
1D Restriction by injection

• Mapping from the fine grid to the coarse grid:

\[ I_{h}^{2h} : \Omega^h \rightarrow \Omega^{2h} \]

• Let \( v^h, v^{2h} \) be defined on \( \Omega^h, \Omega^{2h} \). Then

\[ I_{h}^{2h} v^h = v^{2h} \]

where \( v_{i}^{2h} = v_{2i}^h \).
1D Restriction by full-weighting

- Let $v^h$, $v^{2h}$ be defined on $\Omega^h$, $\Omega^{2h}$. Then

$$I^2_h v^h = v^{2h}$$

where

$$v_i^{2h} = \frac{1}{4}(v_{2i-1}^h + 2v_{2i}^h + v_{2i+1}^h)$$
The restriction operator $R$ (1D)

- We may regard $I_h^{2h}$ as a linear operator from $\mathbb{R}^{N-1} \rightarrow \mathbb{R}^{N/2-1}$

- e.g., for $N=8$,

$$
\begin{pmatrix}
1/4 & 1/2 & 1/4 \\
1/4 & 1/2 & 1/4 \\
1/4 & 1/2 & 1/4 \\
\end{pmatrix}
\begin{pmatrix}
v_1^h \\
v_2^h \\
v_3^h \\
v_4^h \\
v_5^h \\
v_6^h \\
v_7^h \\
\end{pmatrix}
= 
\begin{pmatrix}
v_1^{2h} \\
v_2^{2h} \\
v_3^{2h} \\
\end{pmatrix}
$$

- $I_h^{2h}$ has rank $\sim \frac{N}{2}$, and thus $\dim(\text{NS}(R)) \sim \frac{N}{2}$
Multilevel preconditioning methods: MG

Procedure $MG$: $u^{(k)} \leftarrow MG \left( u^{(k)}, f^{(k)}, k, \{\nu^{(k)}_j \}_{j=1}^k \right)$;

if $k = 0$, then solve $A^{(0)} u^{(0)} = f^{(0)}$ exactly or by smoothing,
else

$u^{(k)} \leftarrow S_1^{(k)} (u^{(k)}, f^{(k)})$, perform $s_1$ pre-smoothing steps,

Correct the residual:
$r^{(k)} = A^{(k)} u^{(k)} - f^{(k)}$; form the current residual,
$r^{(k-1)} \leftarrow R \left( r^{(k)} \right)$, restrict the residual on the next coarser grid,
$e^{(k-1)} \leftarrow MG \left( 0, r^{(k-1)}, k - 1, \{\nu^{(k-1)}_j \}_{j=1}^{k-1} \right)$;
$e^{(k)} \leftarrow P \left( e^{(k-1)} \right)$; prolong the error from the next coarser to the current grid,
$u^{(k)} = u^{(k)} - e^{(k)}$; update the solution,
$u^{(k)} \leftarrow S_2^{(k)} (u^{(k)}, f^{(k)})$, perform $s_2$ post-smoothing steps.

endif

end Procedure $MG$
One MG step (V-cycle)

The MG W-cycle
Nested iteration

**Procedure** *NI*: \( u^{(\ell)} \leftarrow NI \left( u^{(0)}, \{ f^{(k)} \}_{k=1}^{\ell}, \ell, \{ \nu^{(k)} \}_{k=1}^{\ell} \right) \);

\[ u^{(0)} = A^{(0)}^{-1} f^{(0)}, \]

for \( k = 1 \) to \( \ell \) do

\[ u^{(k)} = \mathcal{P} \left( u^{(k-1)} \right); \]

\[ u^{(k)} \leftarrow MG \left( u^{(k)}, f^{(k)}, k, \{ \nu^{(k)}_{j} \}_{j=1}^{k} \right); \]

endfor

end Procedure *NI*

The so-called *full MG* corresponds to **Procedure** *NI*\((\cdot, \cdot, \ell, \{1, 1, \ldots, 1\})\)

The full MG (*V-cycle*)
A compact formula presenting the MG procedure in terms of a recursively defined iteration matrix:
(i) Let $M^{(0)} = 0$,
(ii) For $k = 1$ to $\ell$, define
\[
M^{(k)} = S^{(k)} s_2 \left( A^{(k)}^{-1} - P^{k}_{k-1} (I - M^{(k-1)})^{-1} R^{k}_{k-1} \right) A^{(k)} S^{(k)} s_1,
\]
where $S^{(k)}$ is a smoothing iteration matrix (assuming $S_1$ and $S_2$ are the same), $R^{k}_{k-1}$ and $P^{k}_{k-1}$ are matrices which transfer data between two consecutive grids and correspond to the restriction and prolongation operators $R$ and $P$, respectively, and $\nu = 1$ and $\nu = 2$ correspond to the $V$- and $W$-cycles.

It turns out that in many cases the spectral radius of $M^{(\ell)}$, $\rho (M^{(\ell)})$, is independent of $\ell$, thus the rate of convergence of the NI method is optimal. Also, a mechanism to make the spectral radius of $M^{(\ell)}$ smaller is to choose $s_1$ and $s_2$ larger. The price for the latter is, clearly, a higher computational cost.
MG ingredients

- smoothers (many different)
  - Jacobi, weighted Jacobi ($\omega_{\text{diag}}(A)$), GS, SOR, SSOR, SPAI
- restriction and prolongation operators
- coarse level matrix (approximation properties)
MG: Rate of convergence and computational complexity

Let one Work Unit (WU) be the cost of one relaxation sweep on the fine-grid.
– Ignore the cost of restriction and interpolation (typically about 20% of the total cost).
– Consider a V-cycle with 1 pre-smoothing and 1 post-smoothing sweep.
– In $d$-dimensions, each coarse grid has about $2^{-d}$ the number of points as the finer grid. – Cost of V-cycle (in WU):

\[
2(1 + 2^{-d} + 2^{-2d} + + 2^{-3d} + \cdots + 2^{-\ell d}) \leq \frac{2}{1 - 2^{-d}}.
\]

– Total storage:

\[
2N^d (1 + 2^{-d} + 2^{-2d} + + 2^{-3d} + \cdots + 2^{-\ell d}) \leq \frac{2N^d}{1 - 2^{-d}}.
\]
Algebraic Multigrid
C-AMG coarsening

- select C-pt with maximal measure
- select neighbors as F-pts
- update measures of F-pt neighbors
C-AMG coarsening

- select C-pt with maximal measure
- select neighbors as F-pts
- update measures of F-pt neighbors
C-AMG coarsening

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- select neighbors as F-pts
- update measures of F-pt neighbors
C-AMG coarsening

- select C-pt with maximal measure
- select neighbors as F-pts
- update measures of F-pt neighbors
C-AMG coarsening

1. Select C-pt with maximal measure.
2. Select neighbors as F-pts.
C-AMG coarsening

- select C-pt with maximal measure
- select neighbors as F-pts
- update measures of F-pt neighbors
C-AMG coarsening

- select C-pt with maximal measure
- select neighbors as F-pts
- update measures of F-pt neighbors
C-AMG coarsening

→ select C-pt with maximal measure

→ select neighbors as F-pts

→ update measures of F-pt neighbors
C-AMG coarsening

- select C-pt with maximal measure
- select neighbors as F-pts
- update measures of F-pt neighbors
C-AMG coarsening is inherently sequential

- select C-pt with maximal measure
- select neighbors as F-pts
- update measures of F-pt neighbors
AMG demo ...

AMG: The ideal prolongation and restriction

Reference: Wiesner, Tuminaro, Wall, Gee
Multigrid transfers for nonsymmetric systems based on Schur complements and Galerkin projections, NLA, 2013

AMG and the Schur complement

\[
\begin{pmatrix}
A_{ff} & A_{fc} \\
A_{cf} & A_{cc}
\end{pmatrix}
\begin{pmatrix}
x_f \\
x_c
\end{pmatrix}
= 
\begin{pmatrix}
b_f \\
b_c
\end{pmatrix}.
\]

Assuming \(A_{ff}\) to be invertible, \(A\) has the corresponding LDU decomposition

\[
\begin{pmatrix}
A_{ff} & A_{fc} \\
A_{cf} & A_{cc}
\end{pmatrix}
= 
\begin{pmatrix}
I & 0 \\
A_{cf}A_{ff}^{-1} & I
\end{pmatrix}
\begin{pmatrix}
A_{ff} & 0 \\
0 & S
\end{pmatrix}
\begin{pmatrix}
I & A_{ff}^{-1}A_{fc} \\
0 & I
\end{pmatrix}
\]

where \(S = A_{cc} - A_{cf}A_{ff}^{-1}A_{fc}\) and is referred to as the Schur complement.
Define

$$R^{opt} = (-A_{cf} A_{ff}^{-1} I), \quad P^{opt} = \begin{pmatrix} -A_{ff}^{-1} A_{fc} \\ I \end{pmatrix} \quad \text{and} \quad \hat{I} = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$ 

One can easily verify that $S = R^{opt} A P^{opt}$,

$$\begin{pmatrix} I & 0 \\ A_{cf} A_{ff}^{-1} & I \end{pmatrix}^{-1} = \begin{pmatrix} \hat{I}^T \\ R^{opt} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & A_{ff}^{-1} A_{fc} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} \hat{I} & \hat{P}^{opt} \end{pmatrix}.$$

Application of the inverses of the three operators in the exact factorization is equivalent to restriction at the $c$-points, followed by solution of two systems: $A_{ff}$ which can be interpreted as relaxation and $R^{opt} A P^{opt}$ which is the coarse correction. Finally, the coarse correction is interpolated and added to the relaxation solution. As this procedure is exact, it converges in one iteration.
Further work:
how to approximate $R^{opt}$, $P^{opt}$ and $S$, or rather the coarse correction $R^{opt}A^{opt}P^{opt}$, which is nothing but $A_{cf}A_{ff}^{-1}A_{fc}$.

We enter the full block factorized preconditioning framework, that can be seen as purely algebraic and not related to MG.
Algebraic Multilevel Iteration Methods (AMLI)

The so-called AMLI methods have been developed by Owe Axelsson and Panayot Vassilevski in a series of papers between 1989 and 1991. These methods were originally developed for elliptic problems and spd matrices, and are the first regularity-free optimal order preconditioning methods.

Sequence of matrices \( \{ A^{(k)} \}_{k=k_0}^\ell \)

\[ N_{k_0} \subset N_{k_0+1} \subset \ldots \subset N_\ell \]

\[ A^{(k)} = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix} \}

\[ N_k \setminus N_{k-1} \]

\[ N_{k-1} \]
$A^{(k)}$ has to approximate $S_{A^{(k+1)}}$ in some way. For instance,

$$A^{(k)} = A^{(k+1)}_{22} - A^{(k+1)}_{21}B^{(k+1)}_{11}A^{(k+1)}_{12}.$$ 

where $B^{(k+1)}_{11}$ is some sparse, positive definite, nonnegative and symmetric approximation of $A^{(k+1)-1}_{11}$. How to split $N_{k+1}$ into two parts: the order $n_k$ of the matrices $A^{(k)}$ should decrease geometrically:

$$\frac{n_{k+1}}{n_k} = \rho_k \geq \rho > 1.$$
\[ M^{(k_0)} = A^{(k_0)}, \]
\[
\text{for } k = k_0, k_0 + 1, \ldots \ell - 1
\]
\[ M^{(k+1)} = \begin{bmatrix}
A_{11}^{(k+1)} & 0 \\
A_{21}^{(k+1)} & \tilde{S}(k)
\end{bmatrix}
\begin{bmatrix}
I_1^{(k+1)} & A_{11}^{(k+1)\,-1} A_{12}^{(k+1)} \\
0 & I_2^{(k+1)}
\end{bmatrix},
\]
\text{endfor}

where \( \tilde{S}(k) \) can be, for instance:
\[ \tilde{S}(k) = A^{(k)} \left[ I - P_\nu(M^{(k)\,-1} A^{(k)}) \right]^{-1}, \]

\( P_\nu(t) \) denotes a polynomial of degree \( \nu \).

We could use some other way of stabilization.
**Forward sweep:**

Solve
\[
\begin{bmatrix}
A_{11}^{(k+1)} & 0 \\
A_{21}^{(k+1)} & \tilde{S}^{(k)}
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix} =
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix},
\]

i.e.

\(F1\) \(w_1 = A_{11}^{(k+1)-1} y_1\),

\(F2\) \(w_2 = \tilde{S}^{(k)-1} (y_2 - A_{21}^{(k+1)}w_1)\).

**Backward sweep:**

Solve
\[
\begin{bmatrix}
I_{1}^{(k+1)} & A_{11}^{(k+1)-1} A_{12}^{(k+1)} \\
0 & I_{2}^{(k+1)}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix},
\]

i.e.

\(B1\) \(x_2 = w_2\),

\(B2\) \(x_1 = w_1 - A_{11}^{(k+1)-1} A_{12}^{(k+1)} x_2\).
Procedure AMLI: \[ u^{(k)} \leftarrow AMLI \left( f^{(k)}, k, \nu_k, \{a_j^{(k)}\}_{j=0}^{\nu_k} \right); \]

\[ [f_1^{(k)}, f_2^{(k)}] \leftarrow f^{(k)}, \]
\[ w_1^{(k)} = B_{11}^{(k)} f_1^{(k)}, \]
\[ w_2^{(k)} = f_2^{(k)} - A_{21}^{(k)} w_1^{(k)}, \]
\[ k = k - 1, \]

if \( k = 0 \) then \( u_2^{(0)} = A^{(0)} w_2^{(1)} \), solve on the coarsest level exactly;
else
\[ u_2^{(k)} \leftarrow AMLI \left( a_{\nu_k}^{(k)} w_2^{(k)}, k, \nu_k, \{a_j^{(k)}\}_{j=0}^{\nu_k} \right); \]

for \( j = 1 \) to \( \nu_k - 1 \):
\[ u_2^{(k)} \leftarrow AMLI \left( A^{(k)} u_2^{(k)} + a_{\nu_k-j}^{(k)} w_2^{(k)}, k, \nu_k, \{a_j^{(k)}\}_{j=0}^{\nu_k} \right); \]
endfor
endif

\[ k = k + 1, \]
\[ u_1^{(k)} = w_1^{(k)} - B_{11}^{(k)} A_{12}^{(k)} u_2^{(k)}, \]
\[ u^{(k)} \leftarrow [u_1^{(k)}, u_2^{(k)}] \]
end Procedure AMLI
One AMLI step ($V$-cycle)

$\nu$-fold $W$-cycle, $[1, 1, 3, 1]$
AMLI: Computational complexity

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<th>Polynomial degree/inner iterations</th>
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<td>( \nu )</td>
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<td>...</td>
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<tr>
<td>( l-m+1 )</td>
<td>( 1 )</td>
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<tr>
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<td>( \nu )</td>
</tr>
<tr>
<td>( l-m-1 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( l-2m+1 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\nu & = C(n_\ell + \cdots + n_{\ell-\mu}) \\
    & + C\nu(n_{\ell-\mu-1} + \cdots + n_{\ell-2\mu-1}) \\
    & + C\nu^2(n_{\ell-2\mu-2} + \cdots + n_{\ell-3\mu-2}) \\
    & \quad \quad \quad + \cdots \\
    & \leq Cn_\ell \left[ 1 + \frac{1}{\rho} + \cdots + \left( \frac{1}{\rho} \right)^\mu \right] \frac{1}{1 - \nu \rho^{-(\mu+1)}},
\end{align*}
\]

where \( 1 < \rho \leq \rho_k = \frac{n_{k+1}}{n_k} \), \( k = 0, 1, \ldots \ell-1 \). Hence

\[
\nu < \rho^{\mu+1}
\]