Networks, Graphs and Matrices

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Networks

Network: a collection of items, called vertices, with connections between then, called edges.

Example: social network. The vertices represent persons and the edges relations, e.g. friendship relations

Networks are studied in mathematical graph theory
Connectedness properties of graphs:

- Are there subgraphs that are only loosely connected?
- Can a graph be divided into two separate graphs by breaking only a few edges?

For large graphs those questions are difficult to answer by visual inspection. Connectedness can be computed: eigenvalues and eigenvectors of the graph Laplacian matrix. This is spectral graph theory

Graphs

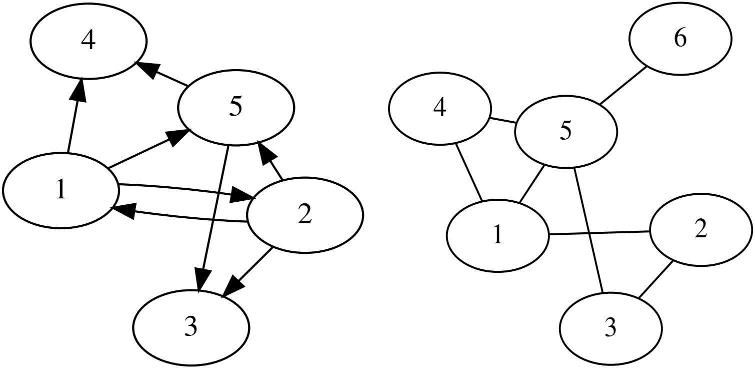
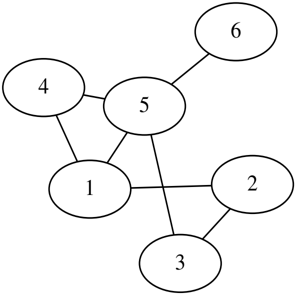


Figure: Directed and undirected graph.

Adjacency matrix

Adjacency matrix A of an undirected graph:

$$a_{ij} = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j, \text{ and there is an edge between vertices } i \text{ and } j. \end{cases}$$

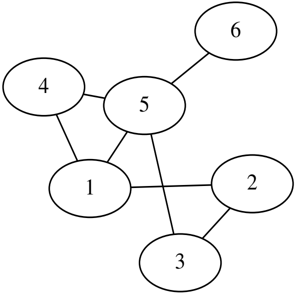


$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Degrees

Degree of a vertex: the number of edges

The degrees of the vertices can be computed by the matrix-vector multiplication $d = Ae$, where e is a vector of all ones:



$$d = Ae = \begin{pmatrix} 3 \\ 2 \\ 2 \\ 2 \\ 4 \\ 1 \end{pmatrix}.$$

d is called the degree vector.

Connectedness and reducibility

A subgraph is a subset of the vertices and corresponding edges.

An undirected graph is called connected if there is no subgraph isolated from the rest of the graph.

Connectedness is equivalent to the concept of irreducibility of a matrix.

A symmetric matrix A is called reducible if there exists a permutation matrix P such that PAP^T is block-diagonal,

$$PAP^T = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

If there is no such permutation, then the matrix A is called irreducible.

Proposition *An undirected graph is connected if and only if its adjacency matrix is irreducible.*

Connectedness

Connectedness properties of graphs:

- Are there subgraphs that are only loosely connected?
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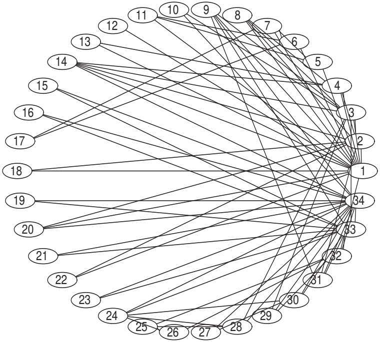


Figure: A rendition of the karate club graph.

Spectral graph partitioning

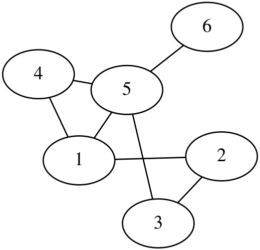
Degree matrix D: diagonal matrix with the degree vector on the diagonal:

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & d_{n-1} & 0 \\ 0 & \cdots & 0 & 0 & d_n \end{pmatrix}.$$

Laplacian matrix for an undirected graph with adjacency matrix A and degree matrix D:

$$L = D - A.$$

Laplacian



$$\begin{pmatrix} 3 & -1 & 0 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ -1 & 0 & -1 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The sum of all elements in each row is equal to zero,

$$Le = De - Ae = 0,$$

so e is an eigenvector corresponding to the eigenvalue 0.

Second smallest eigenvalue

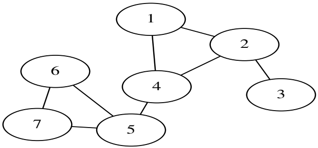
Assume for the moment that we have a disconnected graph with two subgraphs with adjacency matrix and Laplacian

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$$

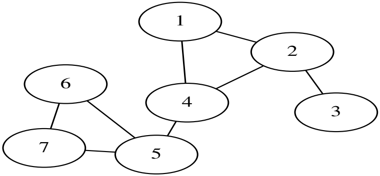
Then the Laplacian has a double eigenvalue equal to zero and eigenvectors

$$v_1 = \begin{pmatrix} e \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ e \end{pmatrix}$$

Connect the subgraphs:



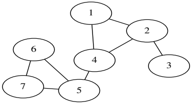
Adjacency matrix

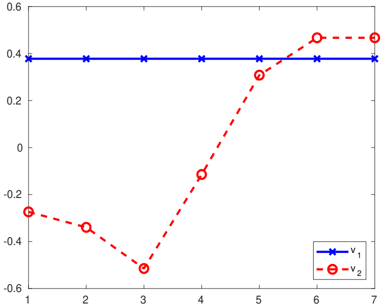


$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Eigenvectors v_1 and v_2

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 0.34$





Eigenvectors of symmetric matrices are orthogonal: $v_1^T v_2 = 0$, so v_2 must have positive and negative entries

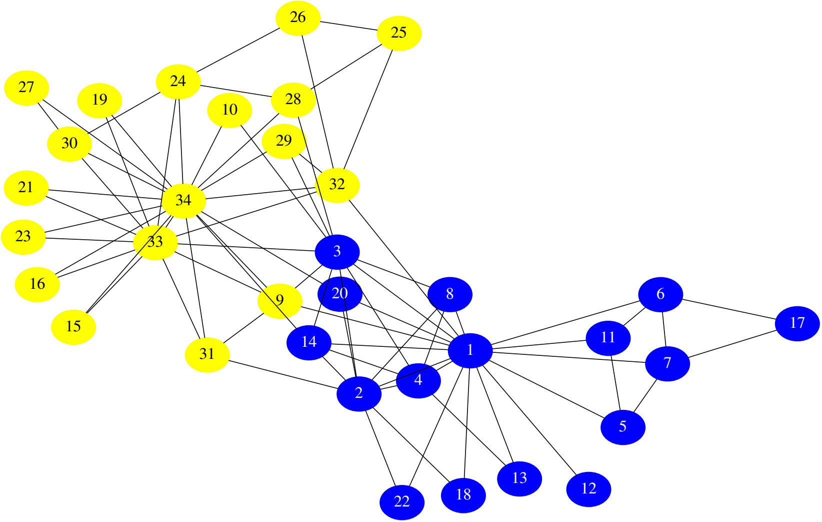
The values of the components of v_2 show how the vertices are located: 3 and 7 are furthest apart. 6 is as far away from 3 as 7. 4 is in the middle.

Spectral partitioning (simplified)

1. Given the Fiedler vector v_2, reorder its elements in ascending order. This defines a permutation P of the integers $\{1, 2, \ldots, n\}$, which induces a reordering of the vertices of the graph.
2. Apply the permutation to the vertices of the graph, and modify the set of edges accordingly. The adjacency matrix of the reordered graph is $\widetilde{A} = PAP^T$.
3. For each partitioning in a neighborhood of the sign change, compute the corresponding cost for partitioning the graph. Choose the partitioning with the smallest conductance.

Cost: The number of edges that are broken.

Karate club graph



References

There is a huge literature on graphs and spectral partitioning. For a brief introduction, see Chapters 10 and 16 of

L. Eldén.
Matrix Methods in Data Mining and Pattern Recognition, Second Edition.
SIAM, 2019.