Singular Value Decomposition

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Singular Value Decomposition

Theorem (SVD)

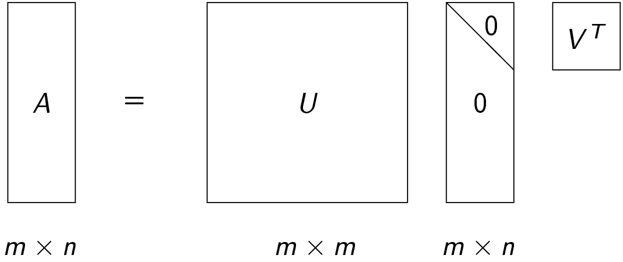
Any $m \times n$ matrix A, with $m \geq n$, can be factorized

$$A = U \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^T, \qquad (1)\text{span}$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal, and $\Sigma \in \mathbb{R}^{n \times n}$ is diagonal,

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n),$$
$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$$

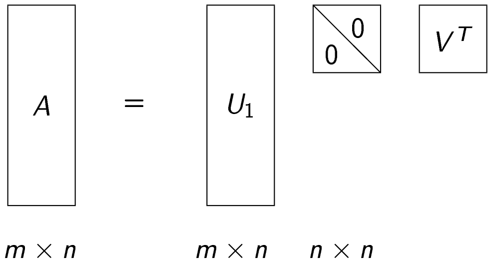
SVD



Thin SVD

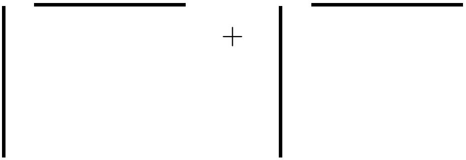
Partition $U = (U_1 \; U_2)$, where $U_1 \in \mathbb{R}^{m \times n}$,

$$A = U_1 \Sigma V^T,$$



SVD Expansion

$$A = \sum_{i=1}^{n} \sigma_i u_i v_i^T =$$



$$+ \quad \cdots$$

SVD of a matrix with full column rank I

```
A =  1     1
     1     2
     1     3
     1     4
 >> [U,S,V]=svd(A)

U = 0.2195   -0.8073    0.0236    0.5472
    0.3833   -0.3912   -0.4393   -0.7120
    0.5472    0.0249    0.8079   -0.2176
    0.7110    0.4410   -0.3921    0.3824

S = 5.7794         0
         0    0.7738
         0         0
         0         0
```

SVD of a matrix with full column rank II

```
V = 0.3220   -0.9467
    0.9467    0.3220
```

Thin SVD

```
>> [U,S,V]=svd(A,0)

U = 0.2195   -0.8073
    0.3833   -0.3912
    0.5472    0.0249
    0.7110    0.4410

S = 5.7794         0
         0    0.7738

V = 0.3220   -0.9467
    0.9467    0.3220
```

Rank deficient matrix I

```
>> A(:,3)=A(:,1)+0.5*A(:,2)

A = 1.0000    1.0000    1.5000
    1.0000    2.0000    2.0000
    1.0000    3.0000    2.5000
    1.0000    4.0000    3.0000

>> [U,S,V]=svd(A,0)

U = 0.2612   -0.7948   -0.5000
    0.4032   -0.3708    0.8333
    0.5451    0.0533   -0.1667
    0.6871    0.4774   -0.1667
```

Rank deficient matrix II

```
S = 7.3944         0         0
         0    0.9072         0
         0         0         0

V = 0.2565   -0.6998    0.6667
    0.7372    0.5877    0.3333
    0.6251   -0.4060   -0.6667
```

Fundamental Subspaces I

The *range of the matrix A:*

$$\mathcal{R}(A) = \{y \mid y = Ax, \text{ for arbitrary } x\}.$$

Assume that A has rank r:

$$\sigma_1 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0.$$

Outer product form:

$$y = Ax = \sum_{i=1}^{r} \sigma_i u_i v_i^T x = \sum_{i=1}^{r} (\sigma_i v_i^T x) u_i = \sum_{i=1}^{r} \alpha_i u_i.$$

The *null-space of the matrix* A:

$$\mathcal{N}(A) = \{x \mid Ax = 0\}.$$

Fundamental Subspaces II

$$Ax = \sum_{i=1}^{r} \sigma_i u_i v_i^T x$$

Any vector $z = \sum_{i=r+1}^{n} \beta_i v_i$ is in the null-space:

$$Az = (\sum_{i=1}^{r} \sigma_i u_i v_i^T)(\sum_{i=r+1}^{n} \beta_i v_i) = 0.$$

Fundamental Subspaces

Theorem (Fundamental subspaces)

1. *The singular vectors* $u_1, u_2, \dots, u_r$ *are an orthonormal basis in* $\mathcal{R}(A)$ *and*

$$\mathrm{rank}(A) = \dim(\mathcal{R}(A)) = r.$$

2. *The singular vectors* $v_{r+1}, v_{r+2}, \dots, v_n$ *are an orthonormal basis in* $\mathcal{N}(A)$ *and*

$$\dim(\mathcal{N}(A)) = n - r.$$

3. *The singular vectors* $v_1, v_2, \dots, v_r$ *are an orthonormal basis in* $\mathcal{R}(A^T)$.
4. *The singular vectors* $u_{r+1}, u_{r+2}, \dots, u_m$ *are an orthonormal basis in* $\mathcal{N}(A^T)$.

Null Space

The third column of V is a basis vector in $N(A)$:

```
>> A*V(:,3)

ans =
    1.0e-15 *
         0
   -0.2220
   -0.2220
         0
```

Eckart-Young Property

Frobenius norm

$$\|A\|_F = \left(\sum_{i,j} a_{ij}^2 \right)^{1/2}$$

Theorem

Assume that the matrix $A \in \mathbb{R}^{m \times n}$ has rank $r > k$. The Frobenius norm matrix approximation problem

$$\min_{\operatorname{rank}(Z)=k} \|A - Z\|_F$$

has the solution

$$Z = A_k = U_k \Sigma_k V_k^T,$$

Principal Component Analysis (PCA) I

Data matrix $\mathbb{R}^{m \times n} \ni X = U \Sigma V^T$

Each column is an observation of a real-valued random vector with mean zero.

The right singular vectors v_i are called principal components directions of X. The vector

$$z_1 = X v_1 = \sigma_1 u_1$$

has the largest sample variance amongst all normalized linear combinations of the columns of X:

$$\mathrm{Var}(z_1) = \mathrm{Var}(X v_1) = \frac{\sigma_1^2}{m}.$$

The normalized variable u_1 is called the normalized first principal component of X. The second principal component is the vector of largest sample variance of the deflated data matrix $X - \sigma_1 u_1 v_1^T$, and so on.

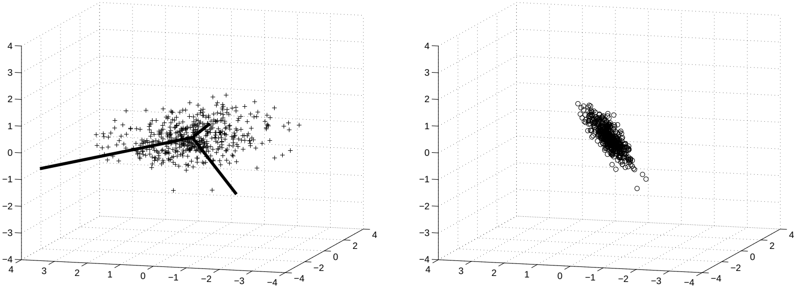


Figure: Cluster of points in $\mathbb{R}^3$ with (scaled) principal components (top). The same data with the contributions along the first principal component deflated (bottom).

Least Squares by SVD I

```
A =   1      1              b = 7.9700
      1      2                 10.2000
      1      3                 14.2000
      1      4                 16.0000
      1      5                 21.2000

>> [U1,S,V]=svd(A,0)

U1 =0.1600   -0.7579
    0.2853   -0.4675
    0.4106   -0.1772
    0.5359    0.1131
    0.6612    0.4035
```

Least Squares by SVD II

```
S = 7.6912         0
         0    0.9194

V = 0.2669   -0.9637
    0.9637    0.2669
```

```
>> x=V*(S\(U1'*b))

x = 4.2360
    3.2260
```

Least Squares by SVD, in R I

```
> A.svd<-svd(A)
> A.svd
$d
[1] 7.6912131 0.9193696

$u
          [,1]       [,2]
[1,] 0.1600071  0.7578903
[2,] 0.2853078  0.4675462
[3,] 0.4106086  0.1772020
[4,] 0.5359094 -0.1131421
[5,] 0.6612102 -0.4034862

$v
          [,1]       [,2]
```

Least Squares by SVD, in R II

```
[1,] 0.2669336  0.9637149
[2,] 0.9637149 -0.2669336

> x=A.svd$v %*% diag(1/A.svd$d) %*% t(A.svd$u) %*% b
> x
      [,1]
[1,] 4.236
[2,] 3.226
>
```

Linear dependence – SVD

Theorem

Let the singular values of A satisfy

$$\sigma_1 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0.$$

Then the rank of A is equal to r.

Rank = the number of linearly independent columns of A.

Linear dependence I

```
A=[1 1; 1 2; 1 3; 1 4]
singval=svd(A)

% Third col=linear combination of first two
A1=[A A(:,1)+0.5*A(:,2)]
singval1=svd(A1)

A =   1      1
      1      2
      1      3
      1      4
```

Result:

Linear dependence II

```
singval = 5.7794
          0.7738

A1 = 1.0000    1.0000    1.5000
     1.0000    2.0000    2.0000
     1.0000    3.0000    2.5000
     1.0000    4.0000    3.0000

singval1 = 7.3944
           0.9072
                0
```

Almost linear dependence I

```
A2=[A A(:,1)+0.5*A(:,2)+0.0001*randn(4,1)]
singval2=svd(A2)

--------------------------------------

A2 = 1.0000    1.0000    1.4999
     1.0000    2.0000    2.0001
     1.0000    3.0000    2.5000
     1.0000    4.0000    3.0001

singval2 = 7.3944
           0.9072
           0.0001
```