Matrices and Statistics with Applications
Pseudoinverses

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The inverse of a nonsingular matrix

Nothing easier:
If A is a square nonsingular matrix, then A^{-1} is a matrix of the same size as A, such that

$$A^{-1}A = AA^{-1} = I.$$

Properties:

I1 $(A^{-1})^{-1} = A$

I2 $(A^T)^{-1} = (A^{-1})^T$

I3 $(A^*)^{-1} = (^{-1})^*$

I4 $(AB)^{-1} = B^{-1}A^{-1}$

I5 If $A\mathbf{v} = \lambda\mathbf{v}$ and $A^{-1}\mathbf{w} = \mu\mathbf{w}$ then $\mu = 1/\lambda$.

A definition of a generalized inverse

Any matrix, satisfying

$$AXA = A.$$

Example: Solvability of a linear system $A\mathbf{x} = \mathbf{b}$.
Let $\mathbf{b}$ be in the range of A, i.e., there exist a vector $\mathbf{h}$, such that $\mathbf{b} = A\mathbf{h}$.
If X is a generalized inverse of A, then $\mathbf{x} = X\mathbf{b}$.
If $AXA = A$, then $A\mathbf{x} = AX\mathbf{b} = AXA\mathbf{h} = A\mathbf{h} = \mathbf{b}$

Generalized / Pseudo- inverses

- The Moore-Penrose Pseudoinverse
- The Drazin inverse
- Weighted generalized inverses, group inverses
- The Bott-Duffin inverse (for constrained matrices)

Moore-Penrose Pseudoinverse I

The Moore-Penrose pseudoinverse A^+ is defined for any matrix and is unique. Moreover, it brings notational and conceptual clarity to the study of solutions to arbitrary systems of linear equations and linear Least Squares problems.
Consider $A \in \mathbb{R}_r^{m,n}$. The subscript r denotes the rank of A.

Moore-Penrose Pseudoinverse II

Theorem (Penrose, 1956)

Let $A \in \mathbb{R}_r^{m,n}$. Then $G = A^+$ if and only if

- P1 $AGA = A$
- P2 $GAG = G$
- P3 $(AG)^* = AG$
- P4 $(GA)^* = GA$

Furthermore, A^+ always exists and is unique.

The theorem is not constructive but gives criteria that can be checked.

Moore-Penrose Pseudoinverse III

Example:
Let $A \in \mathbb{R}_r^{m,n}$. Then, from $A = U\Sigma V^T$ we find $A^+ = V\Sigma^+ U^T$, where $\Sigma^+ = \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix}$.

Moore-Penrose Pseudoinverse IV

Properties:

- $A^+ = (A^TA)^+A^T = A^T(AA^T)^+$
- $(A^T)^+ = (A^+)^T$
- $(A^+)^+ = A$
- $(A^TA)^+ = A^+(A^T)^+ = (A^T)^+A^+$
- $\mathcal{R}(A^+) = \mathcal{R}(A^T) = \mathcal{R}(A^+A) = \mathcal{R}(A^TA)$
- $\mathcal{N}(A)^+ = \mathcal{N}(AA^+) = \mathcal{N}((AA^T)^+) = \mathcal{N}(AA^T) = \mathcal{N}(A^T)$

Moore-Penrose Pseudoinverse V

For linear systems $A\mathbf{x} = \mathbf{b}$ with non-unique solutions (such as under-determined systems), the pseudoinverse may be used to construct the solution of minimum Euclidean norm $\|\mathbf{x}\|_2$ among all solutions.
If $A\mathbf{x} = \mathbf{b}$ is consistent, the vector $\mathbf{x} = A^{+}\mathbf{b}$ is a solution, and satisfies $\|\mathbf{z}\|_2 \leq \|\mathbf{x}\|_2$ for all solutions.

Uniqueness of the Moor-Penrose inverse I

Let $A \in \mathbb{R}_r^{m,n}$. Assume that there are two matrices that satisfy the conditions:

$$\begin{array}{ll} AA^+A = A & ABA = A \\ A^+AA^+ = A^+ & BAB = B \\ (AA^+)^* = AA^+ & (AB)^* = AB \\ (A^+A)^* = A^+A & (BA)^* = BA \end{array}$$

Let $M_1 = AB - AA^+ = A(B - A^+)$. By the hypothesis, M_1 is self-adjoint (since it is the difference of two self-adjoint matrices) and

$$\begin{aligned} (M_1)^2 &= (AB - AA^+)A(B - A^+) \\ &= (ABA - AA^+A)(B - A^+) = (A - A)(B - A^+)A = 0. \end{aligned}$$

Uniqueness of the Moor-Penrose inverse II

Since M_1 is self-adjoint, the fact that $M_1^2 = 0$ implies that $M_1 = 0$, since for all x one has $\|M_1 x\|^2 = (M1x, M1x) = (x, (M1)^2 x) = 0$, implying $M_1 = 0$. This showed that $AB = AA^+$.

Following the same steps we can prove that $BA = A^+A$ (consider the self-adjoint matrix $M2 := BAA + A$ and proceed as above). Thus, $A^+ = A^+AA^+ = A^+(AA^+) = A^+AB = (A^+A)B = BAB = B$, thus A^+ is unique.

The Drazin Inverse

Defined for a square matrix.
Let A be a square matrix. The index k of A is the least nonnegative integer k such that $rank(A^{k+1}) = rank(A^k)$.
The Drazin inverse of A is the unique matrix A^D which satisfies

$$A^{k+1}A^D = A^k, \quad A^D A A^D = A^D, \quad A A^D = A^D A.$$

If A is invertible with inverse A^{-1}, then $A^D = A^{-1}$.